$H^2(G, \mathbb{Z}/2\mathbb{Z})$

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Contents

1 Introduction: space group and point group

A space group SG of a crystalographic lattice are the group of operations that leave the lattice invariant (in physics language, the symmetry group of the lattice). There are in total 230 of space groups for the 230 three-dimensional crystallographic lattices. For any spatial dimensions, one can define the concept of crystallographic lattice and its corresponding space group. The number of space groups for dimension d is

Spatial dimension d	2	3	4
Number of space groups	17	230	4894

For any space group, there is a natural definition of translation group which is a subgroup of a space group. In d-spatial dimensions, the translation group is isomorphic to $\mathbb{Z}^d \equiv \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{d \text{ copies of } \mathbb{Z}}$. In fact, the translation group is always a normal subgroup of the space group (for a proof in three-dimensions, see e.g. Dresselhaus, Group Theory: Application to the Physics of Condensed Matter).

Suppose G is a group and $N \lhd H$. It is a general fact that the quotient group H/N need not be isomorphic to any subgroup of G. However, in the case of G being a space group SG and N being the translation group, it is true that the quotient group $SG/\mathbb{Z}^d$ is always isomorphic to a subgroup of SG, which we call PG, the point group of SG. We have $PG \cong SG/\mathbb{Z}^d$, or

$$0 \to \mathbb{Z}^d \to SG \to PG \to 1.$$

Obviously, SG has infinite order and the point group has finite order. In three-dimensions, there are 32 different point groups in the crystallographic sense, which belong to 18 different abstract groups (different point groups may be isomorphic and thus correspond to the same abstract group; however they are different crystallographically because the physical operations are diferent). It is worth noticing the following points:

1. It is true that $SG/\mathbb{Z}^d$ is always isomorphic to a subgroup of SG. However, it is wrong to think of $SG/\mathbb{Z}^d$ as defined by the subset of SG of which the translation part of each element is set to zero. In fact this can be ill-defined, since there may exist elements of SG for which "setting their translation part to zero" would result in an lattice operation outside SG. And this is precisely the case when there are elements of SG who is a fraction of a translation (best illstrated in the space group example in next section, in group relation $S^2T_3 = 1$). The space group can then be roughly divided into two cases, corresponding to this definition is well-defined or ill-defined: the symmorphic space group and the non-symmorphic space group.
2. The symmorphic space groups are the space groups SG satisfying $SG \cong \mathbb{Z}^d \rtimes SG/\mathbb{Z}^d$, i.e. SG is the semi-direct product of the translation group and the space group. Note that there is no way to obtain a space group by taking direct proeuct of translation and point group, since rotation/mirror/glide operators can by no means commute with all the translations. Equivalently, symmorphic space groups are those space groups that contain an isomorphic copy of the point group PG (note that PG by definition is the quotient group $PG := SG/\mathbb{Z}^d$). In this case, the space group contains an isomorphic copy of the point group. The opposite is also true: if the space group is nonsymmorphic (i.e. not symmorphic), then it does not contain an isomorphic copy of the point group. (Semidirect product group $G = H \rtimes Q$ contains an isomorphic copy of $Q = G/H$.)
3. The non-symmorphic space group is when SG is not semi-direct product of the translation group and the space group.
4. In point 2. and 3. symmorphic and non-symmorphic space groups are defined in the mathematical sense. A proof should exist in matching these definitions with physical definition 1.

Below we will proceed to calculate the second group cohomology of a non-symmorphic space group.

2 The No. 227 space group

Here, the symmetry group SG in consideration is the No.227 space group $Fd\bar{3}m$ (No. 227 out of the 230 $3d$ space groups), minimally generated by $\{T_1, T_2, T_3, \overline{C}_6, S\}$, where T_1, T_2, T_3 are the generators of the translation group $\mathbb{Z}^3$, and $\overline{C}_6$ and S are the generators of the cubic group O_h.[1] I.e. we have the structure

$$0 \to \mathbb{Z}^3 \to SG \to O_\mathrm{h} \to 1.$$

One can check (using the relations below) that the O_h action on $\mathbb{Z}^3$ by conjugation is not trivial. The set of relations can be chosen as

$$T_1 T_2 T_1^{-1} T_2^{-1} = 1, \tag{1a}$$
$$T_2 T_3 T_2^{-1} T_3^{-1} = 1, \tag{1b}$$
$$T_3 T_1 T_3^{-1} T_1^{-1} = 1, \tag{1c}$$
$$\overline{C}_6^6 = 1, \tag{1d}$$
$$S^2 T_3^{-1} = 1, \tag{1e}$$
$$\overline{C}_6 T_1 \overline{C}_6^{-1} T_2 = 1, \tag{1f}$$
$$\overline{C}_6 T_2 \overline{C}_6^{-1} T_3 = 1, \tag{1g}$$
$$\overline{C}_6 T_3 \overline{C}_6^{-1} T_1 = 1, \tag{1h}$$
$$S T_1 S^{-1} T_3^{-1} T_1 = 1, \tag{1i}$$
$$S T_2 S^{-1} T_3^{-1} T_2 = 1, \tag{1j}$$
$$S T_3 S^{-1} T_3^{-1} = 1, \tag{1k}$$
$$(\overline{C}_6 S)^4 = 1, \tag{1l}$$
$$(\overline{C}_6^3 S)^2 = 1. \tag{1m}$$

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[1] The cubic group O_h can be written abstractly as $O_\mathrm{h} \simeq S_4 \times \mathbb{Z}_2$, where S_4 is the symmetry group of four elements $\{1, 2, 3, 4\}$, and we regard $\mathbb{Z}_2$ as the symmetry group of two elements $\{+, -\}$. This way, the elements $\overline{c}_6, s \in O_\mathrm{h}$, defined as the conjugate class $\overline{C}_6\mathbb{Z}^3$ and $S\mathbb{Z}^3$, can be written as $\overline{c}_6 = (123)(+-)$, and $s = (14)(+-)$.

Due to the fractionalization of T_3 by S^2 (see the relation $S^2T_3^{-1} = 1$), this space group is non-symmorphic. This can also be confirmed by proving that this space group is not semi-direct product of $\mathbb{Z}^3$ and O_h: if it were semi-direct product of $\mathbb{Z}^3$ and O_h, then the lifting $p\colon O_\mathrm{h} \to SG$, with $p(s) = S$ should be a group homomorphism. However this is clearly contradictary since S has infinite order (due to $S^2 = T_3$) and $(14)(+-)$ has order two.

The relations also offer a short exact sequence

$$0 \to \mathcal{R} \to F \to SG \to 1$$

obtained from the presentation of group $SG \cong F/\mathcal{R}$, where $F = \langle T_1, T_2, T_3, \overline{C}_6, S\rangle$ is the free group of five elements, and $\mathcal{R}$ is the normal closure in F generated by the above 13 relations.

3 Second group cohomology with $\mathbb{Z}_2$ coefficients

We want to compute $H^2(G, \mathbb{Z}_2)$ which arises from a group extension problem

$$0 \to \mathbb{Z}_2 \to \widetilde{G} \to G \to 1,$$

where $\widetilde{G}$ is the group extension of G by $\mathbb{Z}_2$[2].

For any $H^n(G, M)$ arising from $0 \to M \to \widetilde{G} \to G \to 1$ (where M is an abelian group), there is a pre-defined G-module structure on M (i.e. M is a G-module), defined by $G \to \mathrm{End}(M)$, $g \mapsto \gamma_g$, where γ_g is simply conjugation on M: $\gamma_g\colon M \to M$, $m \mapsto gmg^{-1}$.

For a given G-module structure on M, the second group cohomology $H^2(G, M)$ characterzes the inequivalent group extensions we can get; the number of inequivalent extensions corresponds to the order of the cohomology group $H^2(G, M)$. However, keep in mind that two extensions $\widetilde{G}$ and $\widetilde{G}'$ are defined as inequivalent if they cannot fit into a commutative diagram

$$\begin{tikzcd}
 & & \widetilde{G} \arrow[rd, "\pi"] \arrow[dd, dashed, leftrightarrow, "\gamma"] & & \\
0 \arrow[r] & M \arrow[ru, hook, "i"] \arrow[rd, hook, "i'"'] & & G \arrow[r] & 1 \\
 & & \widetilde{G}' \arrow[ru, "\pi'"'] & &
\end{tikzcd}$$

The inequivalence is not a statement about whether $\widetilde{G}$ and $\widetilde{G}'$ are isomorphic. In fact, it can happen that isomorphic groups $\widetilde{G}$ and $\widetilde{G}'$ (or even $\widetilde{G} = \widetilde{G}'$) correpond to different extensions (and hence corresponds to different element of $H^2(G, M)$).

In $H^2(G, \mathbb{Z}_2)$ which we are considering, the G-module structure on $\mathbb{Z}_2$ is the trivial one. I.e. we only consider central extension.

The main tool we will use is the Lyndon-Hochschild-Serre spectral sequence: if we can find a normal group $N \lhd G$, i.e.

$$0 \to N \to G \to G/N \to 1$$

the Lyndon-Hochschild-Serre (LHS) spectral sequence gives, from the first few pages, a five-term exact sequence

$$0 \to H^1(G/N, M^N) \xrightarrow{\mathrm{inf}} H^1(G, M) \xrightarrow{\mathrm{res}} H^1(N, M)^{G/N} \xrightarrow{\mathrm{tg}} H^2(G/N, M^N) \xrightarrow{\mathrm{inf}} H^2(G, M). \tag{2}$$

Here the maps inf, res and tg stand for inflation, restriction and transgression. The maps and cohomology groups have very concrete meanings:

- $H^1(G, M)$ is defined as the $\mathrm{Hom}(G, M)$. It is a vector space of characters $\chi\colon G \to M$. (This definition works only when M is a G-trivial module.)
- The restriction map is defined as $\mathrm{res}\colon \chi \to \chi|_N$, where $\chi|_N\colon N \to M$ is simply χ restricted to N.
- It can be proved that the image of the retriction map is G/N-stable: i.e. the G/N acts trivially on $\chi|_N$. The G/N-action on $\chi \in H^1(N, M)$ is defined as $\chi(n) \to \chi(gng^{-1})$, for any $n \in N$ and any $g \in G/N$. G/N-stable means that $\chi|_N(n) = \chi|_N(gng^{-1})$.

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[2] In this note we will always write $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}_2$.

The above five-term exact sequence can be applied to another short exact sequence: from the presentation of a group

$$0 \to \mathcal{R} \to F \to G \to 1$$

we have

$$0 \to H^1(G, M^{\mathcal{R}}) \xrightarrow{\text{inf}} H^1(F, M) \xrightarrow{\text{res}} H^1(\mathcal{R}, M)^G \xrightarrow{\text{tg}} H^2(G, M^{\mathcal{R}}) \xrightarrow{\text{inf}} H^2(F, M). \tag{3}$$

Note the following:

- We have $M^{\mathcal{R}} = M$ for trivial G-action on M.
- Keep in mind that $\mathcal{R}$ is the normal closure generated by the independent relations defining G. In other words, $\mathcal{R} = \{frf^{-1} | f \in F, r \in \mathcal{R}_0\}$, where we defined $\mathcal{R}_0$ to be the subgroup of $\mathcal{R}$ generated by the independent relations in G. $\mathcal{R}$ and $\mathcal{R}_0$ are all free group, since they are subgroups of F.
- Due to non-trivial G-module structure on $\mathcal{R}$, we do not have $H^1(\mathcal{R}, M)^G \cong H^1(\mathcal{R}, M)$; but instead we have $H^1(\mathcal{R}, M)^G \cong H^1(\mathcal{R}_0, M)$ where $\mathcal{R}_0$ is the subgroup of $\mathcal{R}$ generated by independent relations that define G. However, there is a defining difference: the elements of $H^1(\mathcal{R}, M)^G$ are characters χ defined on $\mathcal{R}$, while the elements of $H^1(\mathcal{R}_0, M)$ are characters χ_0 defined on $\mathcal{R}_0$. The calulations in the examples below largely depend on the fact that χ is define on $\mathcal{R}$. To be more explicit, any of the argument of χ can be written as frf^{-1} where $f \in F$ and r is any of the relation in $\mathcal{R}_0$, and we have $\chi(frf^{-1}) = \chi(r)$.
- Take $M = \mathbb{Z}_2$. Although $\mathcal{R}_0$ is the free group generated by the independent relations of G, which are usually easy to find, the group $H^1(\mathcal{R}_0, \mathbb{Z}_2)$ may not be easy to determine. This is because, the number of generators of $\mathcal{R}_0$ only defines the upper dimension of $H^1(\mathcal{R}_0, \mathbb{Z}_2)$; it may not equal the dimension oh $H^1(\mathcal{R}_0, \mathbb{Z}_2)$, which is really the dimension of vector space formed by the G-stable characters $\chi\colon \mathcal{R} \to M$. To find out the dimension of $H^1(\mathcal{R}_0, M)$, we must apply characters on $\mathcal{R}_0$ to see if some characters are really the same. As we shall see, this is really the only part of the calculation we do not have absolute control of (i.e. we cannot give a proof).
- For $M = \mathbb{Z}_2$ and any character χ, we have $\chi([R, F]R^2) = 0$, i.e. $[R, F]R^2$ sits in the kernel of any χ.
- For finite abelian group $M = \langle 1, m, ..., m^{l-1}\rangle$ with order m and free group $F_n = \langle f_1, ..., f_n\rangle$ (and trivial action), we have $H^1(F_n, M) = M^n$, generated by the characters $\chi_{i\mu}\colon F_n \to M$ with $\chi_{i\mu}(f_j) = \delta_{ij} a^{\mu}$, where $i, j = 1, ..., n$ and $\mu = 0, ..., l-1$. We have $H^{n\geq 2}(F_n, M) = 0$.

We will illustrate these ideas in the examples below.

4 Apply to $G = S_4$: a simply example

We study a simply example: $H^2(S_4, \mathbb{Z}_2)$, where S_4 is the symmetric group (permutation group on four elements). Again action is trivial.

From Adem, Lectures on the Cohomology of Finite Groups, Example 4.4 on page 9 we know that $H^2(S_4, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$, i.e. the Klein four group. Here we check this result by the group presentation exact sequence (3) above. Denote the generators of S_4 as k, r, with group relations $k^3, r^2, (kr)^4$. The group is presented as

$$0 \to \mathcal{R} = \overline{\langle k^3, r^2, (kr)^4\rangle} \to \langle k, r\rangle \to S_4 \to 1.$$

The exact sequence (3) becomes

$$0 \to H^1(S_4, \mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F, \mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R}, \mathbb{Z}_2)^{S_4} \xrightarrow{tg} H^2(G, \mathbb{Z}_2) \to 0,$$

The overline bar reminds that it is the normal closure. We can calculate the relevant places in (3). First, note that $F = \langle k, r\rangle$ so $H^1(F, \mathbb{Z}_2) = \mathbb{Z}_2^2$, generated by χ_k and χ_r, where $\chi_k(k) = 1, \chi_k(r) = 0, \chi_r(k) = 0, \chi_r(r) = 1$. After mapping under restriction, χ_r gets sent to zero (since $\chi_r(k^3) = 0$, $\chi_r(r^2) = 2\chi_r(r) = 0$ and $\chi_r((kr)^4) = \chi_r(r^4) = 4\chi_r(r) = 0$) and χ_k is still non-trivial, therefore im(res) is $\mathbb{Z}_2$ and ker(res) $= \mathbb{Z}_2$; note that ker(inf) $= 0$ so inf is injective, and we have $H^1(S_4, \mathbb{Z}_2) \cong$ im(inf) $=$ ker(res) $\cong \mathbb{Z}_2$. Then, we have to determine $H^1(\mathcal{R}, \mathbb{Z}_2)^{S_4} \cong H^1(\mathcal{R}_0, \mathbb{Z}_2)$. Since $\mathcal{R}_0 = \langle k^3, r^2, (kr)^4\rangle$, $H^1(\mathcal{R}_0, \mathbb{Z}_2)$ has maximal three dimensions; and we are pretty sure that these three dimensions can no loger be reduced, therefore we assert that $H^1(\mathcal{R}_0, \mathbb{Z}_2) = \mathbb{Z}_2^3$ (in fact it must be three dimensional, considering we already know that $H^2(S_4, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$), which is spanned by χ_{k^3}, χ_{r^2} and $\chi_{(kr)^4}$ defined as characters nontrivial only on their respective subscripts. therefore $H^2(S_4, \mathbb{Z}_2) =$ im(tg) $= H^1(\mathcal{R}_0, \mathbb{Z}_2)/$im(res) $= \mathbb{Z}_2^2$, generated by χ_{r^2} and $\chi_{(rk)^4}$; the one being modded out is χ_{k^3} since $\chi_{k^3} = \text{res}(\chi_k) = \chi_k|_{\mathcal{R}}$. This checks the result for S_4. more explicitly, we have

$$H^2(S_4, \mathbb{Z}_2) \cong \langle \chi_k^3, \chi_{r^2}, \chi_{(kr)^4}\rangle / \langle \chi_k\rangle \cong \langle \chi_{r^2}, \chi_{(kr)^4}\rangle \cong \mathbb{Z}_2^2.$$

Note this agrees with the formula given on Adem's paper [3], Example 4.4:

$$H^*(S_4) \cong \mathbb{F}_2[x_1, y_2, c_3]/(x_1 c_3),$$

where x_1, y_2, c_3 are of degree 1, 2, 3, respectively. This formula gives the full ring structure of all cohomology group of S_4. In particular, we have

$$H^{1,2,3}(S_4, \mathbb{Z}_2) = \langle x_1 \rangle, \langle x_1 \cup x_1, y_2 \rangle, \langle x_1 \cup x_1 \cup x_1, x_1 \cup y_2, c_3 \rangle \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3.$$

Note that $[S_4, S_4] = A_4$ and $S_4/[S_4, S_4] = \mathbb{Z}_2$, meaning that $H^1(S_4, \mathbb{Z}_2) = \mathrm{Hom}(S_4/[S_4, S_4], \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.

5 Apply to $G = O_\mathrm{h}$

Since $O_\mathrm{h} \cong S_4 \times \mathbb{Z}_2$, we simply have to use the previous result and Künneth formula. Künneth formula in this case is given by

$$\begin{aligned}
&H^1(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) \cong H^1(S_4, \mathbb{Z}_2) \oplus H^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2^2, \\
&H^2(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) \cong H^2(S_4, \mathbb{Z}_2) \oplus \big(H^1(S_4, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2)\big) \oplus H^2(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2^4, \\
&H^3(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) \cong H^3(S_4, \mathbb{Z}_2) \oplus \big(H^2(S_4, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2)\big) \oplus \big(H^1(S_4, \mathbb{Z}_2) \otimes H^2(\mathbb{Z}_2, \mathbb{Z}_2)\big) \oplus H^3(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2^7,
\end{aligned} \tag{4}$$

Note that $H^2(S_4, \mathbb{Z}_2) = \mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$, so

$$H^2(S_4, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \otimes \mathbb{Z}_2 = (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \times (\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where we used the fact that tensor products are ditributive over direct products (here direct product is the same as direct sum), and that $\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_{\gcd(m,n)}$ (which gives $\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2$), and that $H^n(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for trivial action (see, e.g. Rotman, An Introduction to Homological Algebra, Corollary 9.29, P523). Note the above has been checked in the GAP software. To summarize, we have

$$H^*(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{F}_2[x_1, y_2, c_3]/(x_1 c_3) \otimes \mathbb{F}_2[x_2] = \mathbb{F}_2[x_1, x_2, y_2, c_3]/(x_1 c_3), \tag{5}$$

where

$$H^*(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{F}_2[x_2], \tag{6}$$

$x_{1,2}$ have order 1, y_2 is order 2, and c_3 is order 3.

To count this number, first notice that x_1, y_2 and c_3 has no quotient part. The number of terms that can be constructed can be obtained using `https://math.stackexchange.com/questions/1529121/the-number-pn-of-triplets-x-y-z-x2y3z-n`, there it's mentioned that this problem appears in Polya, Szego "Problems and theorems in analysis" P27, and the answer is $p(n) = \text{Round}\left[\frac{(n+3)^2}{2}\right]$, where Round is to take it to the closest integer. Then, x_1, x_2, y_2 satisfies equation $x + y + 2z = n$ where $x \geq 1$, so $x' + y + 2z = (n-1)$, The answer is $q(n) = \text{Round}[\frac{(n+1)^2}{4}]$ (see `https://math.stackexchange.com/questions/3538889/how-many-pairs-of-x-y-z-for-xy2z-n`). So the final answwre is

$$f(n) = p(n) + q(n) = \text{Round}\left[\frac{(n+3)^2}{2}\right] + \text{Round}\left[\frac{(n+1)^2}{4}\right],$$

$$f(0,1,2,3,4,5,6,7,8) = (1,2,4,7,10,14,19,24,30),$$

These numbers have been checked in Mathematica by the code

`Table[Tally[Flatten[Table[(i + j + 2 k + 3 l == n && (i == 0` $|l == 0)), i, 0, 20, j, 0, 20, k, 0, 20, l, 0, 20]]], n, 0, 20]|$

`Table[Round[(n + 3)^2/12] + Round[(n + 1)^2/4], {n, 0, 50}]`

to be correct, up to $n \leq 20$. The $n = 0, 1, ..., 5$ terms have also been checked in GAP to be correct.

To see the structure of $H^2(O_\mathrm{h}, \mathbb{Z}_2)$ more clearly, we again use a group presentation analysis. Name the extra generator in $\mathbb{Z}_2 = O_\mathrm{h}/S_4$ t (corresponding to spatial inversion), then we have

$$0 \to \mathcal{R} = \overline{\langle k^3, r^2, (kr)^4, u^2, [t,k], [t,r] \rangle} \to \langle k, r, t \rangle \to S_4 \times \mathbb{Z}_2 \to 1,$$

and we have

$$0 \to H^1(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F_3, \mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R}, \mathbb{Z}_2)^{S_4 \times \mathbb{Z}_2} \xrightarrow{tg} H^2(S_4 \times \mathbb{Z}_2, \mathbb{Z}_2) \to 0.$$

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[3] See `http://homepages.math.uic.edu/~bshipley/ConMcohomology1.pdf`, Lectures on the Cohomology of Finite Groups, Alejandro Adem

Now out of the three independent characters in $H^1(F_3,\mathbb{Z}_2)$, which are χ_k, χ_r and χ_t, only χ_k survives in the image of restriction, i.e. $\mathrm{im}(\mathrm{res})=\mathbb{Z}_2$, therefore $H^1(O_\mathrm{h},\mathbb{Z}_2)\cong\mathrm{im}(\mathrm{inf})=\ker(\mathrm{res})=\mathbb{Z}_2^2$.

Then we need to determine the dimension of $H^1(\mathcal{R},\mathbb{Z}_2)^{O_\mathrm{h}}$. This is not a trivial procedure. The six characters χ_{k^3}, χ_{r^2}, $\chi_{(kr)^4}$, χ_{t^2}, $\chi_{[t,k]}$ and $\chi_{[t,r]}$. May not be all independent. Suppose $\chi\colon O_\mathrm{h}\to\mathbb{Z}_2$ is any character. Then in fact we have $\chi(k^3)=\chi([t,k])$. This can be seen from the follows:

- Denote $[t,k]=\omega_{tk}$ for simplicity. Then we have $tkt^{-1}k^{-1}=\omega_{tk}$, or $tkt^{-1}=\omega_{tk}k$, or $tk^3t^{-1}=(\omega_{tk}k)^3=\omega_{tk}k^3(k^{-2}\omega_{tk}k^2)(k^{-1}\omega_{tk}k)$, therefore apply character χ and keep in mind the property that $\chi(frf^{-1})=\chi(r)$, then we have $\chi(k^3)=\chi(\omega_{tk})+\chi(k^3)+\chi(\omega_{tk})+\chi(\omega_{tk})$, therefore we must have $\chi(\omega_{tk})=0$, i.e. any character χ acts trivially on ω_{tk}.

We see that any character is trivial on $[t,k]$. Therefore this shows that $H^1(\mathcal{R},\mathbb{Z}_2)^{O_\mathrm{h}}$ can have at most five dimensions, and therefore $H^2(O_\mathrm{h},\mathbb{Z}_2)\cong H^1(\mathcal{R},\mathbb{Z}_2)^{S_4\times\mathbb{Z}_2}/\mathrm{im}(\mathrm{res})$ has at most four dimensions. This is already the correct answer from (25).

The structure of $H^2(O_\mathrm{h},\mathbb{Z}_2)$ is

$$H^2(O_\mathrm{h},\mathbb{Z}_2)\cong\langle\chi_{k^3},\chi_{r^2},\chi_{(kr)^4},\chi_{t^2},\chi_{[t,r]}\rangle/\langle\chi_k\rangle\cong\langle\chi_{r^2},\chi_{(kr)^4},\chi_{t^2},\chi_{[t,r]}\rangle\cong\mathbb{Z}_2^4$$

where the last step is because $\chi_k|_\mathcal{R}=\chi_{k^3}$.

This example shows that, as mentioned before, quite often the non-trivial part of the calculation is to determine the dimension of $H^1(\mathcal{R},M)^G$.

We can use another generator set for O_h:

$$0\to\mathcal{R}=\overline{\langle s^2,c^6,(cs)^4,(c^3s)^2\rangle}\to\langle s,c\rangle\to O_\mathrm{h}\to 1,$$

and we have

$$0\to H^1(O_\mathrm{h},\mathbb{Z}_2)\xrightarrow{\mathrm{inf}}H^1(F_2,\mathbb{Z}_2)\xrightarrow{\mathrm{res}}H^1(\mathcal{R},\mathbb{Z}_2)^{O_\mathrm{h}}\xrightarrow{tg}H^2(O_\mathrm{h},\mathbb{Z}_2)\to 0.$$

Compare to the group presentation above, we simply have $c=kt$ and $s=rt$.

Now neither of the two independent characters in $H^1(F_2,\mathbb{Z}_2)$, which are χ_c and χ_s, survives to the image of restriction. This can be checked by noticing that $H^1(O_\mathrm{h},\mathbb{Z}_2)=\mathbb{Z}_2^2$, and $H^1(F_2,\mathbb{Z}_2)=\mathbb{Z}_2^2$, therefore inf is isomorphism therefore res must be trivial. This is in fact saying that both $H^1(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)\xrightarrow{\mathrm{inf}}H^1(F_2,\mathbb{Z}_2)$ and $H^1(\mathcal{R},\mathbb{Z}_2)\xrightarrow{tg}H^2(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)$ are isomorphism; the former isomorphism can be explicitly obtained: note $\mathrm{inf}\colon H^1(O_\mathrm{h},\mathbb{Z}_2)\to H^1(F_2,\mathbb{Z}_2)$ is injective; in fact $H^1(O_\mathrm{h},\mathbb{Z}_2)$ is simply the homomorphism group $O_\mathrm{h}\to\mathbb{Z}_2$ and is generated by χ_c and χ_s, hence isomorphic to $\mathbb{Z}_2^2$. On the other hand, $H^1(F_2,\mathbb{Z}_2)=\mathbb{Z}_2^2$ so the inflation map is isomorphic.

Now we need to determine $H^1(\mathcal{R},\mathbb{Z}_2)^{S_4\times\mathbb{Z}_2}$, which we know has upper dimension four. In fact, this must be the dimension of $H^1(\mathcal{R},\mathbb{Z}_2)^{S_4\times\mathbb{Z}_2}$ since we already know $H^2(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)$ is four dimensional. Therefore we explicitly have

$$H^2(O_\mathrm{h},\mathbb{Z}_2)\cong\langle\chi_{s^2},\chi_{c^6},\chi_{(cs)^4},\chi_{(c^3s)^2}\rangle. \tag{7}$$

6 Apply to space group $G=Fd\overline{3}m$

Let us label the thirteen relations defined in Eqs. (1) by thirteen ω's:

$$T_1T_2T_1^{-1}T_2^{-1}=\omega_1, \tag{8a}$$
$$T_2T_3T_2^{-1}T_3^{-1}=\omega_2, \tag{8b}$$
$$T_3T_1T_3^{-1}T_1^{-1}=\omega_3, \tag{8c}$$
$$\overline{C}^6=\omega_C, \tag{8d}$$
$$S^2T_3^{-1}=\omega_S, \tag{8e}$$
$$\overline{C}_6T_1\overline{C}_6^{-1}T_2=\omega_{C1}, \tag{8f}$$
$$\overline{C}_6T_2\overline{C}_6^{-1}T_3=\omega_{C2}, \tag{8g}$$
$$\overline{C}_6T_3\overline{C}_6^{-1}T_1=\omega_{C3}, \tag{8h}$$
$$ST_1S^{-1}T_3^{-1}T_1=\omega_{S1}, \tag{8i}$$
$$ST_2S^{-1}T_3^{-1}T_2=\omega_{S2}, \tag{8j}$$
$$ST_3S^{-1}T_3^{-1}=\omega_{S3}, \tag{8k}$$
$$(\overline{C}_6S)^4=\omega_{CS}, \tag{8l}$$
$$(\overline{C}_6^3S)^2=\omega_{SC}, \tag{8m}$$

Therefore $\mathcal{R}_0 = \langle \omega_1, \omega_2, \omega_3, \omega_C, \omega_S, \omega_{C1}, \omega_{C2}, \omega_{C3}, \omega_{S1}, \omega_{S2}, \omega_{S3}, \omega_{CS}, \omega_{SC} \rangle$ and $\mathcal{R} = \overline{\mathcal{R}_0}$. We then have

$$0 \to H^1(SG, \mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F_5, \mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R}, \mathbb{Z}_2)^{SG} \xrightarrow{\text{tg}} H^2(SG, \mathbb{Z}_2) \xrightarrow{\text{inf}} 0, \tag{9}$$

again we have $H^1(F_5, \mathbb{Z}_2) = \mathbb{Z}_2^5$, generated by χ_{T_1}, χ_{T_2}, χ_{T_3}, $\chi_{\overline{C}_6}$ and χ_S. When restricted to $\mathcal{R}$, $\chi_{\overline{C}_6}$ and χ_S will become tivial, since all the relations in $\mathcal{R}_0$ (and hence in $\mathcal{R}$) have either even numbers of $\overline{C}_6$ or S, or $\chi_o(\overline{C}_6)$ and $\chi_o(S)$ cancel with $\chi_o(\overline{C}_6^{-1})$ and $\chi_o(S^{-1})$, for $o = \overline{C}_6$ or S. It can be seen from the thirteen relations that χ_{T_1}, χ_{T_2} and χ_{T_3} are not trivial on $\mathcal{R}$. Therefore

$$\text{im(res)} \cong \langle \chi_{T_1}, \chi_{T_2}, \chi_{T_3} \rangle \cong \mathbb{Z}_2^3.$$

Next, we need to find $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$, which amounts to finding the dimension m as in $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG} \cong \mathbb{Z}_2^m$. We know that the upper dimension is 13, spanned by thirteen χ_ω's corresponding to the thirteen ω's; however not all these thirteen characters χ_ω's are independent, and we will reduce number in independent characters by the following:

- For any character χ, we must have $\chi(\omega_{S3}) = 0$. This is because $\chi(\omega_{S3}) = \chi(ST_3S^{-1}T_3^{-1}) = \chi(S(T_3S^{-2})S^{-1}S^2T_3^{-1}) = \chi(S\omega_S^{-1}S^{-1}\omega_S) = \chi(S\omega_S^{-1}S^{-1}) + \chi(\omega_S) = \chi(\omega_S) + \chi(\omega_S) = 0$.

- For any character χ, we have $\chi(\omega_{C1}\omega_{C2}\omega_{C3}) = 0$. To prove this, notice that $\chi(\omega_{C3}^{-1}) = \chi(\overline{C}_6T_1\omega_{C3}^{-1}T_1^{-1}\overline{C}_6^{-1})$, $\chi(\omega_{C2}^{-1}) = \chi(\overline{C}_6^{-1}T_3\omega_{C2}^{-1}T_3^{-1}\overline{C}_6)$, therefore $\chi(\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}) = \chi(\overline{C}_6T_1\omega_{C3}^{-1}T_1^{-1}\overline{C}_6^{-1}\omega_{C1}\overline{C}_6^{-1}T_3\omega_{C2}^{-1}T_3^{-1}\overline{C}_6) = \chi(\overline{C}_6^2T_3^{-1}\overline{C}_6^{-3}T_3^{-1}\overline{C}_6$ $\chi(T_3^{-1}\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3) = \chi(S^2T_3^{-1}\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3S^{-2}) = \chi(\omega_S\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3S^{-2}) = \chi(\omega_S) + \chi(\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3S^{-2}) = \chi(\omega_S) + \chi(T_3^{-1}\overline{C}_6^{-3}S^{-2}\overline{C}_6^3) = \chi(\omega_S) + \chi(S^2T_3^{-1}\overline{C}_6^{-3}S^{-2}\overline{C}_6^3S^{-2}) = \chi(\omega_S) + \chi(\omega_S\overline{C}_6^{-3}S^{-2}\overline{C}_6^3S^{-2}) = \chi(\overline{C}_6^{-3}S^{-2}\overline{C}_6^3S^{-2}) = \chi(S^{-2}\overline{C}_6^{-3}S^{-2}\overline{C}_6^3) = \chi(S^{-2}\overline{C}_6^{-3}S^{-2}\overline{C}_6^{-3}) + \chi(\omega_C) = \chi(\overline{C}_6^3S^2\overline{C}_6^3S^2) + \chi(\omega_C) = \chi(S\overline{C}_6^3S^2\overline{C}_6^3S) + \chi(\omega_C) = \chi(S^{-1}\overline{C}_6^{-3}S\overline{C}_6^3S^2) + \chi(\omega_{SC}) + \chi(\omega_C) = \chi(\overline{C}_6^{-3}S\overline{C}_6^3S) + \chi(\omega_{SC}) + \chi(\omega_C) = -\chi(\omega_C) + \chi(\overline{C}_6^3S\overline{C}_6^3S) + \chi(\omega_{SC}) + \chi(\omega_C) = 2\chi(\omega_{SC}) = 0$, therefore $\chi(\omega_{C1}\omega_{C2}\omega_{C3}) = 0$, since the image of χ is in $\mathbb{Z}_2$.

- For any character χ, we have $\chi(\omega_2) = \chi(\omega_3)$. Note that we have $\chi(\omega_{C2}^{-1}) = \chi(T_3^{-1}\overline{C}_6T_2^{-1}\overline{C}_6^{-1}) = \chi(\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_2^{-1})$, therefore $\chi(\omega_{C2}^{-1}\omega_2) = \chi(\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3T_2^{-1}T_3^{-1}) = \chi(T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3T_2^{-1})$, and using $\chi(\omega_{C2}) = \chi(T_2\overline{C}_6^{-1}T_3\overline{C}_6)$, we have $\chi(\omega_{C2}^{-1}\omega_2\omega_{C2}) = \chi(T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3T_2^{-1}T_2\overline{C}_6^{-1}T_3\overline{C}_6) = \chi(T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3\overline{C}_6^{-1}T_3\overline{C}_6)$. Similarly, we have $\chi(\omega_{C3}^{-1}\omega_3) = \chi(\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_1^{-1}T_1T_3^{-1}T_1^{-1}T_3) = \chi(\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}T_1^{-1}T_3)$, and $\chi(\omega_{C3}^{-1}\omega_3\omega_{C3}) = \chi(T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}T_1^{-1}T_1\overline{C}_6T$ $\chi(T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3\overline{C}_6^{-1}),\chi(T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3\overline{C}_6^{-1}T_3\overline{C}_6)$, we see that $\chi(\omega_{C2}^{-1}\omega_2\omega_{C2}) = \chi(\omega_{C3}^{-1}\omega_3\omega_{C3})$, or $\chi(\omega_2) = \chi(\omega_3)$.

- For any character χ, we have $\chi(\omega_1) = \chi(\omega_2)$. To see this, first we have $\chi(\omega_{C2}\omega_{C3}) = \chi(\overline{C}_6^{-1}T_3\overline{C}_6T_2T_1\overline{C}_6T_3\overline{C}_6^{-1}) = \chi(T_2T_1\overline{C}_6T_3\overline{C}_6^{-2}T_3\overline{C}_6)$, and similarly we have $\chi(\omega_{C3}^{-1}\omega_{C2}^{-1}) = \chi(\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_2^{-1}T_1^{-1}\overline{C}_6T_3^{-1}\overline{C}_6^{-1}) = \chi(\overline{C}_6T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6T_2^{-1}T_1^{-1}$ therefore $\chi(\omega_1) = \chi(\omega_{C3}^{-1}\omega_{C2}^{-1}\omega_1\omega_{C2}\omega_{C3}) = \chi(\overline{C}_6T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6T_2^{-1}T_1^{-1}T_1T_2T_1^{-1}T_2^{-1}T_2T_1\overline{C}_6T_3\overline{C}_6^{-2}T_3\overline{C}_6) = \chi(\overline{C}_6T_3^{-1}\overline{C}_6^{-2}T_3^{-1}$ $\chi(\overline{C}_6^2T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6^2T_3\overline{C}_6^{-2}T_3)$. Then, we use the second point above, where we just proved as an intermediate step that $\chi(\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}) = \chi(T_3^{-1}\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3) = \chi(T_3^{-1}\overline{C}_6^3T_3^{-1}\overline{C}_6^{-3})$: we have $\chi(\omega_1\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}) = \chi(\overline{C}_6^2T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6^2T_3\overline{C}_6^{-2}T_3T_3$ $\chi(\overline{C}_6^{-1}T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6^2T_3\overline{C}_6T_3^{-1}) = \chi(\overline{C}_6^{-1}T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6^{-2}T_3^{-1}\overline{C}_6^3) = \chi(T_3^{-1}\overline{C}_6^{-3}T_3^{-1}\overline{C}_6^3\overline{C}_6^{-1}T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3) = \chi(\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}\overline{C}_6^{-1}T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3)$, therefore $\chi(\omega_1) + \chi(\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}) = \chi(\omega_{C3}^{-1}\omega_{C1}\omega_{C2}^{-1}) + \chi\overline{C}_6^{-1}T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3)$, or $\chi(\omega_1) = \chi(\overline{C}_6^{-1}T_3\overline{C}_6T_3^{-1}\overline{C}_6^{-1}T_3^{-1}\overline{C}_6T_3)$. Compair this with the the result for $\chi(\omega_2)$ obtained in the last point, we see that $\chi(\omega_1) = \chi(\omega_2) = \chi(\omega_3)$.

These four constraints for a general character χ shows that in fact the upper independent dimension for $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$ is $13 - 4 = 9$. Besides these four, we did not find other constraints for χ, and we are almost sure that we indeed have $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG} = \mathbb{Z}_2^9$. (Although we do not prove the dimensionality nine is correct, we do have strong evidence that this is true, and this is what we mean by "almost sure." See below.)

Therefore, we believe that we have

$$H^2(SG, \mathbb{Z}_2) \cong H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}/\text{im(res)} \cong \mathbb{Z}_2^9/\mathbb{Z}_2^3 \cong \mathbb{Z}_2^6.$$

In orer to better see the structure of this six-dimensional group, let's do some subsequent analysis. First, let us see how the image of restriction, im(res), is quotiented out from $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$. This is due to that χ_{T_1}, $\chi_{T_2}$and χ_{T_3} must be able to be expressed as linear combination of some of the characters in $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$. Now, we make the choice of the generators of $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$ to be $\chi_{\omega_1}, \chi_{\omega_C}, \chi_{\omega_S}, \chi_{\omega_{C1}}, \chi_{\omega_{C2}}, \chi_{\omega_{S1}}, \chi_{\omega_{S2}}, \chi_{\omega_{CS}}$ and $\chi_{\omega_{SC}}$, which acts on $\omega_{1,C,S,C1,C2,S1,S2,CS,SC}$ nontrivially only their respective subscripts. We know that χ_{T_1,T_2,T_3} acts trivially on $\omega 1$, ω_C, ω_{SC} and ω_S, therefore we only need to check the action on the rest six ω's: we see that $\chi_{T_1} = \chi_{\omega_{C1}}$, $\chi_{T_2} = \chi_{\omega_{C1}} + \chi_{\omega_{C2}}$ and $\chi_{T_3} = \chi_{\omega_S} + \chi_{\omega_{C_2}} + \chi_{\omega_{S1}} + \chi_{\omega_{S2}}$.

	ω_S	ω_{C1}	ω_{C2}	ω_{S1}	ω_{S2}
χ_{T_1}	0	1	0	0	0
χ_{T_2}	0	1	1	0	0
χ_{T_3}	1	0	1	1	1

Therefore a more explicit way to write the group cohomology group is

$$\boxed{H^2(SG,\mathbb{Z}_2) \cong \langle \chi_{\omega_1}, \chi_{\omega_C}, \chi_{\omega_S}, \chi_{\omega_{S1}}, \chi_{\omega_{S2}}, \chi_{\omega_{SC}}, \chi_{\omega_{CS}} \rangle / \langle \chi_{\omega_S} + \chi_{\omega_{S1}} + \chi_{\omega_{S2}} \rangle \cong \mathbb{Z}_2^6.} \tag{10}$$

Let us yet give another analysis using the point group strucutre for the space group. From

$$0 \to \mathbb{Z}^3 \to SG \to O_\mathrm{h} \to 1$$

we have the long exact sequence obtained from Eq. (2)

$$0 \to H^1(O_\mathrm{h},\mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(SG,\mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathbb{Z}^3,\mathbb{Z}_2)^{O_\mathrm{h}} \xrightarrow{\text{tg}} H^2(O_\mathrm{h},\mathbb{Z}_2) \xrightarrow{\text{inf}} H^2(SG,\mathbb{Z}_2).$$

Again $H^1(\mathbb{Z}^3,\mathbb{Z}_2) \cong \mathbb{Z}_2^3$, whose generators are precisely χ_{T_1,T_2,T_3}. In fact they are not O_h stable characters: this can be explicitly seen:

- If χ_{T_i} is O_h stable for $i = 1,2,3$, then we must have $1 = \chi_{T_i}(T_i) = \chi_{T_i}(\overline{C}_6 T_i \overline{C}_6^{-1}) = \chi_{T_i}(T_{i+1}) = 0$ ($i+3$ understood as i), a contradition.

Therefore $H^1(\mathbb{Z}^3,\mathbb{Z}_2)^{O_\mathrm{h}} = 0$ and both the restriction and transgression map are trivial. This suggests that $\text{inf}\colon H^1(O_\mathrm{h},\mathbb{Z}_2) \to H^1(SG,\mathbb{Z}_2)$ is isomorphism. This can be shown explicitly: $H^1(SG,\mathbb{Z}_2)$ can be computed using the group presentation exact sequence (9): in there $H^1(SG,\mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F_5,\mathbb{Z}_2)$ is injective, and we have $H^1(SG,\mathbb{Z}_2) \cong \text{im(inf)} = \text{ker(res)} = \langle \chi_S, \chi_{\overline{C}_6} \rangle \cong \mathbb{Z}_2^2$, i.e. $H^1(SG,\mathbb{Z}_2)$ is generated by χ_S and $\chi_{\overline{C}_6}$. Another way to see this is that $H^1(SG,\mathbb{Z}_2)$ is simply all the homomorphism group $SG \to \mathbb{Z}_2$ and must be generated by the characters on the generator of SG, $\chi_{T_1,T_2,T_3,\overline{C}_6,S}$. Apply them to the group relations (1): from $S^2 T_3^{-1} = 1$ we must have $\chi_{T_3} = 0$; and to $\overline{C}_6 T_2 \overline{C}_6^{-1} T_3 = 1$ and $\overline{C}_6 T_3 \overline{C}_6^{-1} T_1 = 1$, we must have $\chi_{T_1} = \chi_{T_2} = 0$. There is no more constraints on χ_S or $\chi_{\overline{C}_6}$ so that they are kept in $H^1(SG,\mathbb{Z}_2)$. Similarly, from the example of computing $H^2(O_\mathrm{h},\mathbb{Z}_2)$, we have $H^1(O_\mathrm{h},\mathbb{Z}_2) \cong \text{ker(res)} = \langle \chi_r, \chi_t \rangle \cong \mathbb{Z}_2^2$. Therefore, $\text{inf}\colon H^1(O_\mathrm{h},\mathbb{Z}_2) \to H^1(SG,\mathbb{Z}_2)$ is isomorphism.

The only piece to be understood is

$$0 \to H^2(O_\mathrm{h},\mathbb{Z}_2) \xrightarrow{\text{inf}} H^2(SG,\mathbb{Z}_2).$$

This shows that the inflation map is injective, therefore all nontrivial elements of $H^2(O_\mathrm{h},\mathbb{Z}_2)$ must be embedded in $H^2(SG,\mathbb{Z}_2)$. This can be seen to be true by comparing the results (7) and (10).

7 Towards simplification of calculation

From the examples of S_4, O_h and space group $Fd\overline{3}m$, we see the power of LHS spectral sequence in obtaining the second cohomology group. The maps and terms in the LHS five-term exact sequence have very concrete meanings, allowing us to analysis the cohomology structure to great detail. We summarize that there are usually two ways of using the LHS exact sequence, one for a group with its normal group, or for the group presentation. A combination of these two ways can often fully determine the cohomology group. For the latter way, we simply uses the LHS exact sequence

$$0 \to H^1(SG,\mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F,\mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R},\mathbb{Z}_2)^{SG} \xrightarrow{\text{tg}} H^2(SG,\mathbb{Z}_2) \to 0 \tag{11}$$

to determine

$$H^2(SG,\mathbb{Z}_2) \cong H^1(\mathcal{R},\mathbb{Z}_2)^{SG}/\text{im(res)}.$$

For our purpose, of course, we would like to compute the cohomology group of all space groups. From the example of $Fd\overline{3}m$ we see the hardest part is to determine $H^1(\mathcal{R},\mathbb{Z}_2)^{SG}$: its dimension can be at most the number of independent relations forming $\mathcal{R}_0$, and we must be careful in knocking down some of them to find the independent characters. The procedure has been illustrated in detail for $Fd\overline{3}m$ where we knocked down four dimensions $\chi(\omega_{S3}) = 0$, $\chi(\omega_{C1}\omega_{C2}\omega_{C3}) = 0$, $\chi(\omega_1) = \chi(\omega_2) = \chi(\omega_3)$; the idea is to iteratively use the property of χ:

- $\chi\colon \mathcal{R} \to \mathbb{Z}_2$ is homomorphism;
- χ is SG invariant, therefore for any $g \in SG$ and $\omega \in \mathcal{R}$ we have $\chi(g\omega g^{-1}) = \chi(\omega)$.

As was the case for $Fd\bar{3}m$, the dimension knocking-down can become quite involved. However, if we recall what we did there, we were only using the multiplication of SG to multiply around a bunch of elements; and the complication was due to that χ is defined on $\mathcal{R}$, and not any "words" in F would make sense in χ.

Simplification of calculation can be achieved along this line of thoughts: we can write down the dimension knocking-down equations before χ is applied, and only apply χ at the very end. When applying χ, we just have to make sure that both side of the equation are elements of $\mathcal{R}$ on which χ is well-defined. By definition, this can always be achieved. Therefore, whenever such an equation is written down, say, $A = B$, it is in the sense that $\chi(A) = \chi(B)$. This leads to the following idea:

- A congruence equation $A \equiv B$ in $\mathcal{R}$ modding out $[F, R]R^2$ will give $\chi(A) = \chi(B)$, since for any character χ, $[F, \mathcal{R}]\mathcal{R}^2 \subseteq \ker(\chi)$.

Therefore, to determine $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$, we just have to follow the procedures below:

- For any space group SG, suppose the generators are $T_1, T_2, \cdots$. Use $R_1, R_2, \cdots$ to label them. The $R_1, R_2, \cdots$ represent the cosets $T_1[F, \mathcal{R}]\mathcal{R}^2, T_2[F, \mathcal{R}]\mathcal{R}^2, \cdots$
- Write all the independent relations (which are generators of $\mathcal{R}_0$) in terms of the $R_1, R_2, \cdots$ symbols, and place them on the left hand side of equations; write Ω with corresponding subscripts as the right hand side of these equations. The Ω's belong to the same coset as the corresponding left hand side relations.
- The Ω's commute with all the R symbols, since ΩR and $R\Omega$ belong to the same coset.
- We have $\Omega^2 \equiv 0$, since $\Omega^2 \in [F, R]R^2$.
- The $\Omega_1, \Omega_2 \cdots$ are in fact dual to $\chi_1, \chi_2 \cdots$. This is due to the fact that, a finite dimensional vector space is isomorphic to its dual space, after a set of basis is specified. In our case, we have chosen the (overcomplete) basis as $\chi_1, \chi_2, \cdots$ and the corresponding (overcomplete) dual basis as $\Omega_1, \Omega_2, \cdots$.
- Use these congruence equations to find all constraints on Ω's (there is not a controlled way to find all constraints but usually this can be achieved if one is careful enough). The remaining independent ones are the (dual) generators of $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$.

this completes our simplification of the formalism for calculating $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$.

For calculating im(res), there is also some simplification one can make (only when the cohomology has $\mathbb{Z}_2$ coefficients): note that whichever character kept in im(res) must act nontrivially on the relations. To be concrete, let's choose $H^1(F, \mathbb{Z}_2)$ to be generated by $\chi_{T_1}, \chi_{T_2}, \cdots$ which are the characters acting nontrivially only on their respective subscripts. Then im(res) must be the χ's whose corresponding subscript appears odd number of times in the relations $\mathcal{R}_0$. However, im(res) are to be quotiented out, suggesting the following:

- Whenver there exists equations in which an R appears odd number of times, we subtract one dimension from $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$ in order to arrive at $H^2(SG, \mathbb{Z}_2)$. The Ω associated to one of these equations–in which an R appears odd number of times–can be eliminated in $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$, in order to arrive at $H^2(SG, \mathbb{Z}_2)$.

This completes the simplification of im(res).

8 Simplified calculation: example on space group $Fd\bar{3}m$

We apply the algorithm in last section to the space group $SG = Fd\bar{3}m$ to find $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$. Denote the cosets containing $T_1, T_2, T_3, \overline{C}_6 = C, S$ by R_1, R_2, R_3, R_C, R_S. The congruence equations are

$$R_1 R_2 R_1^{-1} R_2^{-1} \equiv \Omega_1, \tag{12a}$$

$$R_2 R_3 R_2^{-1} R_3^{-1} \equiv \Omega_2, \tag{12b}$$

$$R_3 R_1 R_3^{-1} R_1^{-1} \equiv \Omega_3, \tag{12c}$$

$$R_C^6 \equiv \Omega_C, \tag{12d}$$

$$R_S^2 R_3^{-1} \equiv \Omega_S, \tag{12e}$$

$$R_C R_1 R_C^{-1} R_2 \equiv \Omega_{C1}, \tag{12f}$$

$$R_C R_2 R_C^{-1} R_3 \equiv \Omega_{C2}, \tag{12g}$$

$$R_C R_3 R_C^{-1} R_1 \equiv \Omega_{C3}, \tag{12h}$$

$$R_S R_1 R_S^{-1} R_3^{-1} R_1 \equiv \Omega_{S1}, \tag{12i}$$

$$R_S R_2 R_S^{-1} R_3^{-1} R_2 \equiv \Omega_{S2}, \tag{12j}$$
$$R_S R_3 R_S^{-1} R_3^{-1} \equiv \Omega_{S3}, \tag{12k}$$
$$(R_C R_S)^4 \equiv \Omega_{CS}, \tag{12l}$$
$$(R_C^3 R_S)^2 \equiv \Omega_{SC}, \tag{12m}$$

remember that Ω commutes with any R elements in the congruence equations and that $\Omega^2 \equiv 0$.

First from Eq. (12e) and (12k) we have

$$R_3 \equiv R_S^2 \Omega_S^{-1} \equiv R_S^2 \Omega_S \tag{13}$$

and

$$\Omega_{S3} \equiv 1. \tag{14}$$

Then from Eq. (12g) and (12h) we have

$$\begin{aligned} R_2 &\equiv R_C^{-1} R_3^{-1} R_C \Omega_{C2}, &(15\text{a})\\ R_1 &\equiv R_C R_3^{-1} R_C^{-1} \Omega_{C3}, &(15\text{b}) \end{aligned}$$

and plug these into Eq. (12f) we get

$$R_C^3 R_3^{-1} R_C^{-3} R_3^{-1} \equiv \Omega_{C1} \Omega_{C2} \Omega_{C3}. \tag{16}$$

On the other hand, square of Eq. (12m) gives, using Eq. (12d),

$$R_C^3 R_S^2 R_C^{-3} R_S^2 \equiv 1, \tag{17}$$

compare with Eq. (16) and, by use of Eq. (13), we get

$$\Omega_{C1} \Omega_{C2} \Omega_{C3} \equiv 1. \tag{18}$$

Now plug Eq. (15) into Eqs. (12b) and (12c) we get

$$\Omega_2 \equiv \Omega_3 \tag{19}$$

and

$$R_C R_3^{-1} R_C^{-1} R_3 \equiv R_3 R_C R_3^{-1} R_C \Omega_3. \tag{20}$$

Then from Eq. (12a) we get $R_C R_3^{-1} R_C^{-2} R_3^{-1} R_C \equiv R_C^{-1} R_3^{-1} R_C^2 R_3^{-1} R_C^{-1} \Omega_1$, and after we insert Eqs. (16) and (18) we get

$$R_C R_3^{-1} R_C^{-1} R_3 \equiv R_3 R_C R_3^{-1} R_C \Omega_1, \tag{21}$$

thus we must have

$$\Omega_1 \equiv \Omega_2 \equiv \Omega_3. \tag{22}$$

We believe these are all the constraints in Ω's. And we have

$$H^1(\mathcal{R}, \mathbb{Z}_2)^{SG} \cong \langle \Omega_1, \Omega_C, \Omega_S, \Omega_{C1}, \Omega_{C2}, \Omega_{S1}, \Omega_{S2}, \Omega_{CS}, \Omega_{SC} \rangle.$$

Then deal with im(res). Note that R_1 appear once in the equation for Ω_{C1} and equation for Ω_{C3}, R_2 appear odd number of times in the equation for Ω_{C1} and Ω_{C2}, and R_3 appear odd nmber of times in the equation for Ω_S, the equation for Ω_{C2}, the equation for Ω_{C3}, the equation for Ω_{S1} and Ω_{S2}. Therefore according to the algorthim, we can elimiate Ω_{C1}, Ω_{C2}, and one of Ω_S, Ω_{CS} and Ω_{SC} from $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$ in order to arrive at the answer for $H^2(SG, \mathbb{Z}_2)$. Therefore we can write

$$H^2(SG, \mathbb{Z}_2) \cong \langle \Omega_1, \Omega_C, \Omega_S, \Omega_{S1}, \Omega_{S2}, \Omega_{CS}, \Omega_{SC} \rangle / \langle \Omega_S \Omega_{S1} \Omega_{S2} \rangle.$$

This result is precisely the result we obtained in Eq. (10).

The algorithm examplified in this section is exactly the Projective Symmetry Group calculation in condensed matter physics. In physics, we also consider the cohomology group $H^2(SG \times \mathbb{Z}_2, \mathbb{Z}_2)$, where the summand $\mathbb{Z}_2$ denotes time reversal symmetry. Using Künneth formula we have

$$\begin{aligned} H^2(SG \times \mathbb{Z}_2, \mathbb{Z}_2) &\cong H^2(SG, \mathbb{Z}_2) \oplus \big(H^1(SG, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2)\big) \oplus H^2(\mathbb{Z}_2, \mathbb{Z}_2) \\ &\cong \mathbb{Z}_2^6 \oplus (\mathbb{Z}_2^2 \otimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \\ &\cong \mathbb{Z}_2^6 \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \otimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \\ &\cong \mathbb{Z}_2^9. \end{aligned}$$

9 Another example: space group $R\overline{3}m$ (incomplete)

The space group $SG = R\overline{3}m$ is the No. 166 space group, generated by $T_1, T_2, T_3, \overline{C}_6$ and D, defined by the relations

$$T_1T_2T_1^{-1}T_2^{-1} = 1, \tag{23a}$$

$$T_2T_3T_2^{-1}T_3^{-1} = 1, \tag{23b}$$

$$T_3T_1T_3^{-1}T_1^{-1} = 1, \tag{23c}$$

$$\overline{C}_6^6 = 1, \tag{23d}$$

$$D^2 = 1, \tag{23e}$$

$$\overline{C}_6T_1\overline{C}_6^{-1}T_2 = 1, \tag{23f}$$

$$\overline{C}_6T_2\overline{C}_6^{-1}T_1^{-1}T_2^{-1} = 1, \tag{23g}$$

$$\overline{C}_6T_3\overline{C}_6^{-1}T_3 = 1, \tag{23h}$$

$$DT_1DT_2^{-1} = 1, \tag{23i}$$

$$DT_2DT_1^{-1} = 1, \tag{23j}$$

$$(DT_3)^2 = 1, \tag{23k}$$

$$(\overline{C}_6D)^2 = 1, \tag{23l}$$

The point group is $D_{3\mathrm{d}} \cong \mathrm{Dih}_6 \cong \mathrm{Dih}_3 \times \mathbb{Z}_2$, where the Dihedral group Dih_6 has order 12 (we are using the geometry convention). We give a presentation $\mathrm{Dih}_3 = \langle r, s | r^3 = d^2 = (sd)^2 = 1\rangle$, and call the extra generator in Dih_6 σ. We have $\overline{C}_6 \equiv c \equiv \sigma r$, $r = \overline{C}_6^2$ and $\sigma = \overline{C}_6^3$. σ corresponds to inversion, r the generator of three-fold rotation, and $D = d$ a two-fold rotation about an horizontal axis. Therefore, Dih_6 can be presented as $\mathrm{Dih}_6 = \langle r, d, \sigma | r^3 = d^2 = (\sigma d)^2 = \sigma^2 = [r, \sigma] = [d, \sigma] = 1\rangle = \langle c, d | c^6 = d^2 = (cd)^2 = 1\rangle$ (these two presentations can be found on Wikipedia).

First we write the exact sequence obtained from group presentation

$$0 \to H^1(SG, \mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F, \mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R}, \mathbb{Z}_2)^{SG} \xrightarrow{\text{tg}} H^2(SG, \mathbb{Z}_2) \to 0$$

we have $\mathrm{im(res)} \cong \langle \chi_{T_1}, \chi_{T_2}\rangle \cong \mathbb{Z}_2^2$, $H^1(F, \mathbb{Z}_2) \cong \langle \chi_{T_1}, \chi_{T_2}, \chi_{T_3}, \chi_{\overline{C}_6}, \chi_D\rangle \cong \mathbb{Z}_2^5$ and $H^1(SG, \mathbb{Z}_2) \cong \langle \chi_{T_3}, \chi_{\overline{C}_6}, \chi_D\rangle \cong \mathbb{Z}_2^3$ (we get $\chi_{T_1} = 0$ from equation $\overline{C}_6T_2\overline{C}_6^{-1}T_1^{-1}T_2^{-1} = 1$ or $DT_1DT_2^{-1} = 1$ or $D_T2DT_1^{-1} = 1$, and $\chi_{T_2} = 0$ from $\overline{C}_6T_1\overline{C}_6^{-1}T_2 = 1$ or $DT_1DT_2^{-1} = 1$ or $DT_2DT_1^{-1} = 1$), which are easy to check.

Then we work on $H^1(\mathcal{R}, \mathbb{Z}_2)^{SG}$. Following the simplified procedures, we label the twelve relations by $\Omega_{1,2,3,C,D,C1,C2,C3,D1,D2,D3,C}$
We have $T_2 \equiv \overline{C}_6T_1^{-1}\overline{C}_6^{-1}$, $T_1 \equiv T_2^{-1}\overline{C}_6T_2\overline{C}_6^{-1}$,

This section is not complete. Perhaps some discussion is needed.

10 Cohomology for free abelian group and $\mathbb{Z}_2^n$

Method 1: See `https://math.stackexchange.com/questions/4234008/about-the-cohomology-groups-of-a-free-abelian`
When G acts trivially on M, the cohomology of G agrees with the cohomology of the classifying space of G. For $G = \mathbb{Z}^n$, we have $K(\mathbb{Z}^n, 1) = T^n$ the n-torus, then

$$H^k(\mathbb{Z}^n, M) \cong H^k(T^n, M) \cong M^{\binom{n}{k}} \tag{24}$$

. This means that $H^k(\mathbb{Z}, M) = 1, 0, ...$ for $k = 1, 2, ...$, $H^k(\mathbb{Z}^2, M) = 2, 1, 0, ...$ for $k = 1, 2, 3, ...$, and $H^k(\mathbb{Z}^3, M) = 3, 3, 1, 0, ...$ for $k = 1, 2, 3, 4, ...$

From this it is also easy to see the cohomology ring structure. Setting $M = \mathbb{Z}/2\mathbb{Z}$, It is

$$\mathbb{F}_2[x_1, ..., x_n]/(x_i^2, x_ix_j + x_jx_i).$$

Method 2: Let us look at a very simple example: $H^2(\mathbb{Z}^3, \mathbb{Z}_2)$. The three dimensional translation group $\mathbb{Z}^3$ is generated by T_1, T_2, T_3, this group has three independent group relations $T_iT_{i+1}T_i^{-1}T_{i+1}^{-1} = 1$ for $i = 1, 2, 3$ (note that $T_3T_1T_3^{-1}T_1^{-1} = 1$ cannot be derived from the other two). And we do have $H^1(\mathbb{Z}^3, \mathbb{Z}_2) = \mathbb{Z}_2^3$ and $H^2(\mathbb{Z}^3, \mathbb{Z}_2) = \mathbb{Z}_2^2$.

If we want to use

$$0 \to \mathcal{R} \to F_3 \to \mathbb{Z}^3 \to 1$$

to calculate $H^2(\mathbb{Z}^3, \mathbb{Z}_2)$, that is we want to calculate

$$0 \to H^1(\mathbb{Z}^3, \mathbb{Z}_2) \xrightarrow{\text{inf}} H^1(F_3, \mathbb{Z}_2) \xrightarrow{\text{res}} H^1(\mathcal{R}, \mathbb{Z}_2)^{\mathbb{Z}^3} \xrightarrow{tg} H^2(\mathbb{Z}^3, \mathbb{Z}_2) \to 0,$$

using $H^1(F_3, \mathbb{Z}_2) = \mathrm{Hom}(F_3/[F_3, F_3], \mathbb{Z}_2) = \mathrm{Hom}(\mathbb{Z}^3, \mathbb{Z}_2)$, the inf arrow is isomorphism so that the tg map is also isomorphism.

Then we only have to calculate $H^1(\mathcal{R}, \mathbb{Z}_2)^{\mathbb{Z}^3}$. This is clearly $\mathbb{Z}_2^3$, generated by the characters for the three relations. So $H^2(\mathbb{Z}^3, \mathbb{Z}_2) = \mathbb{Z}_2^3$.

However, if we choose $\mathcal{R}$ to be generated by three relations $\omega_1 = T_1T_2T_1^{-1}$, $\omega_2 = T_2T_3T_2^{-1}T_3^{-1}$ and $\omega_3 = T_3T_1T_3^{-1}T_1^{-1}$, then for any character $\chi\colon \mathcal{R} \to \mathbb{Z}_2$, we must have $\chi(\omega_1\omega_2\omega_3) = 0$. However this is very hard to prove using the properties of χ. May I know if such a proof exist? This may need ore discussion.

Method 3: Alternatively, Künneth formula in this case is given by

$$\begin{aligned} &H^1(\mathbb{Z}^3, \mathbb{Z}_2) \cong H^1(\mathbb{Z}, \mathbb{Z}_2) \oplus H^1(\mathbb{Z}, \mathbb{Z}_2) \oplus H^1(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2^3, \\ &H^2(\mathbb{Z}^3, \mathbb{Z}_2) \cong H^2(\mathbb{Z}^2, \mathbb{Z}_2) \oplus \big(H^1(\mathbb{Z}^2, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}, \mathbb{Z}_2)\big) \oplus H^2(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2^2 \otimes \mathbb{Z}_2) = \mathbb{Z}_2^3, \end{aligned} \tag{25}$$

note that $\mathbb{Z}_2^2 \otimes \mathbb{Z}_2 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \otimes \mathbb{Z}_2) = (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \mathbb{Z}_2^2$, where we have used the fact that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/(\gcd(m, n)\mathbb{Z})$ and distributivitiy of tensor product over direct sum.

On the other hand, let's consider $H^2(\mathbb{Z}_2^n, \mathbb{Z}_2)$ with trivial action. We know that $H^*(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{F}_2[x]$, so $H^n(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $n \geq 0$. It is also easy to see that $H^1(\mathbb{Z}_2^n, \mathbb{Z}_2) = \mathbb{Z}_2^n$, using Kunneth formula. Then for H^2, we have $H^2(\mathbb{Z}_2^n, \mathbb{Z}_2) = H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}_2) \oplus (H^1(\mathbb{Z}_2^{n-1}, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2)) \oplus H^2(\mathbb{Z}_2, \mathbb{Z}_2)$. Note that $(H^1(\mathbb{Z}_2^{n-1}, \mathbb{Z}_2) \otimes H^1(\mathbb{Z}_2, \mathbb{Z}_2)) = \mathbb{Z}_2^{n-1} \otimes \mathbb{Z}_2 = (\oplus_{n-1 \text{ copies}} \mathbb{Z}_2) \otimes \mathbb{Z}_2 = \oplus_{n-1 \text{ copies}} (\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \oplus_{n-1 \text{ copies}} \mathbb{Z}_2 = \mathbb{Z}_2^{n-1}$, and that $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, so that

$$H^2(\mathbb{Z}_2^n, \mathbb{Z}_2) = H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}_2) \oplus \mathbb{Z}_2^n = \cdots = \mathbb{Z}_2^{n(n+1)/2}.$$

11 Spectral Sequences

An excellent note can be found at `http://www.math.mcgill.ca/goren/SeminarOnCohomology/infres.pdf`.

At the E_1 page of a spectral sequence, we have

$$\begin{tikzcd}
E_1^{0,3} \arrow[r] & E_1^{1,3} \arrow[r] & E_1^{2,3} \arrow[r] & E_1^{3,3} \arrow[r] & E_1^{4,3} \arrow[r] & {} \\
E_1^{0,2} \arrow[r] & E_1^{1,2} \arrow[r] & E_1^{2,2} \arrow[r] & E_1^{3,2} \arrow[r] & E_1^{4,2} \arrow[r] & {} \\
E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & E_1^{3,1} \arrow[r] & E_1^{4,1} \arrow[r] & {} \\
E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & E_1^{3,0} \arrow[r] & E_1^{4,0} \arrow[r] & {}
\end{tikzcd}$$

At the E_2 page of a spectral sequence, we have

$$\begin{tikzcd}
E_2^{0,3} \arrow[drr, "d_2"] & E_2^{1,3} \arrow[drr, "d_2"] & E_2^{2,3} \arrow[drr, "d_2"] & E_2^{3,3} \arrow[drr, "d_2"] & E_2^{4,3} & \\
E_2^{0,2} \arrow[drr, "d_2"] & E_2^{1,2} \arrow[drr, "d_2"] & E_2^{2,2} \arrow[drr, "d_2"] & E_2^{3,2} \arrow[drr, "d_2"] & E_2^{4,2} & {} \\
E_2^{0,1} \arrow[drr, "d_2"] & E_2^{1,1} \arrow[drr, "d_2"] & E_2^{2,1} \arrow[drr, "d_2"] & E_2^{3,1} \arrow[drr, "d_2"] & E_2^{4,1} & {} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & {}
\end{tikzcd}$$

where the two red elements are the stabilized terms at the E_2 page. At the E_3 page, more elements are stabilized (in red):

$$\begin{tikzcd}
E_3^{0,3} \arrow[rrrdd, "d_3"] & E_3^{1,3} \arrow[rrrdd, "d_3"] & E_3^{2,3} \arrow[rrrdd, "d_3"] & E_3^{3,3} & E_3^{4,3} & \\
E_3^{0,2} \arrow[rrrdd, "d_3"] & E_3^{1,2} \arrow[rrrdd, "d_3"] & E_3^{2,2} \arrow[rrrdd, "d_3"] & E_3^{3,2} & E_3^{4,2} & \\
{\color{red}E_3^{0,1}} & {\color{red}E_3^{1,1}} & {\color{red}E_3^{2,1}} & E_3^{3,1} & E_3^{4,1} & \\
{\color{red}E_3^{0,0}} & {\color{red}E_3^{1,0}} & {\color{red}E_3^{2,0}} & E_3^{3,0} & E_3^{4,0} &
\end{tikzcd}$$

where for the Lyndon-Hochschild-Serre spectral sequence, we have
We have $H^n(G,M) \to E_\infty^{0,n}$ surjective, and $E_\infty^{n,0} \to H^n(G,M)$ injective.

$$\begin{aligned}
H^1(G,M) &= E_\infty^{1,0} \oplus E_\infty^{0,1}, \\
H^2(G,M) &= E_\infty^{2,0} \oplus E_\infty^{1,1} \oplus E_\infty^{0,2},
\end{aligned}$$

where the stabilization happens at

$$\begin{aligned}
E_\infty^{1,0} &= E_2^{1,0} = \ker(d_1\colon E_1^{1,0} \to E_1^{2,0})/\mathrm{im}(d_1\colon E_1^{0,0} \to E_1^{1,0}), \\
E_\infty^{0,1} &= E_3^{0,1} = \ker(E_2^{0,1} \to E_2^{2,0}),
\end{aligned} \tag{26}$$

$$\begin{aligned}
E_\infty^{0,2} &= E_4^{0,2} = \ker(d_3\colon E_3^{0,2} \to E_3^{3,0}), \\
E_\infty^{1,1} &= E_3^{1,1} = \ker(d_2\colon E_2^{1,1} \to E_2^{3,0}), \\
E_\infty^{2,0} &= E_3^{2,0} = E_2^{2,0}/\mathrm{im}(d_2\colon E_2^{0,1} \to E_2^{2,0})
\end{aligned} \tag{27}$$

where $E_3^{0,2} = \ker(d_2\colon E_2^{0,2} \to E_2^{2,1})$, and $E_3^{3,0} = \mathrm{im}(d_2\colon E_2^{1,1} \to E_2^{3,0})$.
We can derive the LHS five term exact sequence: we have

$$0 \to E_\infty^{1,0} \to H^1(G,M) \to E_\infty^{0,1} \to 0,$$

$$0 \to E_2^{1,0} \to H^1(G,M) \to E_3^{0,1} \to 0,$$

or

$$0 \to E_2^{1,0} \to H^1(G,M) \to \ker(E_2^{0,1} \to E_2^{2,0}) \to 0,$$

on the other hand we have $E_\infty^{2,0} = E_3^{2,0} \to H^2(G,M)$ injective so

$$0 \to E_3^{2,0} \to H^2(G,M)$$

exact or

$$0 \to E_2^{2,0}/\mathrm{im}(d_2\colon E_2^{0,1} \to E_2^{2,0}) \to H^2(G,M)$$

exact. Combine these two, we get

$$0 \to E_2^{1,0} \to H^1(G,M) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(G,M)$$

exact.

In LHS spectral sequence, we have

$$E_2^{p,q} = H^p(G/N, H^q(N,M)),$$

especially, we have $E_2^{0,n} = H^0(G/N, H^n(N,M)) = H^n(N,M)^{G/N}$, and $E_2^{n,0} = H^n(G/N, H^0(N,M)) = H^n(G/N, M^N)$.

For

$$0 \to N \to G \to Q \to 0,$$

and A an abelian group, we have $Q \cong G/N$, and

$$
\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & & \vdots & \iddots \\
q=3 & H^3(N,A)^Q & H^1(Q,H^3(N,A)) & H^2(Q,H^3(N,A)) & H^3(Q,H^3(N,A)) & H^4(Q,H^3(N,A)) & \cdots \\
q=2 & H^2(N,A)^Q & H^1(Q,H^2(N,A)) & H^2(Q,H^2(N,A)) & H^3(Q,H^2(N,A)) & H^3(Q,H^3(N,A)) & \cdots \\
q=1 & H^1(N,A)^Q & H^1(Q,H^1(N,A)) & H^2(Q,H^1(N,A)) & H^3(Q,H^1(N,A)) & H^2(Q,H^3(N,A)) & \cdots \\
q=0 & A & H^1(Q,A^N) & H^2(Q,A^N) & H^3(Q,A^N) & H^4(Q,A^N) & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots
\end{array}
\tag{28}
$$

In our case, we have

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
0 & 0 & 0 & 0 & \cdots \\
H^3(\mathbb{Z}^3,M)^{O_\mathrm{h}} & H^1(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & \cdots \\
H^2(\mathbb{Z}^3,M)^{O_\mathrm{h}} & H^1(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & \cdots \\
H^1(\mathbb{Z}^3,M)^{O_\mathrm{h}} & H^1(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & \cdots \\
(M^{\mathbb{Z}^3})^{O_\mathrm{h}} & H^1(O_\mathrm{h},M^{\mathbb{Z}^3}) & H^2(O_\mathrm{h},M^{\mathbb{Z}^3}) & H^3(O_\mathrm{h},M^{\mathbb{Z}^3}) & \cdots
\end{array}
$$

where $M=\mathbb{Z}/2\mathbb{Z}$. the action of SG on M is trivial so $M^{\mathbb{Z}^3}=M$ and $M^{O_\mathrm{h}}=M$. We also have $H^1(\mathbb{Z}^3,M)^{O_\mathrm{h}}=0$, $H^{1,2,3}(O_\mathrm{h},M)=\mathbb{Z}_2^2,\mathbb{Z}_2^4,\mathbb{Z}_2^6$ from previous sections. So now we have

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
0 & 0 & 0 & 0 & \cdots \\
H^3(\mathbb{Z}^3,M)^{O_\mathrm{h}} & H^1(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^3(\mathbb{Z}^3,M)) & \cdots \\
H^2(\mathbb{Z}^3,M)^{O_\mathrm{h}} & H^1(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^2(\mathbb{Z}^3,M)) & \cdots \\
0 & H^1(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & H^2(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & H^3(O_\mathrm{h},H^1(\mathbb{Z}^3,M)) & \cdots \\
\mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^4 & \mathbb{Z}_2^7 & \cdots
\end{array},
$$

further using $H^2(\mathbb{Z}^3,M)=M^3$ and $H^1(\mathbb{Z}^3,M)=M^3$, we have

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z}_2^{O_\mathrm{h}} & H^1(O_\mathrm{h},\mathbb{Z}_2) & H^2(O_\mathrm{h},\mathbb{Z}_2) & H^3(O_\mathrm{h},\mathbb{Z}_2) & \cdots \\
(\mathbb{Z}_2^3)^{O_\mathrm{h}} & H^1(O_\mathrm{h},\mathbb{Z}_2^3) & H^2(O_\mathrm{h},\mathbb{Z}_2^3) & H^3(O_\mathrm{h},\mathbb{Z}_2^3) & \cdots \\
0 & H^1(O_\mathrm{h},\mathbb{Z}_2^3) & H^2(O_\mathrm{h},\mathbb{Z}_2^3) & H^3(O_\mathrm{h},\mathbb{Z}_2^3) & \cdots \\
\mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^4 & \mathbb{Z}_2^7 & \cdots
\end{array},
$$

Then we can at least obtain $E_\infty^{2,0}=E_3^{2,0}=E_2^{2,0}/\mathrm{im}(d_2\colon E_2^{0,1}\to E_2^{2,0})=\mathbb{Z}_2^4$.

Note that $H^1(\mathbb{Z}^3,\mathbb{Z}/2\mathbb{Z})=\langle\chi_1,\chi_2,\chi_3\rangle$, where $\chi_i(T_j)=\delta_{ij}$. We now need $\langle\chi_1,\chi_2,\chi_3\rangle^{O_\mathrm{h}}$: the only nontrivial one stabilized by $\overline{C}_6$ is $\chi_1+\chi_2+\chi_3$, but this one clearly is not stabilized by S, so $H^1(\mathbb{Z}^3,M)^{O_\mathrm{h}}=0$.

Note that $H^2(\mathbb{Z}^3,\mathbb{Z}/2\mathbb{Z})=\langle\chi_1\cup\chi_2,\chi_1\cup\chi_3,\chi_2\cup\chi_3\rangle$,

Note that effectively, from $\overline{C}_6T_i\overline{C}_6^{-1}=T_{i+1}^{-1}, i=1,2,3$, and $ST_iS^{-1}=T_i^{-1}T_3, i=1,2$ and $ST_3S^{-1}=T_3$ we see that $1=(\chi_1\cup\chi_3+\chi_2\cup\chi_3)(T_1,T_3)\xrightarrow{\overline{C}_6}(\chi_1\cup\chi_3+\chi_2\cup\chi_3)(T_2,T_1)=0$, so that any $\chi_1\cup\chi_2$ or $\chi_1\cup\chi_3+\chi_2\cup\chi_3$ does not work. So our only chance is $\chi_1\cup\chi_2+\chi_2\cup\chi_3+\chi_3\cup\chi_1$: it is obviously stabilized by $\overline{C}_6$; for S, since $(T_1,T_2)\to(T_1^{-1}T_3,T_2^{-1}T_3)$, $(T_i,T_3)\to(T_i^{-1}T_3,T_3)$ for $i=1,2$, so we see that

$$
H^2(\mathbb{Z}^3,\mathbb{Z}/2\mathbb{Z})^{O_\mathrm{h}}=\langle\chi_1\cup\chi_2+\chi_1\cup\chi_3+\chi_2\cup\chi_3\rangle\cong\mathbb{Z}_2.
$$

so $E_2^{0,2}=\mathbb{Z}_2$, then we must have $E_3^{0,2}=\mathbb{Z}_2$ and $E_4^{0,2}=\mathbb{Z}_2$ to match our PSG results (that the translation contributes one fractionalization). If true, this means that the map d_2 in $E_3^{0,2}=\ker(d_2\colon E_2^{0,2}\to E_2^{2,1})$ is a zero map; this further means that $E_3^{2,1}=\ker(d_2\colon E_2^{2,1}\to E_2^{4,0})/\mathrm{im}(d_2\colon E_2^{0,2}\to E_2^{2,1})=\ker(d_2\colon E_2^{2,1}\to E_2^{4,0})$.

$$
1=\chi_1(T_1)\xrightarrow{\overline{C}_6}\chi_1(T_2^{-1})=\chi_1(T_2)=0,
$$

$$
0=\chi_1(T_2)\xrightarrow{\overline{C}_6}\chi_1(T_3^{-1})=0,
$$

$$
0=\chi_1(T_3)\xrightarrow{\overline{C}_6}\chi_1(T_1^{-1})=1,
$$

$$
0=\chi_2(T_1)\xrightarrow{\overline{C}_6}\chi_2(T_2^{-1})=\chi_1(T_2)=1,
$$

$$1 = \chi_2(T_2) \xrightarrow{\overline{C}_6} \chi_2(T_3^{-1}) = 0,$$

$$0 = \chi_2(T_3) \xrightarrow{\overline{C}_6} \chi_2(T_1^{-1}) = 0,$$

$$0 = \chi_3(T_1) \xrightarrow{\overline{C}_6} \chi_3(T_2^{-1}) = \chi_1(T_2) = 0,$$

$$0 = \chi_3(T_2) \xrightarrow{\overline{C}_6} \chi_3(T_3^{-1}) = 1,$$

$$1 = \chi_3(T_3) \xrightarrow{\overline{C}_6} \chi_3(T_1^{-1}) = 0,$$

and

$$1 = \chi_1(T_1) \xrightarrow{S} \chi_1(T_1^{-1}T_3) = \chi_1(T_1) = 1,$$

$$0 = \chi_1(T_2) \xrightarrow{S} \chi_1(T_2^{-1}T_3) = 0,$$

$$0 = \chi_1(T_3) \xrightarrow{S} \chi_1(T_3) = 0,$$

$$0 = \chi_2(T_1) \xrightarrow{S} \chi_2(T_1^{-1}T_3) = 0,$$

$$1 = \chi_2(T_2) \xrightarrow{S} \chi_2(T_2^{-1}T_3) = 1,$$

$$0 = \chi_2(T_3) \xrightarrow{S} \chi_2(T_3) = 0,$$

$$0 = \chi_3(T_1) \xrightarrow{S} \chi_3(T_1^{-1}T_3) = 1,$$

$$0 = \chi_3(T_2) \xrightarrow{S} \chi_3(T_2^{-1}T_3) = 1,$$

$$1 = \chi_3(T_3) \xrightarrow{S} \chi_3(T_3) = 1,$$

so we have

$$\overline{C}_6\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_3, \chi_1, \chi_2),$$

$$S\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_1, \chi_2, \chi_1 + \chi_2 + \chi_3),$$

We start with any map $d\colon O_{\mathrm{h}} \to \langle \chi_1, \chi_2, \chi_3 \rangle$ that satisfies $d(xy) = x.d(y) + d(x)$, for any $x, y \in O_{\mathrm{h}}$. We just need to specify what $\overline{C}_6$ and S map to. There are 64 choices. Denote $d(S) = \sum_{i=1,2,3} s_i\chi_i$ and $d(\overline{C}_6) = \sum_{i=1,2,3} c_i\chi_i$, then calculation shows that $c_1 + c_2 + c_3 = 0$ while $s_3 = 0$, so we are left with 16 choices of parameters c_1, c_2, s_1, s_2. The PDer part: if if $d(x) = x.a - a$ for some $a \in \langle \chi_1, \chi_2, \chi_3 \rangle$, then $d(\overline{C}_6) = \overline{C}_6(i_1\chi_1 + i_2\chi_2 + i_3\chi_3) - i_1\chi_1 + i_2\chi_2 + i_3\chi_3 = (i_2 - i_1)\chi_1 + (i_3 - i_2)\chi_2 + (i_1 - i_3)\chi_3$ while $d(S) = i_3(\chi_1 + \chi_2)$, so that there are one constraint for PDer, therefore $H^1(O_{\mathrm{h}}, \mathbb{Z}_2^3) = \mathbb{Z}_2^3$, generated by $d\colon \overline{C}_6 \mapsto (i_1\chi_1 + i_2\chi_2 + i_3\chi_3), S \mapsto \chi_j$ where $i_1 + i_2 + i_3 = 0$, $j = 1$ or 2, $i_1, i_2, i_3 = 0$ or 1, i.e. the choices are $(i_1, i_2, i_3, j) = (0,0,0,1), (0,1,1,1), (1,0,1,1), (1,1,0,1), (0,0,0,2), (0,1,1,2), (1,0,1,2), (1,1,0,2)$. To summarize, we now have

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z}_2 & H^1(O_{\mathrm{h}}, \mathbb{Z}_2) & H^2(O_{\mathrm{h}}, \mathbb{Z}_2) & H^3(O_{\mathrm{h}}, \mathbb{Z}_2) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2) & \cdots \\
\mathbb{Z}_2 & H^1(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^2(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^3(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2^3) & \cdots \\
0 & \mathbb{Z}_2^3 & H^2(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^3(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2^3) & \cdots \\
\mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^4 & \mathbb{Z}_2^7 & \mathbb{Z}_2^{10} & \cdots
\end{array}.$$

In view of Bill's note, the essence now is to see what form of translations the term $(\bar{g}_1\bar{g}_2)^{*-1}\bar{g}_1^*\bar{g}_2^*$ are of, for all $\bar{g}_{1,2} \in O_{\mathrm{h}}$. This is at most a 48×48 matrix.

Interestingly, there is a lifting for the $T_d \cong S_4$ part in the full space group that is isomorphic to it self: $T_d \lhd Fd\bar{3}m$. However, including inversion masses this up: there is not a subgroup of $Fd\bar{3}m$ that is isomorphic to O_{h}.

Next we write the conjectured E_3 page:

$$E_3\colon \quad \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \cdots \\
? & H^1(O_{\mathrm{h}}, \mathbb{Z}_2) & H^2(O_{\mathrm{h}}, \mathbb{Z}_2) & H^3(O_{\mathrm{h}}, \mathbb{Z}_2) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2) & \cdots \\
\mathbb{Z}_2 & H^1(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^2(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^3(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2^3) & \cdots \\
0 & 0 & ? & H^3(O_{\mathrm{h}}, \mathbb{Z}_2^3) & H^4(O_{\mathrm{h}}, \mathbb{Z}_2^3) & \cdots \\
\mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^4 & \mathbb{Z}_2^4 & ? & \cdots
\end{array}.$$

An element in $B^3(G, A)$ (remember $H^3(G, A) = Z^3(G/A)/B^3(G, A)$) is $\omega\colon G \times G \times G \to A$ (i.e. $\omega \in C^3(G, A)$) such that $d\omega(g_1, g_2, g_3) = g_1.h(g_2, g_3) - h(g_1g_2, g_3) + h(g_1, g_2g_3) - h(g_1, g_2)$ for some $h \in C^2(G, A)$. Similarly, If $f \in B^2(G, A)$ then there is $h \in C^1(G, A)$ s.t. $f(g_1, g_2) = g_1.h(g_2) - h(g_1g_2) + h(g_1)$, while $f \in Z^2(G, A)$ is equivalent to $0 = g_1.f(g_2, g_3) -$

$f(g_1g_2,g_3)+f(g_1,g_2g_3)-f(g_1,g_2)$. Finally, $f\in B^1(G,A)$ means that there is $a_0\in A$ s.t. $f(g_1)=g_1.a_0-a_0$, while $f\in Z^1(G,A)$ means that $0=g_1.f(g_2)-f(g_1g_2)+f(g_1)$.

Since Bill's note says $d_2(\alpha)(g_1,g_2,g_3)=\alpha(g_3)((g_1g_2)^{*-1}g_1^*g_2^*)$, to find those $d_2(\alpha)$ that lives in $B^3(O_\mathrm{h},\mathbb{Z}_2)$ we must solve $\alpha(g_3)((g_1g_2)^{*-1}g_1^*g_2^*)=h(g_2,g_3)-h(g_1g_2,g_3)+h(g_1,g_2g_3)-h(g_1,g_2)$ for some $h\in C^2(O_\mathrm{h},\mathbb{Z}_2)$.

The 1981 paper by Johannes Huebschmann is what we want!! `https://core.ac.uk/download/pdf/82585136.pdf` This paper gives the recipe for the d_2 maps. See the original question on mathoverflow `https://mathoverflow.net/questions/590/differentials-in-the-lyndon-hochschild-spectral-sequence`.

Eq. (2) in the language of spectral sequence, this is

$$0\to E_2^{1,0}\to H^1(G,M)\to E_2^{0,1}\xrightarrow{d_2}E_2^{2,0}\to H^2(G,M),$$

which is in Rotman, P645. Now in our case, since we know $E_2^{1,0}=\mathbb{Z}_2^2$, $E_2^{0,1}=0$, and $E_2^{2,0}=\mathbb{Z}_2^4$, this becomes

$$0\to\mathbb{Z}_2^2\to H^1(G,M)\to 0\xrightarrow{d_2}\mathbb{Z}_2^4\to H^2(G,M),$$

which gives $H^1(G,M)=\mathbb{Z}_2^2$, in agreement with the GAP result.

Using the LHS short exact sequence associated with group presentation, i.e. Eq. (11), we get (note that F is the free group generated by $T_1,T_2,T_3,S,\overline{C}_6$)

$$0\to\mathbb{Z}_2^2\to\mathbb{Z}_2^5\to H^1(\mathcal{R},\mathbb{Z}_2)^{SG}\xrightarrow{\mathrm{tg}}H^2(SG,\mathbb{Z}_2)\to 0,$$

12 The example studied in Johannes Huebschmann

Setup: $N=\mathbb{Z}/2\times\mathbb{Z}/2$, $G=\mathbb{Z}/4\times\mathbb{Z}/2$, $Q=G/N=\mathbb{Z}/2$, $A=\mathbb{Z}/2$, with trivial G-action. Consider $H^2(G,A)$.

Background: $H^1(\mathbb{Z}_2,\mathbb{Z}_2)=\mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$. In fact we have $H^n(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$ for any $n\geq 1$ according to Rotman's book, see the paragraph below Eq. (25); see also `https://groupprops.subwiki.org/wiki/Second_cohomology_group_for_trivial_group_action_of_Z2_on_Z2`. In fact $H^2(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$ precisely corresponds to the only two groups of order 4: the Klein four-group $V=\mathbb{Z}_2\times\mathbb{Z}_2$, and the cyclic group $\mathbb{Z}_4$. The results can be easily generalized to other prime-characteristic fields, see the links within the above url.

$$e\colon 0\to A\to E\xrightarrow{\pi}N\to 1,$$

- Let Γ be a group and A a left Γ-module, ${}^\gamma a$ is the action of γ on A. Denote by $\mathrm{Aut}(\Gamma,A)$ the subgroup of $\mathrm{Aut}(\Gamma)\times\mathrm{Aut}(A)$ that consists of pairis (φ,σ) of automorphisms φ of Γ and σ of A such that $\sigma({}^\gamma a)={}^{\varphi(\gamma)}\sigma(a)$, $\gamma\in\Gamma$, $a\in A$, we call $\mathrm{Aut}(\Gamma,A)$ the group of automorphisms of the pair (Γ,A).
- $\mathrm{Aut}^A(E)$ denotes the group of automorphisms of E which map A to itself.
- Each $\alpha\in\mathrm{Aut}^A(E)$ induces an automorphism
-

In our case, our e is $0\to\mathbb{Z}_2\to E\to\mathbb{Z}^3\to 1$, the corresponding classification is $H^2(\mathbb{Z}^3,\mathbb{Z}_2)=\mathbb{Z}_2^3$, while we have $E_2^{0,2}=H^2(\mathbb{Z}^3,\mathbb{Z}_2)^{O_\mathrm{h}}=(\mathbb{Z}_2^3)^{O_\mathrm{h}}=\mathbb{Z}_2$. We know that $\chi_1\cap\chi_2$ represents the group $E=\langle\bar{T}_1,\bar{T}_2,T_3,x;[\bar{T}_1,\bar{T}_2],[\bar{T}_1,T_3]x,[\bar{T}_2,T_3]x,x^2\rangle$.

12.1 The map $d_2\colon E_2^{0,2}=H^2(N,A)^Q\to E_2^{2,1}=H^2(Q,H^1(N,A))$

Theorem (Theorem 1 of Huebschmann's paper): the rule $e\mapsto\bar{e}$ describes the differential $d_2\colon E_2^{0,2}\to E_2^{2,1}$, where $\bar{e}$ is obtained following the procedure below:

$$\begin{tikzcd}
\text{Step 1:} & e\colon & 0 \arrow[r] & A \arrow[r] \arrow[d, "\zeta"] & E \arrow[r, "\pi"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "i"] & 1 & \in & H^2(N,A)^Q \\
\text{Step3:} & & 0 \arrow[r] & \mathrm{Der}(N,A) \arrow[r] \arrow[d] \arrow[ddr, "\cong"] & \mathrm{Aut}_G(e) \arrow[r] \arrow[d] \arrow[ddr] & G \arrow[r] \arrow[d] \arrow[ddr, "\chi"] & 1 & & \\
\text{Step4:} & \bar{e}\colon & 0 \arrow[r] & H^1(N,A) \arrow[r] & \mathrm{Out}_G(e) \arrow[r] & Q \arrow[r] & 1 & \in & H^2(Q,H^1(N,A)) \\
\text{Step2:} & & & 0 \arrow[r] & \mathrm{Der}(N,A) \arrow[r, "i_e"] & \mathrm{Aut}_G^A(E) \arrow[r, "h_e"] & \mathrm{Aut}_G(N,A) \arrow[rr] & & 1
\end{tikzcd}$$

where

$$\mathrm{Aut}(N,A) = \{(\varphi,\sigma) \in \mathrm{Aut}(N) \times \mathrm{Aut}(A) | \sigma(n.a) = \varphi(n).\sigma(a), n \in N, a \in A\},$$

$$\chi \colon G \to \mathrm{Aut}(N,A), \quad g \mapsto (n \mapsto gng^{-1}, a \mapsto g.a), \quad \mathrm{Aut}_G(N,A) := \mathrm{im}\chi \subset \mathrm{Aut}(N,A),$$

The underlying statement that $\mathrm{Aut}_G(N,A) \subset \mathrm{Aut}(N,A)$ can be easily shown: $\sigma(n.a) = \varphi(n).\sigma(a) \Leftrightarrow g.(n.a) = (gng^{-1}).(g.a)$ which is true.

Denote $\mathrm{Aut}^A(E)$ the group of automorphisms of E that maps A to itself, then

$$\alpha \in \mathrm{Aut}^A(E) \mapsto (\alpha|_A, \alpha|_N) \in \mathrm{Aut}(N,A),$$

again the underlying statement that $\mathrm{Aut}^A(E) \subset \mathrm{Aut}(N,A)$ can be easily proved: $\sigma(n.a) = \varphi(n).\sigma(a) \Leftrightarrow \alpha(n.a) = \alpha(n).\alpha(a)$ which holds (note that the action of $n \in N$ on $a \in A$ exactly becomes the group multiplication rule in the extended group E[4]).

So if we restricts the image of the above map to $\mathrm{Aut}_G(N,A) \subset \mathrm{Aut}(N,A)$ then its preimage defines $\mathrm{Aut}_G^A(E)$ and this defined h_e. Turns out $\ker(h_e) = \mathrm{Der}(N,A)$, which defines i_e. For i_e, it is defined by $i_e \colon d \mapsto \alpha_d$ with $\alpha_d(x) = d(\pi(x)) \cdot x = d(n_x)(a_x,n_x) = (a_x + d(n_x), n_x)$ for $x = (a_x,n_x) \in E$, here $\pi(x) = n_x \in N$. Note that $h_e(\alpha_d) \neq 0$. However, for such a α_d, one can always assuciate a $\tilde{\alpha}_d \in \mathrm{Aut}^A(E)$ which further maps $(a_x + d(n_x), n_x)$ to (a_x, n_x), i.e. $\tilde{\alpha}_d$ is an automorphism of N that is identity on A, and $h_e(\alpha_d) = 0$. If we call $\tilde{i}_e \colon d \mapsto \tilde{\alpha}_d$, then we get the exact sequence $\mathrm{im}\tilde{i}_e = \ker h_e$. As Huebschmann mentioned, one can find the proof in Eilenberg's paper `https://www.ams.org/journals/bull/1949-55-01/S0002-9904-1949-09161-9/S0002-9904-1949-09161-9.pdf`, P12, where Eilenberg's $(G, Q, A_1, Z^1(Q,G))$ is ours $(A, N, \mathrm{Aut}_G^A(E), \mathrm{Der}(N,A))$, and his A_2 is the subgroup of our $\mathrm{aut}_G^A(E)$ that is identity on A.

$\mathrm{Aut}_G(e)$ is defined as the pullback (fibre product) of the two maps $h_e \colon \mathrm{Aut}_G^A(E) \to \mathrm{Aut}_G(N,A)$ and $\chi \colon G \to \mathrm{Aut}_G(N,A)$:

$$\mathrm{Aut}_G(e) := \mathrm{Aut}_G^A(E) \times_{\mathrm{Aut}_G(N,A)} G,$$

it is the unique group that makes the diagram commute. Three other maps:

$$\zeta \colon A \to \mathrm{Der}(N,A), \quad a \mapsto (n \mapsto n.a - a, n \in N) \quad \text{is the inner derivation;}$$

$$\beta \colon E \to \mathrm{Aut}_G(e), \quad x \mapsto (i_x, \pi(x)), \quad i \text{ is the inclusion;}$$

$\beta(E)$ is normal in $\mathrm{Aut}_G(e)$, so we define $\mathrm{Out}_G(e)$ to be the cokernel of β.

For us, G has trivial action on $A = M = \mathbb{Z}_2$, $N = T = \mathbb{Z}^3$, $H^1(N,A) = H^1(\mathbb{Z}^3, \mathbb{Z}_2) = \mathbb{Z}_2^3$, $Q = G/T = O_\mathrm{h}$, and since $H^2(T,A)^Q = (\mathbb{Z}_2^3)^Q = \mathbb{Z}_2$, where the only nontrivial element $\omega(T,T') = x_1 x_2' + x_2 x_3' + x_3 x_1'$ in the mod 2 sense (where we abbreviated $T = T_1^{x_1} T_2^{x_2} T_3^{x_3}$ and similarly for T'), meaning that the group has the form $(a,T)(a',T') = (a + a' + \omega(T,T'), TT')$. Note that $\mathrm{Aut}_G(N,A)$ is nontrivial. In Huebschmann's example $\mathrm{Aut}_G(N,A)$ is trivial, which simplifies things by a lot.

13 Random thoughts

P497: The automorphism group $\mathrm{Aut}(E)$ is a group E is the group whose elements are all the isomorphisms of E with itself and whose operation is composition. An automorphism φ is inner if it is a conjuhation; that is, there is $c \in E$ with $\varphi(e) = c + e - c$ for all $e \in E$ (in additive notation). An automorphism of E is outer if it is not inner. This defines $\mathrm{Inn}(E) \subseteq \mathrm{Aut}(E)$, and $\mathrm{Out}(E) \equiv \mathrm{Aut}(E)/\mathrm{Inn}(E)$.

The book [AM]: $H_\phi^*(G,C)$, where C is the center of some group N, and ϕ is the action of G on C (in [AM]'s language, ϕ actually should be ϕ^C which is is the action of G on N restricted to C). In this book, most of the places ϕ is the identity (meaning the group cohomology is with untwisted coefficient, or equivalently C is a trivial $\mathbb{Z}G$ module). When making connection with the classifying space cohomology, note that it is always assumed that A is a trivial $\mathbb{Z}G$ module. See the Introduction of [AM] or Rotman.

Abelianization of a group G is $G/[G,G]$. For intwisted cohomology with $\mathbb{Z}_2$ coefficients, we have $H^1(SG, \mathbb{Z}_2) = \mathrm{Hom}(SG/[SG,SG], \mathbb{Z}_2)$. The abelianization of all space groups is given in the paper `http://scripts.iucr.org/cgi-bin/paper?S0108767308036222` (but no access to download) and `https://link.springer.com/content/pdf/10.1007/JHEP01(2019)055.pdf`. Turns out that pyrochlore has the abelianzation group $\mathbb{Z}_2^2$. It is easy to check that $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2) = \mathbb{Z}_2^2$, which are $\{1, x, y, x+y\}$, where $x \colon (b,0) \mapsto b$ while $y \colon (0,b) \mapsto b$ for $b = 0, 1$. Then $x + y \colon (1,0) \mapsto 1$,

---

[4] Recall that in the problem of group extension $A \to E \to N$, the essential statement is that any elements in E can be written as (a,n), with group multiplication $(a,n)(b,m) = (a + n.b + f(n,m), nm)$, where the action of $n.b$ is the predefined action of N in A that makes the Abelian group A a N-module, and $f(n,m)$ is factor set satisfies $f(1,m) = f(n,1) = 0$ for any $n, m \in N$, which is a cocycle and lives in $H^2(N,A)$. See Rotman P506. However, note that we have $(0,n)(b,1) = (n.b, n)$ and $(0,n)(b,1)(0,n^{-1}) = (n.b + f(n,n^{-1}), 1)$, and one can always choose $f(n,n^{-1}) = 0$ for all n to make conjugation by $n \in N$ also the action $n.b$, since we can always use coboundary to adjust f: $f(n,n^{-1}) = n.\theta(n^{-1}) - \theta(n)$ where $\theta \colon N \to A$ is any function.

$x+y\colon (0,1)\mapsto 1$, and $x+y\colon (1,1)\mapsto 0$, also $x^2=y^2=(x+y)^2=1$ so this is the Klein four ($\mathbb{Z}_2^2$). Then from the above paper in princple it is eacy to find H^1 for all other space groups.

$H^1_{\text{id}}(227,\mathbb{Z}_2)=\mathbb{Z}_2^2$.

Note that Bieberbach group is defined as torsion-free crystallographic groups. Out of all the three-dimensional crystallographic groups 10 of them are Bieberbach.

Gap has a command `Mod2CohomologyRingPresentation`, but only works for 2-groups.

In the basis $\hat{e}_{1,2,3}$, we have $T_1\colon (r_1,r_2,r_3)\to(r_1+1,r_2,r_3)$, $T_2\colon (r_1,r_2,r_3)\to(r_1,r_2+1,r_3)$, and $T_3\colon (r_1,r_2,r_3+1)$, and

$$\overline{C}_6=\begin{pmatrix}0&0&-1&0\\-1&0&0&0\\0&-1&0&0\\0&0&0&1\end{pmatrix},\quad S=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\1&1&1&1/2\\0&0&0&1\end{pmatrix},$$

Subgroups of 227 that still contain the original translation: see ITC P702. Note that we use the second choice of coordinate so that the origin is the inversion center (this is the merit of this coordinate choice). According to P702, there are three subgroups, $F\bar{4}3m$, $F4_132$, and $Fd\bar{3}1$, which are isomorphic to the groups 216, 210 and 203. The first two (216 and 210) do not contain inversion, while the third contains inversion. Only the first one (216) is symmorphic (semi-direct product), while 210 and 203 are nonsymmorphic.

Note that in ITC, (x,y,z) are the coordinate in the $\hat{x},\hat{y},\hat{z}$ basis of the cubi system. In this basis, $C_3\colon (x,y,z)\to(z,x,y)$ whichi is ITC(5), and $S\colon (x,y,z)\to(y+1/4,x+1/4,-z)$, which is really ITC(13), up to a translation $t(0,1,0)$ (This is the translation that translates $(x,y,z)\mapsto(x+1/2,y,z+1/2)$). We see that 216 does not contain S, while 210 contains S, which makes sense since S is the nonsymmorpic element. Furthermore, ITC(37) $=SI$, up to translation; ITC(38) is the mirror M with mirror plane spanned by $\hat{x}+\hat{y}$ and $\hat{z}$, and using our choice of generators S,I,C_3, we also defined $\Sigma=SI$, $\overline{C}_6=C_3I$, we have (using our PSG paper notation) $M=(12)=\Sigma C_3\Sigma C_3^{-1}\Sigma C_3\colon (x,y,z)\mapsto(y,x,z)$, without even translation. Next, we see that $\Sigma=(37)$ (up to translation) $=(-y+1/4,-x+1/4,z)$ (easily checked) which is a (weird) mirror with mirror plane the $(\hat{x}-\hat{y})\times\hat{z}$ plane, and contains the $(1/8,1/8,0)$ point; and that the two-fold rotation $C_2=\text{ITC}(2)\bar{x}+1/4,\bar{y}+1/4,z=(12)(34)=C_3\Sigma C_3^{-1}\Sigma C_3$ (up to translation), this is the (34)(12) in our notation. Note that (3) is another two-fold rotation along the z axis. We know that $\langle(12)(34),(123)\rangle=A_4$, the order 12 alternating group. To summarize, we now defined

$$\begin{aligned}&\Sigma=SI=C_2'MC_2',\ \text{mirror with mirror plane } (\hat{x}-\hat{y})\times\hat{z} \text{ that contains the point } (1/8,1/8,0),\\&M=\Sigma C_3\Sigma C_3^{-1}\Sigma C_3,\ \text{mirror with plane } (\hat{x}+\hat{y})\times\hat{z} \text{ that contains the origin},\\&C_2=C_3\Sigma C_3^{-1}\Sigma C_3 \text{ two-fold rotation with axis } (1/8,1/8,0)+\hat{z}.\\&C_2'=C_3^{-1}C_2C_3 \text{ two-fold rotation with axis } (1/8,0,1/8)+\hat{y}.\end{aligned}\tag{29}$$

and ITC(2) $=C_2$, ITC(13) $=S$, ITC(5) $=C_3$, ITC(25) $=I$, ITC(37) $=\Sigma$, ITC(38) $=M$.

Now we see that

$$\begin{aligned}F23(196)&=\langle\text{ITC}(1)-(12)\rangle=\langle T_1,T_2,T_3,C_3,C_2\rangle,\\F\bar{4}3m(216)&=\langle\text{ITC}(1)-(12),\text{ITC}(37)-(48)\rangle=\langle T_1,T_2,T_3,C_3,C_2,\Sigma\rangle=\langle T_1,T_2,T_3,C_3,C_2,M\rangle,\\F4_132(210)&=\langle\text{ITC}(1)-(24)\rangle=\langle T_1,T_2,T_3,C_3,C_2,S\rangle,\\Fd\bar{3}1=(Fd\bar{3},203)&=\langle\text{ITC}(1)-(12),\text{ITC}(25)-(36)\rangle=\langle T_1,T_2,T_3,C_3,C_2,I\rangle.\end{aligned}$$

Note that their point groups are, respectively, $T=A_4$, $T_d\cong S_4$, $O\cong S_4$ and $T_h\cong A_4\times\mathbb{Z}_2$.

We can further analyze the subgroup of $F\bar{4}3m(216)$. From ITC P666 we know that this is F23(196), generated by C_3 and C_2, whose point group is T with 12 elements, isomorphic to A_4. In this case, we have

$$H^1(\mathbb{Z}^3,\mathbb{Z}_2)^{A_4}=0$$

also. We know from Milgram, P92 that

$$H^*(A_4,\mathbb{Z}_2)\cong\mathbb{F}_2[a,b]^{\mathbb{Z}/3}\cong\mathbb{F}_2[A,B](C)/(C^3+A^2+B^2+AB)$$

with degree of A,B,C being 3,3,2. The GAP gives that $H^{1:9}$ with dimension $0,1,2,1,2,3,2,3,4$. We have $H^{1:12}(F23,\mathbb{Z}_2)=0,3,6,2,6,10,6,10,14,10,14,18$.

Always use `List([1..3],n->Cohomology(HomToIntegersModP(ResolutionAlmostCrystalGroup(Image(IsomorphismPcpGrou`

We consider the subgroup of F23: R3 (No. 146), symmorphic, point group is C_3, so that $R3=\mathbb{Z}^3\rtimes\mathbb{Z}_3$. The command `List([1..15],n->Cohomology(HomToIntegersModP(ResolutionAlmostCrystalGroup(Image(IsomorphismPcpGroup(SpaceGrou`

Table 1: C_n is cyclic group of order n and D_n is dihedral group of order $2n$. $\rtimes$ means that $H^2(PG,T)$ is trivial, $T=\mathbb{Z}^2$. The column Generator omits the two translations. m_c denotes mirror whose mirror plane is perpendicular to the xy plane and bisects t_1 and t_2; m_p denotes mirror whose mirror plane is perpendicular to the xy plane and contains t_1; s_p is mirror m_p composed with $t_1/2$; s_{pc} is s_p further composed with $t_2/2$; For $p3$, $p3m1$ and $p31m$, the basis t_1 and t_2 has an angle of 120°; m_l is mirror with mirror plane perpendicular to the xy plane and $+30°$ with t_1; m_s is mirror with mirror plane perpendicular to the xy plane and contains t_1. m_y is mirror that flips y coordinate (in $p4g$ and $p4m$ $t_1 \perp t_2$). We see there is no nontrivial direct product between PG and T. The second last column makes use of $H^1(PG,\mathbb{Z}_2)=\mathrm{Hom}(PG,\mathbb{Z}_2)$ and $H^*(S_3,\mathbb{Z}_2)=\mathbb{F}_2[e]$ where $S_3=D_3$, $H^*(D_4,\mathbb{Z}_2)=\mathbb{F}_2[x_1,e_1,y_2]/(x_1e_1)$; the rest of the $H^1(PG,\mathbb{Z}_2)$ are calculated using GAP. The last column records the naive sum of the dimension of previous two columns.

No.	([LZ])	Name	Point Group	$H^2(PG,T)$	Generators	Action: $T\to pTp^{-1}$	$H^1(T,\mathbb{Z}_2)^{PG}$	$H^1(PG,\mathbb{Z}_2)$	dim
1	(1)	$p1$	C_1	$\rtimes$	$\{id\}$	$T\xrightarrow{id}\{t_1,t_2\}$	$\langle\chi_1,\chi_2\rangle$	$\langle id\rangle$	2
2	(5)	cm	D_1(c)	$\rtimes$	$\{m_c\}$	$T\xrightarrow{m_c}\{t_2,t_1\}$	$\langle\chi_1+\chi_2\rangle$	$\mathbb{Z}_2$	2
3	(3)	pm	D_1(p)	$H^2(D_{1,p},\mathbb{Z}^2)=\mathbb{Z}_2$	$\{m_p\}$	$T\xrightarrow{m_p}\{t_1,t_2^{-1}\}$	$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2$	3
4	(4)	pg	D_1(p)	See above	$\{s_p\}$	$T\xrightarrow{s_p}\{t_1,t_2^{-1}\}$	$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2$	$3^{(-1}$
5	(2)	$p2$	C_2	$\rtimes$	$\{i\}$	$T\xrightarrow{m_p}\{t_1^{-1},t_2^{-1}\}$	$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2$	3
6	(9)	cmm	D_2(c)	$\rtimes$	$\{i,m_c\}$		$\langle\chi_1+\chi_2\rangle$	$\mathbb{Z}_2^2$	3
7	(6)	pmm	D_2(p)	$H^2(D_{2,p},\mathbb{Z}^2)=\mathbb{Z}_2^2$	$\{i,m_p\}$		$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2^2$	4
8	(7)	pmg	D_2(p)	See above	$\{i,s_p\}$		$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2^2$	$4^{(-1}$
9	(8)	pgg	D_2(p)	See above	$\{i,s_{pc}\}$	$T\xrightarrow{s_{pc}}\{t_1,t_2^{-1}\}$	$\langle\chi_1,\chi_2\rangle$	$\mathbb{Z}_2^2$	$4^{(-2}$
10	(13)	$p3$	C_3	$\rtimes$	$\{c_3\}$	$T\xrightarrow{c_3}\{t_2,t_1^{-1}t_2^{-1}\}$	$\langle\rangle$	$\langle id\rangle$	0
11	(14)	$p3m1$	D_3(l)	$\rtimes$	$\{c_3,m_l\}$	$T\xrightarrow{m_l}\{t_1t_2,t_2^{-1}\}$	$\langle\rangle$	$\mathbb{Z}_2$	1
12	(15)	$p31m$	D_3(s)	$\rtimes$	$\{c_3,m_s\}$	$T\xrightarrow{m_s}\{t_1,t_1^{-1}t_2^{-1}\}$	$\langle\rangle$	$\mathbb{Z}_2$	1
13	(10)	$p4$	C_4	$\rtimes$	$\{c_4\}$	$T\xrightarrow{c_4}\{t_2,t_1^{-1}\}$	$\langle\chi_1+\chi_2\rangle$	$\mathbb{Z}_2$	2
14	(11)	$p4m$	D_4	$H^2(D_4,\mathbb{Z}^2)=\mathbb{Z}_2$	$\{c_4,m_p\}$		$\langle\chi_1+\chi_2\rangle$	$\mathbb{Z}_2^2$	3
15	(12)	$p4g$	D_4	$H^2(D_4,\mathbb{Z}^2)=\mathbb{Z}_2$	$\{c_4,s_{pc}\}$		$\langle\chi_1+\chi_2\rangle$	$\mathbb{Z}_2^2$	$3^{(-1}$
16	(16)	$p6$	C_6	$\rtimes$	$\{c_6\}$	$T\xrightarrow{c_6}\{t_1t_2,t_1^{-1}\}$	$\langle\rangle$	$\mathbb{Z}_2$	1
17	(17)	$p6m$	D_6	$\rtimes$	$\{c_6,m_s\}$		$\langle\rangle$	$\mathbb{Z}_2^2$	2

gives $1,1,1,0,0,0,0,0,0,0,0,0,0,0,0$, so we have

$$H^n(\mathbb{Z}^3\rtimes\mathbb{Z}_3,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2, & n=1,\\ \mathbb{Z}_2, & n=2,\\ \mathbb{Z}_2, & n=3,\\ 0, & n\geq 4.\end{cases}$$

We have $H^n(\mathbb{Z}_3,\mathbb{Z}_2)=0$ for $n\geq 1$. Note that, according to Weicheng, $H^*(\mathbb{Z}^3,\mathbb{Z}_2)=\mathbb{F}_2[\chi_1,\chi_2,\chi_3]/(\chi_1^2,\chi_2^2,\chi_3^2)$, where $\chi_{1,2,3}$ are grade-1. So we see that

$$H^*(R3,\mathbb{Z}_2)=\mathbb{F}_2[a,b]/(a^2,b^2),\quad \deg(a)=1,\,\deg(b)=2,$$

where $a=\chi_1+\chi_2+\chi_3$ and $b=\chi_1\cup\chi_2+\chi_2\cup\chi_3+\chi_3+\chi_1$. So we have a grade-3 element which is $\chi_1\cup\chi_2\cup\chi_3$. However, R3 is not normal subgroup of F23.

We turn to a normal subgroup of F_{23}. We have $F222(22)=\langle T_1,T_2,T_3,C_2,C_2'\rangle$, point group is D_2, generated by C_2 and C_2'. We have $H^{1:10}(F222,\mathbb{Z}_2)=(4,7,10,14,18,22,26,30,34,38)$. F222 has (normal) subgroup C2(No.5), which has point group C_2. we have $H^{1:10}(C2,\mathbb{Z}_2)=(3,4,4,4,....,4)$. We should be able to guess this one: note that $C_2T_1C_2^{-1}=T_3^{-1}T_2$, $C_2T_2C_2^{-1}=T_3^{-1}T_1$, $C_2T_3C_2^{-1}=T_3^{-1}$. We see that $\chi_1+\chi_2$ is stabilized by C_2, and $\chi_1+\chi_3$. So we see that $H^1(T,\mathbb{Z}_2)^{C_2}=\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle=\mathbb{Z}_2^2$, and we have $H^*(\mathbb{Z}_2,\mathbb{Z}_2)=F[x]$ where x has grade 1. We see that the $d_2\colon E_2^{0,1}\to E_2^{2,0}$ is zero map, giving $H^1(C2,\mathbb{Z}_2)=\mathbb{Z}_2^3$. $C_2\colon(\chi_1,\chi_2,\chi_3)\to(\chi_2,\chi_1,\chi_1+\chi_2+\chi+3)$. According to Weicheng's claim that $H^1(\mathbb{Z}_2,H^1(T,\mathbb{Z}_2))=H^1(\mathbb{Z}_2,\mathbb{Z}_2^p)$ where p is the dimension that is fixed under C_2. We see that $p=2$,

14 Wall paper group

In explicit form (note that each one has its own definition of translation):

$$T_1=\begin{pmatrix}1 & & 1\\ & 1 & \\ & & 1\end{pmatrix},\ T_2=\begin{pmatrix}1 & & \\ & 1 & 1\\ & & 1\end{pmatrix},\ i=\begin{pmatrix}-1 & & \\ & -1 & \\ & & 1\end{pmatrix},\ c_3=\begin{pmatrix} & -1 & \\ 1 & -1 & \\ & & 1\end{pmatrix},\ c_4=\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix},\ c_6=\begin{pmatrix}1 & -1\\ 1 & \end{pmatrix}$$

Table 2: Note: When inputing into gap, $cmm/pmm/pmg/pgg$ should be written as $c2mm/p2mm/p2mg/p2gg$, and $p4m/p4g/p6m$ as $p4mm/p4gm/p6mm$. The column $H^{1:6}(SG, \mathbb{Z}_2)$ records the copies of $\mathbb{Z}_2$ in each $n = 1, 2, ..., 6$ cohomolgoy group. Note we denoted $E_2^{1:6,0} = H^{1:6}(PG, \mathbb{Z}_2)$ and $E_2^{0,1} = H^1(T, \mathbb{Z}_2)^{PG}$. We used $H^{n\geq 1}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. Note that $H^2(SG, \mathbb{Z}_2) = E_\infty^{0,2} \oplus E_\infty^{1,1} \oplus E_\infty^{2,0}$ and $E_\infty^{2,0} = E_3^{2,0}$, $E_\infty^{1,1} = E_3^{1,1}$, and $E_\infty^{0,2} = E_4^{0,2}$.

No.	([LZ])	Name	Point Group	$E_2^{1:6,0}$	$E_3^{2,0}$	$E_3^{0,1} = E_\infty^{0,1}$	$E_2^{0,1}$	$H^{1:6}(SG, \mathbb{Z}_2)$	$E_4^{0,2} \oplus E_3^{1,1}$
1	(1)	$p1$	C_1	(0,0,0,0,0,0)	0	2	$2 = \langle \chi_1, \chi_2 \rangle$	$(2, 1, 0, 0, 0, 0)$	1
2	(5)	cm	D_1(c)	(1,1,1,1,1,1)	1	1	$1 = \langle \chi_1 + \chi_2 \rangle$	$(2, 2, 2, 2, 2, 2)$	1
3	(3)	pm	D_1(p)	(1,1,1,1,1,1)	1	2	$2 = \langle \chi_1, \chi_2 \rangle$	$(3, 4, 4, 4, 4, 4)$	3
4	(4)	pg	D_1(p)	(1,1,1,1,1,1)	0	$1(\mathrm{im} d_2 = 1)$	$2 = \langle \chi_1, \chi_2 \rangle$	$(2, 1, 0, 0, 0, 0)$	1
5	(2)	$p2$	C_2	(1,1,1,1,1,1)	1	2	$2 = \langle \chi_1, \chi_2 \rangle$	$(3, 4, 4, 4, 4, 4)$	3
6	(9)	cmm	D_2(c)	$(2, 3, 4, 5, 6, 7)$	3	1	$1 = \langle \chi_1 + \chi_2 \rangle$	$(3, 5, 7, 9, 11, 13)$	2
7	(6)	pmm	D_2(p)	$(2, 3, 4, 5, 6, 7)$	3	2	$2 = \langle \chi_1, \chi_2 \rangle$	$(4, 8, 12, 16, 20, 24)$	5
8	(7)	pmg	D_2(p)	$(2, 3, 4, 5, 6, 7)$	2	$1(\mathrm{im} d_2 = 1)$	$2 = \langle \chi_1, \chi_2 \rangle$	$(3, 4, 4, 4, 4, 4)$	2
9	(8)	pgg	D_2(p)	$(2, 3, 4, 5, 6, 7)$	1	$0(\mathrm{im} d_2 = 2)$	$2 = \langle \chi_1, \chi_2 \rangle$	$(2, 2, 2, 2, 2, 2)$	1
10	(13)	$p3$	C_3	$(0, 0, 0, 0, 0, 0)$	0	0	$0 = \langle \rangle$	$(0, 1, 0, 0, 0, 0)$	1
11	(14)	$p3m1$	D_3(l)	$(1, 1, 1, 1, 1, 1)$	1	0	$0 = \langle \rangle$	$(1, 2, 2, 2, 2, 2)$	1
12	(15)	$p31m$	D_3(s)	$(1, 1, 1, 1, 1, 1)$	1	0	$0 = \langle \rangle$	$(1, 2, 2, 2, 2, 2)$	1
13	(10)	$p4$	C_4	$(1, 1, 1, 1, 1, 1)$	1	1	$1 = \langle \chi_1 + \chi_2 \rangle$	$(2, 3, 3, 3, 3, 3)$	2
14	(11)	$p4m$	D_4	$(2, 3, 4, 5, 6, 7)$	3	1	$1 = \langle \chi_1 + \chi_2 \rangle$	$(3, 6, 9, 12, 15, 18)$	3
15	(12)	$p4g$	D_4	$(2, 3, 4, 5, 6, 7)$	2	$0(\mathrm{im} d_2 = 1)$	$1 = \langle \chi_1 + \chi_2 \rangle$	$(2, 3, 4, 5, 6, 7)$	1
16	(16)	$p6$	C_6	$(1, 1, 1, 1, 1, 1)$	1	0	$0 = \langle \rangle$	$(1, 2, 2, 2, 2, 2)$	1
17	(17)	$p6m$	D_6	$(2, 3, 4, 5, 6, 7)$	3	0	$0 = \langle \rangle$	$(2, 4, 6, 8, 12)$	1

$$m_c = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \; m_p = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \; m_l = \begin{pmatrix} 1 & & \\ 1 & -1 & \\ & & 1 \end{pmatrix}, \; m_s = \begin{pmatrix} 1 & -1 & \\ & -1 & \\ & & 1 \end{pmatrix}, \; s_p = \begin{pmatrix} 1 & & \frac{1}{2} \\ & -1 & \\ & & 1 \end{pmatrix}, \; s_{pc} = \left(\begin{matrix} 1 \end{matrix} \right.$$

Note that $H^1(T, \mathbb{Z}_2)^{PG} = E_2^{0,1}$, and $H^1(PG, \mathbb{Z}_2) = E_2^{1,0}$. $E_2^{1,0}$ is stablizied already, however $E_2^{0,1}$ is not; we have $H^1(SG, \mathbb{Z}_2) = E_\infty^{0,1} \oplus E_\infty^{1,0} = E_3^{0,1} \oplus E_2^{1,0}$, where

$$E_3^{0,1} = \ker(d_2 \colon E_2^{0,1} \to E_2^{2,0}).$$

Therefore, we further have to check which of the $H^1(T, \mathbb{Z}_2)^{PG}$ gets sent to $H^2(PG, \mathbb{Z}_2)$.

Now using the standard method in GAP:

```
List([1..5],n->Cohomology(HomToIntegersModP(ResolutionAlmostCrystalGroup(Image(IsomorphismPcpGroup(SpaceGroup
```

where we put the name of the wallpaper group in `<ggg>`, such as p1,cm,pm,pg,p2,... etc. Also connecting the data in the previous table, we have

Note in the first table we have given ethe action of point group elements on translations. We now look at the induced action on the character χ_1, χ_2. We have

$$id, m_p, s_p, i, s_{pc} \colon (\chi_1, \chi_2) \mapsto (\chi_1, \chi_2),$$

$$m_c, c_4 \colon (\chi_1, \chi_2) \mapsto (\chi_2, \chi_1).$$

It is not hard to see that we do have $E_2^{0,2} = H^2(T, \mathbb{Z}_2)^{PG} = H^2(T, \mathbb{Z}_2) = \mathbb{Z}_2$, generated by $\chi_1 \cup \chi_2$. Now the question is how to calculate $E_3^{0,2}$ and $E_4^{0,2}$.

15 $H^*(P1(No.1), \mathbb{Z}_2)$

$P1 = \langle T_1, T_2, T_3 \rangle = T \cong \mathbb{Z}^3$ is simply the translation group. We have

$$H^*(T, \mathbb{Z}_2) = \mathbb{F}_2[\chi_1, \chi_2, \chi_3]/(\chi_1^2 = \chi_2^2 = \chi_3^2 = 0). \tag{30}$$

Now we want to check the cocycle condition explicitly. The cocycle conditon for trivial group action is

$$\omega(g_2, g_3) + \omega(g_1, g_2 g_3) = \omega(g_1, g_2) + \omega(g_1 g_2, g_3), \tag{31}$$

where trivial action on $\mathbb{Z}_2$ is assumed. In Weicheng's notes he used the multiplicative notation; for $\mathbb{Z}_2$ this is really additive so we used plus sign.

Using $g = T_1^x T_2^y T_3^z$ and the multiplication rule $g_1 g_2 = T_1^{x_1+x_2} T_2^{y_1+y_2} T_3^{z_1+z_2}$, we write the representative cochain: $n = 1$, the representation cochains are of course $\chi_1(g) = x$, $\chi_2(g) = y$, $\chi_3(g) = z$. When $n = 2$, the representative cochains are $\chi_1\chi_2$ and so on. According to the cocycle condition Eq. (31), we need to check $\omega(g_2, g_3) + \omega(g_1, g_2g_3) = \omega(g_1, g_2) + \omega(g_1g_2, g_3)$, which for $\chi_1\chi_2(g_1, g_2) = \chi_1(g_1)\chi_2(g_2) = xy$ becomes $x_2y_3 + x_1(y_2 + y_3) = x_1y_2 + (x_1 + x_2)y_3$ in the mod 2 sense, which obvious hold. Of course, we can alternatively define $\lambda(g_1, g_2) = \chi_1(g_2)\chi_2(g_1) = y_2x_1$, which gives $\chi_1(g_3)\chi_2(g_2)\chi_1(g_2g_3)\chi_2(g_1) = \chi_1(g_2)\chi_2(g_1)\chi_1(g_3)\chi_2(g_1g_2)$, which is $x_3y_2 + (x_2 + x_3)y_1 = x_2y_1 + x_3(y_1 + y_2)$, which also hold.

We then check that χ_1^2 is a coboundary. Cobounday condition says that, at degree two, there exists $\mu(g)$ s.t.

$$\omega(g_1, g_2) = \mu(g_1) - \mu(g_1g_2) + \mu(g_2). \tag{32}$$

we have $\chi_1^2(g_1, g_2) = x_1x_2$. If we define $\mu(g) = \frac{1}{2}x(x + 1)$, then it can be checked that $\chi_1^2(g_1, g_2) = x_1x_2 = \mu(g_1) - \mu(g_1g_2) + \mu(g_2) = \frac{1}{2}x_1(x_1 + 1) + \frac{1}{2}(x_1 + x_2)(x_1 + x_2 + 1) + \frac{1}{2}x_2(x_2 + 1)$ in the mod 2 sense. Therefore χ_1^2 is a coboundary.

16 $H^*(C2(No.5), \mathbb{Z}_2)$

We consider the space group $C2$ (No.5). This group is very simple, generated by translation and a C_2 rotation, so we have $C2 = T \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2 = \langle C_2 \rangle$. GAP tells that it's cohomology is

$$H^{0,1,2,3,4,\dots}(C2, \mathbb{Z}_2) = (1, 3, 4, 4, 4, ...), \tag{33}$$

where the "4" holds for any $n \geq 2$. This has been verified in GAP using the command `R:=ResolutionSpaceGroup(SpaceGroupIT(3,`
to $n = 15$.

Group relations inherited from $F222$ are

$$C_2T_1C_2 = T_3^{-1}T_2, \quad C_2T_2C_2 = T_3^{-1}T_1, \quad C_2T_3C_2 = T_3^{-1}, \tag{34}$$

but note that in $C2$ the translations are no longer the primitive ones in $F222$. Let us write, following convention

$$\text{I112 \quad UNIQUE AXIS c, \quad CELL CHOICE 3 on P131 of ITC,} \tag{35}$$

Eq. (34) as $T_3 = t(1, 0, 0)$, $T_1 = (1/2, 1/2, 1/2)$, $T_2 = (1/2, -1/2, 1/2)$, $C_2 = (x, y, z) \to (-x, -y, z)$, then the above relations are met.

This induces the action on the charaxter $\chi_{1,2,3}$ (defined by $\chi_i(T_j) = \delta_{ij}$ for $i, j = 1, 2, 3$): $C_2 \colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_2, \chi_1, \chi_1 + \chi_2 + \chi_3)$. We make use of

$$\begin{aligned} H^*(T, \mathbb{Z}_2) &= \mathbb{F}_2[\chi_1, \chi_2, \chi_3]/(\chi_1^2 = \chi_2^2 = \chi_3^2 = 0), \\ H^*(\mathbb{Z}_2, \mathbb{Z}_2) &= \mathbb{F}_2[x], \end{aligned} \tag{36}$$

where x and $\chi_{1,2,3}$ all have degree 1.

Define $K_1 := H^1(T, \mathbb{Z}_2) = \mathbb{Z}_2^3 = \langle \chi_1, \chi_2, \chi_3 \rangle$ and $K_2 := H^2(T, \mathbb{Z}_2) = \mathbb{Z}_2^3 = \langle \chi_1 \cup \chi_2, \chi_2 \cup \chi_3, \chi_3 \cup \chi_1 \rangle$, $Q = \langle C_2 \rangle \cong \mathbb{Z}_2$, then we have

$$\begin{aligned}
&Z^{p\geq 1}(Q, H^{q=1}(T, \mathbb{Z}_2)) = K_1^Q = \langle \chi_1 + \chi_2, \chi_1 + \chi_3 \rangle \cong \mathbb{Z}_2^2, \\
&Z^{p\geq 1}(Q, H^{q=2}(T, \mathbb{Z}_2)) = K_2^Q = \langle \chi_1 \cup \chi_2, \chi_1 \cup \chi_3 + \chi_2 \cup \chi_3 \rangle \cong \mathbb{Z}_2^2, \\
&Z^{p\geq 1}(Q, H^{q=3}(T, \mathbb{Z}_2)) = K_3^Q = \langle \chi_1 \cup \chi_2 \cup \chi_3 \rangle \cong \mathbb{Z}_2, \\
&B^{p\geq 1}(Q, H^{q=1}(T, \mathbb{Z}_2)) = \langle C_2.\chi_1 - \chi_1, C_2.\chi_2 - \chi_2, C_2.\chi_3 - \chi_3 \rangle = \langle \chi_1 + \chi_2 \rangle \cong \mathbb{Z}_2, \\
&B^{p\geq 1}(Q, H^{q=2}(T, \mathbb{Z}_2)) = \langle C_2.(\chi_1 \cup \chi_2) - \chi_1 \cup \chi_2, C_2.(\chi_2 \cup \chi_3) - \chi_2 \cup \chi_3, C_2.(\chi_1 \cup \chi_3) - \chi_1 \cup \chi_3 \rangle = \langle \chi_1 \cup \chi_2 + \chi_2 \cup \chi_3 + \chi_1 \cup \chi_3 \rangle \\
&B^{p\geq 1}(Q, H^{q=3}(T, \mathbb{Z}_2)) = \langle C_1.(\chi_1 \cup \chi_2 \cup \chi_3) - \chi_1 \cup \chi_2 \cup \chi_3 \rangle = \{0\},
\end{aligned} \tag{37}$$

So

$$\begin{aligned} E_2^{p\geq 1,q} &= H^p(Q, H^1(T, \mathbb{Z}_2)) = \mathbb{Z}_2^2/\mathbb{Z}_2 = \mathbb{Z}_2, \\ E_2^{p\geq 1,q} &= H^p(Q, H^2(T, \mathbb{Z}_2)) = \mathbb{Z}_2^2/\mathbb{Z}_2 = \mathbb{Z}_2, \\ E_2^{p\geq 1,q} &= H^p(Q, H^3(T, \mathbb{Z}_2)) = \mathbb{Z}_2, \end{aligned} \tag{38}$$

where the last one is because $\mathbb{Z}_2 \to \mathbb{Z}_2$ only has trivial action. The first and second uses the fact that (see also the argument that leads to Eq. (245))

Using these, as well as $H^0(Q, H^i(T, \mathbb{Z}_2)) = H^0(Q, K_i) = K_i^Q = \mathbb{Z}_2$ for $i = 1, 2, 3$, and $H^0(Q, H^0(T, A)) = A = \mathbb{Z}_2$, we already have all the places of the $E_2^{p,q}$ page of the LHS spectral sequence:

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=4 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q=3 & 1 & 1 & 1 & 1 & 1 & \cdots \\
q=2 & 2 & 1 & 1 & 1 & 1 & \cdots \\
q=1 & 2 & 1 & 1 & 1 & 1 & \cdots \\
q=0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots
\end{array} \tag{39}$$

where the number m at place (p, q) denotes $H^p(Q, H^q(T, \mathbb{Z}_2)) = \mathbb{Z}_2^m$. Crucially, now, we note that the sum of diagonals already matches the cohomology dimension in Eq. (33). Now, if any of the $d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$ maps is nontrivial, the dimension of $E_3^{p,q}$ will be smaller than $E_2^{p,q}$, so that the cohomology dimension of Eq. (33) cannot be matched. This means that all the d_2 maps are actually zero, and that $E_2 = E_\infty$, i.e. the cohomology collapses at E_2 page. We have

$$H^*(C2, \mathbb{Z}_2) = \mathbb{F}_2[a_1, a_2, b_1, b_2, c, x]/R,$$

where we have in mind that $a_1 = \chi_1 + \chi_2$, $a_2 = \chi_1 + \chi_3$, $b_1 = \chi_1 \cup \chi_2$, $b_2 = \chi_1 \cup \chi_3 + \chi_2 \cup \chi_3 = a_1a_2 - b_1$, where $\chi_1^2 = \chi_2^2 = \chi_3^2 = 0$. a_1, a_2, x are degree one, and b_1, b_2 are degree two, and c is degree three.

Note that since the element $\chi_1 + \chi_2$ and the element $\chi_1 \cup \chi_2 + \chi_2 \cup \chi_3 + \chi_3 + \chi_1$ are mod out, we have $a_1x = 0$ and $(b_1 + b_2)x = 0$.

16.1 Ring relations: first try

With all these preparations, we are in the position to look at the relation R that needs to be quotient out. We follow Weicheng's paper and obtain the relations by restricting to subgroups. First, $n = 1$, the cohomology dimension is 3, so we have (a_1, a_2, x) with no relation among them. Then, $n = 2$ the cohomology dimension is 4, while there are 7 degree two terms we can write down:

$$a_1^2, a_2^2, a_1a_2, a_2x, b_1, b_2, x^2.$$

So there must be three relations. What this means is that three terms can be written in terms of the other four terms (the mod-2 rank is 4).

$a_1^2 = \chi_1^2 + \chi_2^2$, $a_2^2 = \chi_1^2 + \chi_3^2$, $a_1a_2 = \chi_1^2 + b_1 + b_2$, so we have, in the translation subgroup T, $a_1^2 = a_2^2 = 0$, and $a_1a_2 = b_1 + b_2$. Obviously, in the group $C2$, they represent three independent relations. Now we choose to write a_1^2, a_2^2 and b_2 in terms of a_1a_2, a_2x, b_1, x^2, the dimension matching means there cannot be relations solely in terms of these four terms. Note that the expression for a_1^2 cannot involve a_1a_2

Restricting to the subgroup $\langle C_2T_1T_2^{-1}\rangle \cong \mathbb{Z}_2$, we have $a_1 = 0$ and $a_2 = x$ nonzero, since it must be one dimensional, the four dimensional space now becomes one dimensional, with basis $a_2^2 + a_2x + x^2$; the other basis b_1 vanishes. Now since we know in $C2$ x^2 is a basis, and we do need to get a relation out of this, so we must have that $a_2^2 = a_2x$. Then, it is easy to see that $a_1^2 = 0$.

(till here, tided up!)

Now, we restrict to $\langle C_2T_3\rangle \cong \mathbb{Z}_2$, which has cohomology $\mathbb{F}_2[x']$ for some x'. We see that restricting to this group, $a_1 = b_1 = 0$ and $x' = a_2 = x$. Now x' cannot be quotient out, so in order to have a trivial relation when restricting to this subgroup, the only possible way to produce a nontrivial relation (but becomes trivial in the subgroup) is to set $p_2 = p_6 = 1$ (note the other terms vanish), which is

$$a_2^2 = a_2x.$$

Then, restricting to the subgroup $\langle C_2T_1T_2^{-1}\rangle \cong \mathbb{Z}_2$ which has cohomology $\mathbb{F}_2[x'']$, we need to identify $x'' = x = a_2$ while $a_1 = 0$. now, again in order to produce a trivial relation (since x'' cannot be quotient out) the only possibility is to set $p_2 = p_6 = 1$ (note the other terms vanish) and we get

$$a_2^2 = a_2x,$$

which gives nothing new.

Now, restricting to $\langle T_1T_2\rangle \cong \mathbb{Z}$ which has vanishing 2nd cohomology, we see that $a_1 = x = 0$ when restricting to this subgroup, while only b_1 and a_2^2 are nonvanishing and they are equal, so the possible relation is $b_1 = 0$ or $a_2^2 = 0$. In order to reduce the number of relations (≤ 3), we conjucture the relation is $a_2^2 = 0$, which is the reedunction of $a_2^2 = a_2x$ when $x = 0$.

We then use the group $\langle C_2T_1T_2\rangle \cong \mathbb{Z}$, which has vanishing 2nd cohomology. We have $a_1 = 0$ and $x = a_2$. Note this also means when restricting to this map we must have $x^2 = 0$ (although $x \neq 0$) and $b_1 = 0$.

Then, restricting to the subgroup $\langle T_3\rangle \cong \mathbb{Z}$ which has zero second cohomology. In this case we have $a_1 = b_1 = x = 0$, so we must have $a_2^2 = 0$, which is consistent with the above condition that $a_2^2 = a_2x$.

Finally, use the fact that $\langle C_2T_2\rangle \cong \mathbb{Z}$ which has vanishing 2nd cohomology, and we have $a_2 = b_1 = 0$, $x = a_1$, so must have the relation

$$a_1^2 = 0,$$

This is because: now the only possible terms in the relation ($R = 0$) are x^2,xa_1 and a_1^2. However, we have shown above that x^2 cannot appear in a relation, so the only possibility is $xa_1 = 0$, $a_1^2 = 0$, or $a_1^2 + xa_1 = 0$. But we know the last one is forbidden, because restricting to the subgroup some relation must exist to nullify the remaining terms x^2, xa_1, a_1^2 (although now they are all identified), but $a_1^2 + xa_1 = 0$ becomes a trivial relation. Then, the only possibilities are a_1^2 and xa_1. But we must have

$$a_1^2 = 0,$$

since this is the only choice that is compatible with restricting to the subgroup $\langle T_1\rangle$ (setting $x = 0$ must give $a_1^2 = 0$).

So now we have found all the three relations, to summarize:

$$R_2 = (a_2^2 = a_2x, a_1^2 = 0, a_1x = 0),$$

and so the order 2 terms are

$$n = 2\colon (b_1, a_1a_2, a_2x, x^2).$$

Now we look at degree terms. The one we can write down are $x^3, a_2x^2, b_1x, b_1a_1, b_1a_2$. Since the $(p, q) = (0, 3)$ term is b_1a_2, this term must be kept, while $b_1a_2 = 0$. Then $n = 3\colon (b_1a_2, b_1x, a_1x^2, x^3)$ matches the cohomology dimension.

Finally, we look at order four, which may still have nontrivial relation. Now the terms we can writen down are $b_1^2, b_1a_2x, b_1x^2, a_2x^3, x^4$, so we need to find one relation to match the cohomology dimension. Restricting to $\langle C_2\rangle \cong \mathbb{Z}_2$ says x^4 cannot appear in any relation. Again restricting to $\langle T_1T_2\rangle \cong \mathbb{Z}_2$, only the term b_1^2 appear, which must vanish so we have $b_1^2 = 0$. Note it is possible that b_1^2 is the linear combination of other four terms, but the import point is that b_1^2 does not appear.

To summarize, we propose that

$$H^*(C2, \mathbb{Z}_2) \overset{??}{\sim} \mathbb{F}_2[a_1, a_2, x, b_1]/(a_1^2 = a_1x = 0, a_2^2 = a_2x; b_1a_1 = 0; b_1^2 = 0 \Rightarrow b_1^2 = b_1a_2x!! \text{ see below}), \tag{40}$$

where

$$n = 1\colon (a_1, a_2, x); \quad n = 2\colon (b_1, a_1a_2, a_2x, x^2); \quad n = 3\colon (b_1a_2, b_1x, a_2x^2, x^3); \quad n = 4\colon (b_1a_2x, b_1x^2, a_2x^3, x^4); \tag{41}$$

so we see that for general n ($n \geq 3$), we have

$$n\colon (b_1a_2x^{n-3}, b_1x^{n-2}, a_2x^{n-1}, x^n).$$

We wonder if we could think of them in terms of the original $\chi_{1,2,3}$ symbol

$$n = 1\colon (\chi_1{+}\chi_2, \chi_1{+}\chi_3, x), \quad n = 2\colon (\chi_1\chi_2, (\chi_1{+}\chi_2)\chi_3, (\chi_1{+}\chi_2)x, x^2), \quad n \geq 3\colon (\chi_1\chi_2(\chi_1{+}\chi_2)x^{n-3}, \chi_1\chi_2x^{n-2}, (\chi_1{+}\chi_3)x^{n-1}, x^n),$$

where keep in mind that when setting $x = 0$ we should recover the original ring relation of the translation group cohomology $\chi_1^2 = \chi_2^2 = \chi_3^2 = 0$. This is actually not a good way to understand them and may lead to very wrong results. See below.

16.2 Ring relations: representative cochain

Now let's explicitly look for the cocycles. For $g_i = T_1^{x_i}T_2^{y_i}T_3^{z_i}C_2^{c_i}$, $i = 1, 2$, $x_i, y_i, z_i \in \mathbb{Z}$, $c_i = 0, 1$. For later use we write the multiplication rule

$$\begin{aligned} g_1g_2 &= T_1^{x_1}T_2^{y_1}T_3^{z_1}C_2^{c_1}T_1^{x_2}T_2^{y_2}T_3^{z_2}C_2^{c_2} \\ &= T_1^{x_1}T_2^{y_1}T_3^{z_1}(T_3^{-1}T_2)^{c_1x_2}T_1^{(1-c_1)x_2}(T_3^{-1}T_1)^{c_1y_2}T_2^{(1-c_1)y_2}T_3^{-c_1z_2}T_3^{(1-c_1)z_2}C_2^{c_1+c_2} \\ &= T_1^{x_1+c_1y_2+(1-c_1)x_2}T_2^{y_1+c_1x_2+(1-c_1)y_2}T_3^{z_1-c_1x_2-c_1y_2-c_1z_2+(1-c_1)z_2}C_2^{c_1+c_2}, \end{aligned} \tag{42}$$

Now we write the representative cochain.

For $n = 1$: $a_1(g) = x + y$ and $a_2(g) = x + z$ and $x(g) = c$ (the right hand side are in the mod 2 sense). It is easily checked that these are invariant in the sense that $\omega(g) = \omega(hgh^{-1})$ for all $g, h \in C2$.

For $n=2$: Write the translation group in the polycyclic notation, $g=(x,y,z)$. we have $C_2gC_2=(y,x,-x-y-z)$, So we do see that $b(g_1,g_2)=x_1y_2$ is invariant under C_2, since $b(C_2g_1C_2,C_2g_2C_2)=y_1x_2$, which differ from x_1y_2 just by a coboundary $\omega(g_1,g_2)=\mu(g_1)+\mu(g_2)-\mu(g_1g_2)$ with $\mu(g)=x+y$. This means that $b\in H^2(T,\mathbb{Z}_2)^Q$. Compare to the spectral sequence, we have $H^2(T,\mathbb{Z}_2)^Q=\langle b,a_1a_2\rangle$. We know the spectral sequence stabilizes at E_2 page, this means that there is a final element $b_1\in H^2(C2,\mathbb{Z}_2)$, which reduces to b when its argument $g_1,g_2\in G$ is restricted in T. The full form of b_1 is still to be found. To summarize,

$$b_1(g_1,g_2)=x_1y_2+\text{something},\quad a_1a_2(g_1,g_2)=(x_1+y_1)(x_2+z_2),\quad a_2x(g_1,g_2)=(x_1+z_1)c_2,\quad x^2(g_1,g_2)=c_1c_2.$$

Now we want to check that they are cocycle, meaning that $\omega(g_2,g_3)+\omega(g_1,g_2g_3)=\omega(g_1,g_2)+\omega(g_1g_2,g_3)$, where trivial action on $\mathbb{Z}_2$ is assumed. For detail, see Weicheng's appendix. Let's first check a_1a_2,a_2x and x^2:

For a_1a_2:

$$\begin{aligned}&(x_2+y_2)(x_3+z_3)+(x_1+y_1)(x_2+c_2y_3+(1-c_2)x_3+z_2-c_2x_3-c_2y_3-c_2z_3+(1-c_2)z_3)\\&=(x_1+y_1)(x_2+z_2)+(x_1+c_1y_2+(1-c_1)x_2+y_1+c_1x_2+(1-c_1)y_2)(x_3+z_3)\end{aligned}\tag{43}$$

which reduces to $2c_2(x_1+y_1)(x_3+z_3)=0$ which always holds in the mod 2 sense.

For a_2x: we have

$$(x_2+z_2)c_3+(x_1+z_1)(c_2+c_3)=(x_1+z_1)c_2+(x_1+c_1y_2+(1-c_1)x_2+z_1-c_1x_2-c_1y_2-c_1z_2+(1-c_1)z_2)c_3,$$

which reduces to $2c_1c_3(x_2+z_2)=0$ which always holds in the mod 2 sense.

For x^2: $c_2c_3+c_1(c_2+c_3)=c_1c_2+(c_1+c_2)c_3$, which holds.

Finally we check b_1. For b_1, the cocycle condition gives

$$(x_2y_3)+(x_1(y_2+c_2x_3+(1-c_2)y_3))=(x_1y_2)+((x_1+c_1y_2+(1-c_1)x_2)y_3),$$

which reduces to

$$c_2x_1(x_3-y_3)=c_1(y_2-x_2)y_3,$$

this clearly shows that b_1 alone is not a cocycle. By inspecting the cochain expression for B_{xy}'s in Weicheng's paper, we find that the following is a cocycle:

$$b_1=(1-c_1)x_1y_2+c_1x_2(x_1+y_2),\tag{44}$$

We checked in Mathematica that the corresponding cocycle equation becomes $(1-c_1)c_1(x_2-y_2)(x_3-y_3)=0$, which always holds since $c_1=0,1$. this has the desired property that when $c_1=0$ this gives $b_1=\chi_1\chi_2$. When $c_1\neq 0$, clearly we can no longer view b_1 as $\chi_1\chi_2$.

Let's check a_1x: $a_1x(g_1,g_2)=(x_1+y_1)c_2$, we have the cocycle $(x_2+y_2)c_3+(x_1+y_1)(c_2+c_3)=(x_1+y_1)c_2+(x_1+c_1y_2+(1-c_1)x_2+y_1+c_1x_2+(1-c_1)y_2)c_3$, which reduces to zero. It is very easy to show that a_1x is a coboundary: take

$$\mu(g)=(x+y)(1+c)+z,$$

then

$$a_1x(g_1,g_2)=(x_1+y_1)c_2=\mu(g_1)+\mu(g_2)-\mu(g_1g_2),$$

showing explicitly that a_1x is a coboundary. Let's also show that $(x_1+y_1)c_2$ and $c_1(x_2+y_2)$ differ by a coboundary: set $\mu_0(g)=(x+y)c$, then we have $\mu_0(g_1)+\mu_0(g_1g_2)+\mu_0(g_2)=(x_1+y_1)c_1+(x_1+c_1y_2+(1-c_1)x_2+y_1+c_1x_2+(1-c_1)y_2)(c_1+c_2)+(x_2+y_2)c_2=(x_1+y_1)c_2+(x_2+y_2)c_1$.

We can explicitly show that $a_1^2(g_1,g_2)=(x_1+y_1)(x_2+y_2)$ is a coboundary: define $\mu_1(g)=\frac{1}{2}(x+y)(x+y+1)$, then it is easy to show that $a_1^2(g_1,g_2)=\mu_1(g_1)-\mu_1(g_1g_2)+\mu_1(g_2)$. So a_1^2 is a coboundary. The essence here is that although c_1 modifies the exponent of T_1 and T_2 in g_1g_2, the sum of these two exponents is still $x_1+x_2+y_1+y_2$, as if c_1 were not there. So a_1^2 is a coboundary just as in the pure translation group $P1$.

For $a_2^2(g_1,g_2)=(x_1+z_1)(x_2+z_2)$, the sum of the exponents of T_1 and T_3 in g_1g_2 is $x_1+x_2+z_1+z_2-2c_1(x_2+z_2)$. We are tempted to use $\mu_2(g)=\frac{1}{2}(x+z)(x+z+1)$ to show that $a_2^2(g_1,g_2)=(x_1+z_1)(x_2+z_2)=\mu_2(g_1)-\mu_2(g_1g_2)+\mu_2(g_2)$ but the RHS gives an additional term which is $c_1(x_2+z_2)$. This is easily understood: note that $\frac{1}{2}(1\cdot 2)=1$ but $\frac{1}{2}(3\cdot 4)=6$, which have the opposite parity. Note that the additional term, $c_1(x_2+z_2)$, is exactly $a_2x(g_2,g_1)$, which differ from $a_2x(g_1,g_2)$ by a coboundary. This shows that $a_2^2+a_2x$ is a coboundary, which checks the relation $a_2^2=a_2x$. To summarize, we have $\mu_2(g_1)-\mu_2(g_1g_2)+\mu_2(g_2)=(x_2+z_2)(x_1+z_1-c_1)=a_2^2(g_1,g_2)-xa_2(g_1,g_2)$.

This actually also provides a way to show that a_1x is a coboundary. Define $\mu_3(g)=\frac{1}{2}(y+z)(y+z+1)$, we have $\mu_3(g_1)-\mu_3(g_1g_2)+\mu_3(g_2)=(c_1+y_1+z_1)(y_2+z_2)=(x(a_1+a_2)+(a_1+a_2)^2)(g_1,g_2)$, this means that $xa_1+xa_2=a_1^2+a_2^2$, i.e. $xa_1=0$. More rigorously: define

$$\mu_4(g)=(x+y)(x+z)+\frac{1}{2}(x+y)(x+y+1)+\frac{1}{2}(x+z)(x+z+1)+\frac{1}{2}(y+z)(y+z+1),\tag{45}$$

then we have

$$\mu_4(g_1) - \mu_4(g_1 g_2) + \mu_4(g_2) = c_1(x_2 + y_2),$$

therefore $a_1 x$ is a coboundary. Now, all the relations and cocycles at degree two have been found and checked.

Now, all the possible third degree terms are $a_1 b_1, a_2 b_1, a_2 x^2, b_1 x, x^3$. We know the cohomology dimension is 4, so we need to find one relation. $a_2 b_1$ cannot vanish, because when restricting to subgroup $P1 = T = \mathbb{Z}^3$ $a_2 b_1$ becomes the $\chi_1 \chi_2 \chi 3$. We also know from the spectral sequence that the last three terms do not vanish and that one of $a_1 b_1$, $a_2 b_1$ vanishes. So it can only be $a_1 b_1$. To conclude, $a_1 b_1$ is a coboundary.

Now, we go to degree four terms. We know that Eq. (41) holds. By exhausting all combinations, we see that we must have one relation about b_1^2. Since the basis of $n = 4$ is $b_1 a_2 x, b_1 x^2, a_2 x^3$ and x^4, Now the task is to find $j_{1,2,3}$ as in

$$b_1^2 = j_1 b_1 a_2 x + j_2 b_1 x^2 + j_3 a_2 x^3,$$

note there is no x^4 term since when restricting to $\langle C_2 \rangle$ the RHS is just x^4 while the LHS has $b_1 = 0$.

Now we restrict to the subgroup $\langle T_1 T_2^{-1} C_2 \rangle \cong \mathbb{Z}_2$. Note that its group has element of the form $g = T_1^s T_2^{-s} C_2^s$, and $b_1(g_1, g_2) = (1 - s_1) s_1 s_2 + s_1 s_2 (s_1 - s_2) = s_1 s_2 - s_1 s_2^2 = s_1 s_2 (s_2 + 1)$ which is always even, therefore $b_1 = 0$. On the other hand, both a_2 and x are nonvanishing on this subgroup, so we must have $j_3 = 0$.

We cannot use $\langle T_3 C_2 \rangle \cong \mathbb{Z}_2$ since b_1 vanishes on it.

Now, we restrict to the group generated by $\langle T_1 T_2^{-1}, C_2 \rangle$. We have $C_2 T_1 T_2^{-1} C_2 = T_2 T_1^{-1}$, therefure denoting $T_4 := T_1 T_2^{-1}$ then we have $\langle T_4, C_2 \rangle \cong p1m$, see Weicheng's appendix, $p1m$ is the one-dimensional group generated by translation T and mirror M with $MTM = T^{-1}$, and its mod 2 cohomology ring is $\mathbb{F}_2[t, m]/(t^2 = tm)$. Now, restricting to $\langle T_1 T_2^{-1}, C_2 \rangle$ we have $a_1 = 0$. For any $g_i = T_1^{s_i} T_2^{-s_i} C_2^{c_i}$, $i = 1, 2$, we have $b_1(g_1, g_2) = (1 - c_1) s_1 s_2 + c_1 s_2 (s_1 + s_2) = s_1 s_2 + c_1 s_2^2 = s_1 s_2 + c_1 s_2$, in the mod 2 sense. We also have $x^2(g_1, g_2) = c_1 c_2$, and $a_2 x(g_1, g_2) = s_1 c_2$. Now, we know that $s_1 s_2$ is a coboundary (setting $\mu(g) = \frac{1}{2} s(s + 1)$) and that $s_1 c_2$ differ from $s_2 c_1$ by a coboundary. Therefore $b_1(g_1, g_2) = c_1 s_2 = c_2 s_1 = a_2 x$, so we have fully identified the generators a_1, a_2, x, b_1 with t, m: $a_1 = 0$, $a_2 = t$, $x = m$, $b_1 = a_2 x$. We have $b_1^2 = a_2^2 x^2 = t^2 m^2 = tm^3$, $b_1 a_2 x = a_2^2 x^2 = tm^3$, and $b_1 x^2 = a_2 x^3 = tm^3$. This means that, in the original group of $C2$, we either have $b_1^2 = b_1 a_2 x$, or $b_1^2 = b_1 x^2$.

Now, in order to distinguish $b_1^2 = b_1 a_2 x$, or $b_1^2 = b_1 x^2$, we really have to find a group where a_2 and x acts differently. We now turn to the group $\langle T_1 T_3^{-1}, C_2 \rangle$. Note that $C_2 T_1 T_3^{-1} C_2 = T_2$, and $C_2 T_2 C_2 = T_1 T_3^{-1}$, so if we denote $T_1 T_3^{-1}$ and T_2 as two translations T_5 and T_6, we see that this group $\langle T_5, T_6, C_2 \rangle$ is exactly the walll paper group cm, which is defined by $cm = \langle T_1, T_2, M \rangle$ with $MT_1M = T_2$, $MT_2M = T_1$. For $g_i = T_1^{s_i} T_3^{-s_i} C_2^{c_i}$ with $i = 1, 2$, we have $g_1 g_2 = T_1^{s_1 + (1 - c_1) s_2} T_2^{c_1 s_2} T_3^{-s_1 - (1 - c_1) s_2} C_2^{c_1 + c_2}$, which also labels the most general form of the group element of cm. therefore the most general form, of course, is $(T_1 T_3^{-1})^x T_2^y C_2^c$. For $g_i = T_1^{p_i} T_2^{q_i} T_3^{-p_i} C_2^{c_i}$ with $i = 1, 2$, we have $a_2 = 0$, $a_1 = p_i + q_i$, $x = c_i$, and $b_1(g_1, g_2) = (1 - c_1) p_1 q_2 + c_1 p_2 (p_1 + q_2)$.

According to Weicheng, the mod2 group cohomology ring of cm is $\mathbb{F}_2[A_{x+y}, A_m, B_{xy}]/(A_{x+y}^2 = 0, A_{x+y} A_m = 0, B_{xy} A_{x+y} = 0, B_{xy}^2 = 0)$. It is easy to see that his A_{x+y}, A_m and B_{xy} exactly map to our a_1, x. Also compare the expression of $b_1(g_1, g_2)$ with his expression of $\lambda_1(g_1, g_2)$ in Eq. (E27), we see that our b_1 is his B_{xy}. He has $B_{xy}^2 = 0$, meaning that we should have $b_1^2 = 0$.

16.3 Final result

$$H^*(C2, \mathbb{Z}_2) = \mathbb{F}_2[a_1, a_2, x, b_1]/(a_1^2 = a_1 x = 0, a_2^2 = a_2 x; b_1 a_1 = a_1 a_2 x = 0; b_1^2 = 0), \tag{46}$$

Write the most general group element of $C2 = \langle T_1, T_2, T_3, C_2 \rangle$ as $g_i = T_1^{x_i} T_2^{y_i} T_3^{z_i} C_2^{c_i}$, $i = 1, 2$, $x_i, y_i, z_i \in \mathbb{Z}$, $c_i = 0, 1$, we have representative cochains (which are cocycles)

$$\begin{aligned} a_1(g) &= x + y, \\ a_2(g) &= x + z, \\ x(g) &= c, \\ b_1(g_1, g_2) &= (1 - c_1) x_1 y_2 + c_1 x_2 (x_1 + y_2). \end{aligned} \tag{47}$$

In degree 1 the independent ones are a_1, a_2, x;
In degree 2 the independent ones are $x^2, a_1 a_2, a_2 x, b_1$; where $a_1^2 = a_1 x = a_2(a_2 + x) = 0$;
In degree 3 the independent ones are $x^3, a_2 x^2, b_1 x, b_1 a_2$, where $a_1 a_2 x = 0$ and $b_1 a_1 = 0$.
Note that we have

$$\tau|_{C2} = b_1 x, \quad \gamma|_{C2} = b_1 a_2. \tag{48}$$

17 $H^*(F222(No.22), \mathbb{Z}_2)$

$F222 = \langle T_1, T_2, T_3, C_2, C_2' \rangle$, where C_2' is two-fold rotation about the axis $(1/8, 0, 1/8) + \hat{y}$, while C_2 is two-fold rotation about the axis $(1/8, 1/8, 0) + \hat{z}$. Point group wise, C_2 is a permutation $(12)(34)$, while C_2' is a permutation $(13)(24)$. Note that C_2' and C_2 commute with each other. We have $C_2T_1C_2 = T_3^{-1}T_2$, $C_2T_2C_2 = T_3^{-1}T_1$, $C_2T_3C_2 = T_3^{-1}$, and $C_2'T_1C_2' = T_2^{-1}T_3$, $C_2'T_2C_2' = T_2^{-1}$, $C_2'T_3C_2' = T_2^{-1}T_1$. From this we have

$$C_2' \colon (\chi_1, \chi_2, \chi_3, \chi_c) \mapsto (\chi_3, \chi_1 + \chi_2 + \chi_3, \chi_1, \chi_c),$$

where χ_c is the character for C_2, which we denoted by x in the above. Define $K_n = H^n(C2, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$ for $n = 1$ and $\cong \mathbb{Z}_2^4$ for $n \geq 2$. We need to compute (with abuse of notation we denote $\mathbb{Z}_2 = \langle C_2' \rangle$ by C_2')

$$\begin{aligned}
&Z^{p\geq 1}(C_2', H^{q=1}(C2, \mathbb{Z}_2)) = K_1^{C_2'} = \langle \chi_1 + \chi_2, \chi_1 + \chi_3, x \rangle = K_1, \\
&Z^{p\geq 1}(C_2', H^{q=2}(C2, \mathbb{Z}_2)) = K_2^{C_2'} = \langle \chi_1\chi_2 + \chi_2\chi_3 + \chi_1\chi_3, (\chi_1 + \chi_3)x, x^2 \rangle = \mathbb{Z}_2^3, \\
&Z^{p\geq 1}(C_2', H^{q\geq 3}(C2, \mathbb{Z}_2)) = K_{q\geq 3}^{C_2'} = \langle \chi_1\chi_2(\chi_1 + \chi_3)x^{q-3}, \chi_1\chi_2x^{q-2}, (\chi_1 + \chi_3)x^{q-1}, x^q \rangle = K_{q\geq 3}, \\
&B^{p\geq 1}(C_2', H^{q=1}(C2, \mathbb{Z}_2)) = \{0\}, \\
&B^{p\geq 1}(C_2', H^{q=2}(C2, \mathbb{Z}_2)) = \langle C_2'.\chi_1\chi_2 - \chi_1\chi_2, C_2'.(\chi_1 + \chi_2)\chi_3 - (\chi_1 + \chi_3)\chi_3 \rangle = \langle \chi_1\chi_2 + \chi_2\chi_3 + \chi_1\chi_3 \rangle \cong \mathbb{Z}_2, \\
&B^{p\geq 1}(C_2', H^{q\geq 3}(C2, \mathbb{Z}_2)) = \{0\},
\end{aligned} \tag{49}$$

We see that, using $H^p = B^p/Z^p$,

$$\begin{aligned}
E_2^{p\geq 1, q=1} &= H^{p\geq 1}(C_2', H^{q=1}(C2, \mathbb{Z}_2) \cong \mathbb{Z}_2^3, \\
E_2^{p\geq 1, q=2} &= H^{p\geq 1}(C_2', H^{q=2}(C2, \mathbb{Z}_2) \cong \mathbb{Z}_2^2, \\
E_2^{p\geq 1, q\geq 3} &= H^{p\geq 1}(C_2', H^{q\geq 3}(C2, \mathbb{Z}_2) \cong \mathbb{Z}_2^4,
\end{aligned} \tag{50}$$

Then the $E_2^{p,q}$ has the form

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=4 & 4 & 4 & 4 & 4 & 4 & \cdots \\
q=3 & 4 & 4 & 4 & 4 & 4 & \cdots \\
q=2 & 3 & 2 & 2 & 2 & 2 & \cdots \\
q=1 & 3 & 3 & 3 & 3 & 3 & \cdots \\
q=0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots
\end{array} \tag{51}$$

Gap shows that we have

$$H^{0:15}(F222, \mathbb{Z}_2) = (1, 4, 7, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58), \tag{52}$$

which already agrees with the above E_2 form (for the d-th diagonal with $d \geq 3$, we have $1 + 3 + 2 + 4(n-2) = 4n - 2$. So all the d_2 maps are zero, and the E_2 already collapses to E_∞. Therefore, we have

$$H^*(F222, \mathbb{Z}_2) = \mathbb{F}_2[a_1, a_2, c_1, c_2, x, y]/R, \tag{53}$$

where $a_{1,2}, x, y$ are degree one and c_1, c_2 are degree three; $a_1 = \chi_1 + \chi_2$, $a_2 = \chi_1 + \chi_3$, $c_1 = \chi_1\chi_2\chi_3 = a_2b_1$, $c_2 = \chi_1\chi_2x = b_1x$.

Now we work on the relation R. Again, at degree one we have $n = 1$: $(a_1, a_2, x; y)$ which matches the cohomology dimension, so no relation among them. The degree two terms are $(a_1a_2, a_1x, x^2; a_1y, a_2y, xy; y^2)$, which are all the terms one can write down, and so no new relation among them (besides the ones inherited from F222). Then at third order: $n = 3$: $(c_1, c_2, a_1x^2, x^3; a_1xy, x^2y; a_1y^2, a_2y^2, xy^2; y^3) \equiv v_3$, here $c_1 = b_1a_1 = \chi_1\chi_2(\chi_1 + \chi_2)$ (note that b_1 is no longer present), and $c_2 = b_1x = \chi_1\chi_2x$.

To summarize:

$$n = 1\colon (a_1, a_2, x; y); \quad n = 2\colon (a_1a_2, a_2x, x^2; a_1y, a_2y, xy; y^2);$$

$$n = 3\colon (c_1, c_2, a_2x^2, x^3; a_2xy, x^2y; a_1y^2, a_2y^2, xy^2; y^3) \equiv v_3;$$

$$v_{n+1} = \text{Concatenate}[(c_1x^{n-2}, c_2x^{n-2}, a_1x^n, x^{n+1}), v_ny],$$

the last recursive relation can be easily understood in terms of the spectral sequence.

The relations: $R = (a_1a_2y = 0, \ldots)$. Using symmetry, we conjecture that $a_2^2 = a_2x$, and $a_1^2 = a_1y$. $a_1x + a_2y = 0$. We have in mind that $c_1 = b_1a_2 = \chi_1\chi_2(\chi_1 + \chi_3)$. Below, under C_3, we have $\chi_1\chi_2(\chi_1 + \chi_3) \to \chi_2\chi_3(\chi_2 + \chi_1) = \chi_1\chi_2(\chi_3 + \chi_1) + \chi_1^2\chi_2 + \chi_2^2\chi_3$.

Another way to count: at degree n, there are $n+1$ terms that involves only x and y. For $n \geq 3$, due to c_1 and c_2, there are $2(n-2)$ terms that involves x and y. There are also n terms of the form $a_1 x^{n-i} y^{i-1}$, where we have considered the relation $a_1 x = a_2 y$ and $a_1^2 = a_1 y$ and $a_2^2 = a_2 x$. A term $a_1 x^{n-i} y^{i-1}$ can be converted to $a_2 x^{n-i-1} y^i$, unless the term is of the form $a_1 y^{n-1}$ or $a_2 x^{n-1}$, So there are $n-1+1+1 = n+1$ terms. Collecting all these we get $n+1+2(n-2)+n+1 = 4n-2$, which are the correct dimension. Note the counting does not work for $n \leq 2$ because there is no $c_{1,2}$ at $n = 1, 2$. The counting for $n = 1, 2$ have been enumerated above. Now, the only missing relation is $c_1^2 = ?$ and $c_2^2 = ?$. To get this relation we turn to the representative cochain. See below.

17.1 Ring relations: representative cochain

Now we know that it is no longer proper to regard the elements with degree $n \geq 2$ as products of degree one characters. Write the most general form of group elements of $F222 = \langle T_1, T_2, T_3, C_2, C_2' \rangle$ as $g = T_1^x T_2^y T_3^z C_2^c C_2'^{c'}$, we have

$$\begin{aligned}
g_1 g_2 &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} C_2'^{c_1'} T_1^{x_2} T_2^{y_2} T_3^{z_2} C_2^{c_2} C_2'^{c_2'} \\
&= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} (T_2^{-1} T_3)^{c_1' x_2} T_1^{(1-c_1')x_2} T_2^{-c_1' y_2 + (1-c_1') y_2} (T_2^{-1} T_1)^{c_1' z_2} T_3^{(1-c_1') z_2} C_2^{c_2} C_2'^{c_1' + c_2'} \\
&= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} T_1^{(1-c_1')x_2 + c_1' z_2} T_2^{-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2} T_3^{c_1' x_2 + (1-c_1') z_2} C_2^{c_2} C_2'^{c_1' + c_2'} \\
&= T_1^{x_1} T_2^{y_1} T_3^{z_1} (T_3^{-1} T_2)^{c_1((1-c_1')x_2 + c_1' z_2)} T_1^{(1-c_1)((1-c_1')x_2 + c_1' z_2)} (T_3^{-1} T_1)^{c_1(-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2)} \\
&\qquad \cdot T_2^{(1-c_1)(-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2)} T_3^{(1-2c_1)(c_1' x_2 + (1-c_1') z_2)} C_2^{c_1 + c_2} C_2'^{c_1' + c_2'} \\
&= T_1^{x_1 + (1-c_1)((1-c_1')x_2 + c_1' z_2) + c_1(-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2)} \\
&\quad \cdot T_2^{y_1 + c_1((1-c_1')x_2 + c_1' z_2) + (1-c_1)(-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2)} \\
&\quad \cdot T_3^{z_1 - c_1((1-c_1')x_2 + c_1' z_2) - c_1(-c_1' x_2 - c_1' y_2 + (1-c_1') y_2 - c_1' z_2) + (1-2c_1)(c_1' x_2 + (1-c_1') z_2)} \\
&\quad \cdot C_2^{c_1 + c_2} C_2'^{c_1' + c_2'}.
\end{aligned} \tag{54}$$

Denote $g_3 = g_1 g_2$, then we have

$$x_3 + y_3 = x_1 + y_1 + x_2 + y_2, \quad x_3 + z_3 = x_1 + z_1 + x_2 + z_2, \quad y_3 + z_3 = y_1 + z_1 + y_2 + z_2.$$

We have checked that $a_1 a_2, a_2 x, x^2; a_1 y, a_2 y, xy; y^2$ are all cocycles. $b_1(g_1, g_2) = (1-c_1)x_1 y_2 + c_1 x_2 (x_1 + y_2)$ is no longer a cocycle.

Restricting $g_1, g_2 \in C2$, then we have $\tilde{b}(g_1, g_2) \equiv C_2'.b_1(g_1, g_2) \equiv b_1(C_2' g_1 C_2', C_2' g_2 C_2') = (-1+c_1) z_1 (x_2 + y_2 + z_2) - c_1 z_2 (x_2 + y_2 - z_1 + z_2)$. We have checked in Mathematica that $\tilde{b}(g_1, g_2)$ is still a cochain in $H^*(C2, \mathbb{Z}_2)$, this means that it can be expressed by $a_1 a_2, a_2 x$ and x^2. So we see that although Eq. (49) is inaccurate (since we should really not use $\chi_{1,2,3}$ which are nonsensical concepts), the counting of the dimensions in spectral sequence is actually correct. So now we have found all the seven cocycles at degree $n = 2$. Although strictly speaking we still need to show they are not coboudaries, we have done multiple side checks that confirm $a_1 a_2, a_2 x, x^2; a_1 y, a_2 y, xy; y^2$ are really generating the cocycles.

A few relations: First of all, setting $\mu_1(g) = \frac{1}{2}(x+z)(x+z+1) + (x+z)c$, we can check that

$$(a_2^2 - a_2 x)(g_1, g_2) = (x_1 + z_1)(x_2 + z_2) - (x_1 + z_1) c_2 = \mu_1(g_1) - \mu_1(g_1 g_2) + \mu_1(g_2) \tag{55}$$

in the mod 2 sense, confirming that the relation $a_2^2 = a_2 x$ still holds in F222. Second, setting $\mu_2(g) = \frac{1}{2}(x+y)(x+y+1) + (x+y)c'$, we can check that

$$(a_1^2 - a_1 y)(g_1, g_2) = (x_1 + y_1)(x_2 + y_2) - (x_1 + y_1) c' = \mu_2(g_1) - \mu_2(g_1 g_2) + \mu_2(g_2) \tag{56}$$

in the mod 2 sense, so we have $a_1^2 = a_1 y$ holds in F222. These two relation are of course symmetric. Then, using the $\mu_4(g)$ function defined in Eq. (45), we have

$$\mu_4(g_1) - \mu_4(g_1 g_2) + \mu_4(g_2) = c_1(x_2 + y_2) + c_1'(x_2 + z_2) = (x a_1 + y a_2)(g_1, g_2), \tag{57}$$

therefore we do have the relation $x a_1 = y a_2$.

Now we go to degree $n = 3$. Using brutal force computation (see Mathematica), we find two cocycles (it is easy to explicitly check that these two are indeed cocycles):

$$\begin{aligned}
c_1(g_1, g_2, g_3) &= (x_2 y_1 + x_1 y_2 + c_1(x_1 + y_1)(x_2 + y_2) + c_1'(x_1 + x_2 + y_1 + y_2)(x_2 + z_2))(x_3 + y_3), \\
c_2(g_1, g_2, g_3) &= (x_2 z_1 + x_1 z_2 + c_1'(x_1 + z_1)(x_2 + z_2) + c_1(x_1 + x_2 + z_1 + z_2)(x_2 + y_2))(x_3 + z_3).
\end{aligned} \tag{58}$$

We have $b_1(g_1, g_2) = (1 - c_1)x_1y_2 + c_1x_2(x_1 + y_2)$ in C2 (i.e. when setting $y = 0$, and by symmetry we also have $b_2(g_1, g_2) = (1 - c_1')x_1z_2 + c_1'x_2(x_1 + z_2)$ when setting $x = 0$. If we switch off c_1' in $c_{1,2}$: note $x_2y_1 + x_1y_2 + c_1(x_1 + y_1)(x_2 + y_2) = (b_1(g_1, g_2) + x_2y_1 + c_1(y_1y_2 + y_1x_2 - x_2y_2))$.

Q: Next step question: are $c_{1,2}$ dependent cocycles? if so, how do they reduce to b_1a_2 and b_1x when $y = 0$?

The above shows that

$$\omega(g_1, g_2) = x_2z_1 + x_1z_2 + c_1(x_1 + x_2 + z_1 + z_2)(x_2 + y_2)$$

is a coboundary, using $\mu(g) = xz = \frac{1}{2}(x + z)(x + z + 1) - \frac{1}{2}x(x + 1) - \frac{1}{2}z(z + 1)$. Therefore what we found in Eq. (58) are just coboundaries.

17.2 Solving by definition

According to the original paper of Hochschild-Serre, for $0 \to N \to G \to Q \to 0$, the action of $Q = G/N$ on $H^n(N, A)$ is: for each cocycle represented by the cochain $f\colon G \times G \times \cdots \times G \to A$, we have, for $g \in G$, $(g(f))(g_1, ..., g_n) = g.f(g^{-1}gg_1, ..., g^{-1}gg_n)$. There are other choices, for example, $g \in G$, $(g(f))(g_1, ..., g_n) = g.f(g_1, ..., g_n)$. These choices lead to different spectral sequence. the direct consequence of using the second choice of G/N action is that, according to Rotman, if A is G-trivial then $H^n(S, A)$ is G/S trivial. This means that the first column of $E_2^{p,q}$ will just be $H^n(S, A)$ since $H^n(S, A)^{S/A} = H^n(S, A)$ under trivial G/S action, and this is bad for us (no restricting the dimension at all; the restriction will only appear at later pages through the d_2 maps, and this further means that more differential maps will be nontrivial, which is not ideal for us).

So, to summarize, we define the action of $Q = G/N$ on $H^n(N, A)$ as: for each cocycle represented by the cochain $f\colon G \times G \times \cdots \times G \to A$, we have, for $g \in G$,

$$(g(f))(g_1, ..., g_n) = g.f(g^{-1}gg_1, ..., g^{-1}gg_n).$$

Using the polycyclic notation that $g = (x, y, z, c, 0)$, then we have

$$C_2'gC_2' = (z, -x - y - z, x, c).$$

Remember that the generators if $H^*(C2, \mathbb{Z}_2)$ are $a_1(g) = x + y$, $a_2(g) = x + z$, $x(g) = c$, and $b_1(g_1, g_2) = (1 - c_1)x_1y_2 + c_1x_2(x_1 + y_2)$. We see that a_1, a_2, c_1 are invariant under the action of C_2' (now regarded as an element of the quotient group $F222/C2$). On the other hand, let's check b_1:

$$b_1(C_2'g_1C_2', C_2'g_2C_2') = (1 - c_1)z_1(x_2 + y_2 + z_2) + c_1(x_2 + y_2 + z_1 + z_2)z_2.$$

After some trial and error, we find that we have

$$b_1(C_2'g_1C_2', C_2'g_2C_2') - b_1(g_1, g_2) = (a_1a_2)(g_1, g_2) + \mu(g_1, g_2),$$

where $\mu(g_1, g_2)$ is a coboundary with $\mu(g_1, g_2) = \theta(g_1) + \theta(g_2) - \theta(g_1g_2)$, with

$$\theta(g) = xy + yz + xz + \frac{1}{2}x(x - 1) + \frac{1}{2}z(z - 1),$$

and indeed, this confirms the $q = 2$ row of the spectral sequence (51). We see that it is very important to have a good feeling (really, by trial and error) about getting various terms using the coboundary condition. The above shows that

$$c_2'a_1a_2 = 0.$$

Recall that $H^2(C2, \mathbb{Z}_2) = (b_1, a_1a_2, a_2x, x^2)$.

while it is not easy to check whether $b_1(C_2'g_1C_2', C_2'g_2C_2') = b_1(g_1, g_2)$ up to coboundary, we now that out of $H^2(C2, \mathbb{Z}_2) = (b_1, a_1a_2, a_2x, x^2)$ that a_1a_2, a_2x and x^2 are all invariant under C_2', so combined with our conjectured spectral sequence b_1 is killed.

Then, go to $n = 3$: we have $H^3(C2, \mathbb{Z}_2) = \langle b_1a_2, b_1x, a_2x^2, x^3\rangle$. According to our conjectured spectral sequence, We must show they are all invariant under the C_2' action. Note that

$$C_2'(b_1a_2) = C_2'(b_1)C_2'(a_2) = (b_1 + a_1a_2)(a_2) = b_1a_2,$$

and

$$C_2'(b_1x) = (b_1 + a_1a_2)(x) = b_1x,$$

so we see that they are both invariant. This proves the $q = 3$ row of the spectral sequence.

Now we are in the position of writing down the ring structure. We have

$$n=1\colon (a_1,a_2,x;y);\quad n=2\colon (a_1a_2,a_2x,x^2;a_1y,a_2y,xy;y^2);$$
$$n=3\colon (c_1,c_2,a_2x^2,x^3;a_2xy,x^2y;a_1y^2,a_2y^2,xy^2;y^3)\equiv v_3;$$
$$v_{n+1}=\text{Concatenate}[(c_1x^{n-2},c_2x^{n-2},a_1x^n,x^{n+1}),v_ny],$$

where we know that $c_{1,2}$ reduce to b_1a_2 and b_1x when $c_2'=0$.
Confirmed representative cocycles: $x(g)=c$, $y(g)=c'$.

$$\begin{aligned} a_1(g)&=x+y,\\ a_2(g)&=x+z,\\ x(g)&=c,\\ y(g)&=c', \end{aligned} \tag{59}$$

among the 10 possibilities $a_1^2,a_2^2,a_1a_2,a_1x,a_1y,a_2x,a_2y,x^2,xy,y^2$, we have checked above in Eqs. (55) and (56) and (57) that $a_2^2=a_2x$ and $a_1^2=a_1y$ and $a_1x=a_2y$, which reduces three of the 10. So we do have full knowledge (ring relation, generators and representative cochains) for the cocycles in $H^{1,2}(F222,\mathbb{Z}_2)$.

Spectral sequence for the E_2 page of the $\mathbb{Z}_2$ cohomology for $F222$: (we have $E_2=E_\infty$)

$q=3$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$q=2$	**1**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$q=1$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
$q=1$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$E_2^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$	$p=12$	$p=13$
H_{tot}^{p-1}	–	1	4	7	10	14	18	22	26	30	34	38	42	46

(60)

17.3 Degree 3 cocycle expression

We have

$$C_{\gamma'}(g_1,g_2,g_3)|_{F222}=C_\tau(g_1,g_2,g_3)|_{F222}+C_\gamma(g_1,g_2,g_3)|_{F222}, \tag{61}$$

where

$C_\tau(g_1,g_2,g_3)|_{F222}=c_2c_3c_1'x_1+c_1c_3c_2'x_1+c_2c_3c_2'x_1+c_3c_2'x_1+c_1c_3c_1'c_2'x_1+c_3c_1'c_2'x_1+c_1c_2c_1'c_3'x_1+c_2c_3c_1'c_3'x_1+c_1c_2'c_3'x_1+$ $c_2c_2'c_3'x_1+c_1c_3c_2'c_3'x_1+c_2c_3c_2'c_3'x_1+c_3c_2'c_3'x_1+\frac{1}{2}c_2c_3c_1'(x_1+1)x_1+\frac{1}{2}c_2c_3c_2'(x_1+1)x_1+\frac{1}{2}c_2c_3'(x_1+1)x_1+\frac{1}{2}c_1c_2c_3'(x_1+1)x_1+$ $\frac{1}{2}c_2c_3c_3'(x_1+1)x_1+(x_1+1)x_1+c_1c_3x_2x_1+c_2c_3c_1'x_2x_1+c_2c_3c_2'x_2x_1+c_1c_3c_1'c_2'x_2x_1+c_3c_1'c_2'x_2x_1+c_2c_3'x_2x_1+c_1c_3c_3'x_2x_1+$ $c_2c_3c_3'x_2x_1+c_1c_1'c_3'x_2x_1+c_2c_1'c_3'x_2x_1+c_3c_1'c_3'x_2x_1+c_1'c_3'x_2x_1+c_1c_2'c_3'x_2x_1+c_1c_2c_2'c_3'x_2x_1+c_1c_3c_2'c_3'x_2x_1+c_1c_1'c_2'c_3'x_2x_1+$ $c_2c_1'c_2'c_3'x_2x_1+c_3c_1'c_2'c_3'x_2x_1+c_1c_2x_3x_1+c_2x_3x_1+c_1c_2c_3x_3x_1+c_2c_1'x_3x_1+c_1c_2c_3c_1'x_3x_1+c_2c_2'x_3x_1+c_1c_3c_2'x_3x_1+$ $c_2c_3c_2'x_3x_1+c_3c_2'x_3x_1+c_1c_2c_3'x_3x_1+c_2c_3'x_3x_1+c_2c_3c_3'x_3x_1+c_1c_2c_1'c_3'x_3x_1+c_2c_1'c_3'x_3x_1+c_2c_3c_1'c_3'x_3x_1+c_1c_3c_2'c_3'x_3x_1+$ $c_2c_3c_2'c_3'x_3x_1+c_3c_2'c_3'x_3x_1+c_2c_3c_1'y_1x_1+c_2c_3c_2'y_1x_1+c_3c_2'y_1x_1+c_3c_1'c_2'y_1x_1+c_1c_2c_3'y_1x_1+c_2c_3'y_1x_1+c_2c_3c_3'y_1x_1+$ $c_2c_1'c_3'y_1x_1+c_3c_2'c_3'y_1x_1+c_1c_3y_2x_1+c_2c_3c_1'y_2x_1+c_1c_3c_2'y_2x_1+c_2c_3c_2'y_2x_1+c_3c_2'y_2x_1+c_1c_2c_3'y_2x_1+c_2c_3'y_2x_1+c_2c_3c_3'y_2x_1+$ $c_1c_1'c_3'y_2x_1+c_1c_2c_1'c_3'y_2x_1+c_1c_3c_1'c_3'y_2x_1+c_3c_1'c_3'y_2x_1+c_1c_2'c_3'y_2x_1+c_1c_2c_2'c_3'y_2x_1+c_1c_3c_2'c_3'y_2x_1+c_3c_2'c_3'y_2x_1+c_1c_2y_3x_1+$ $c_2y_3x_1+c_1c_2c_3y_3x_1+c_2c_1'y_3x_1+c_1c_2c_3c_1'y_3x_1+c_2c_2'y_3x_1+c_1c_3c_2'y_3x_1+c_2c_3c_2'y_3x_1+c_3c_2'y_3x_1+c_2c_3'y_3x_1+c_2c_1'c_3'z_1x_1+$ $c_1c_2'c_3'z_1x_1+c_2c_2'c_3'z_1x_1+c_1c_3z_2x_1+c_3z_2x_1+c_1c_3c_1'z_2x_1+c_3c_1'z_2x_1+c_1c_3c_2'z_2x_1+c_3c_2'z_2x_1+c_1c_3c_1'c_2'z_2x_1+c_3c_1'c_2'z_2x_1+$ $c_1c_3'z_2x_1+c_1c_2c_3'z_2x_1+c_2c_3'z_2x_1+c_1c_3c_3'z_2x_1+c_3c_3'z_2x_1+c_1c_1'c_3'z_2x_1+c_1c_2c_1'c_3'z_2x_1+c_1c_3c_1'c_3'z_2x_1+c_3c_1'c_3'z_2x_1+c_1'c_3'z_2x_1+$ $c_1c_2'c_3'z_2x_1+c_2c_2'c_3'z_2x_1+c_1c_1'c_2'c_3'z_2x_1+c_2c_1'c_2'c_3'z_2x_1+c_3c_1'c_2'c_3'z_2x_1+c_3'z_2x_1+c_1c_2z_3x_1+c_2c_3z_3x_1+c_1c_2c_1'z_3x_1+c_2c_1'z_3x_1+$ $c_2c_3c_1'z_3x_1+c_1c_3c_2'z_3x_1+c_2c_3c_2'z_3x_1+c_3c_2'z_3x_1+c_1c_2c_3'z_3x_1+c_2c_3c_3'z_3x_1+c_1c_2c_1'c_3'z_3x_1+c_2c_1'c_3'z_3x_1+c_2c_3c_1'c_3'z_3x_1+$ $c_1c_3c_2'c_3'z_3x_1+c_2c_3c_2'c_3'z_3x_1+c_3c_2'c_3'z_3x_1+c_2c_3c_1'c_2'x_2+c_3c_1'c_2'x_2+c_1c_1'c_3'x_2+c_2c_1'c_3'x_2+c_1c_2c_2'c_3'x_2+c_1c_1'c_2'c_3'x_2+c_2c_1'c_2'c_3'x_2+$ $c_3c_1'c_2'c_3'x_2+\frac{1}{2}c_1c_3x_2(x_2+1)+\frac{1}{2}c_2c_3c_1'x_2(x_2+1)+\frac{1}{2}c_1c_3c_2'x_2(x_2+1)+\frac{1}{2}c_1c_3c_3'x_2(x_2+1)+c_1c_2c_1'c_2'x_3+c_2c_1'c_2'x_3+c_2c_3c_1'c_2'x_3+$ $c_1c_2c_1'c_3'x_3+c_1c_3c_2'c_3'x_3+c_2c_1'c_2'c_3'x_3+c_3c_1'c_2'c_3'x_3+c_1c_1'x_2x_3+c_2c_3c_1'x_2x_3+c_1c_3c_2'x_2x_3+c_1c_2c_1'c_2'x_2x_3+c_1c_3c_3'x_2x_3+$ $c_2c_1'c_3'x_2x_3+c_2c_3c_1'c_3'x_2x_3+c_1c_2'c_3'x_2x_3+c_1c_2c_2'c_3'x_2x_3+c_2c_1'c_2'c_3'x_2x_3+c_3c_1'c_2'c_3'x_2x_3+\frac{1}{2}c_2c_1'x_3(x_3+1)+\frac{1}{2}c_2c_3c_1'x_3(x_3+$ $1)+\frac{1}{2}c_1c_2'x_3(x_3+1)+\frac{1}{2}c_1c_3c_2'x_3(x_3+1)+\frac{1}{2}c_1c_1'c_2'x_3(x_3+1)+c_2c_3c_1'y_1+c_1c_3c_2'y_1+c_2c_3c_2'y_1+c_1c_3c_1'c_2'y_1+c_1c_2c_1'c_3'y_1+$ $c_2c_1'c_3'y_1+c_2c_3c_1'c_3'y_1+c_1c_2'c_3'y_1+c_2c_2'c_3'y_1+c_1c_3c_2'c_3'y_1+c_2c_3c_2'c_3'y_1+c_1c_3x_2y_1+c_2c_3c_1'x_2y_1+c_2c_3c_2'x_2y_1+c_1c_3c_1'c_2'x_2y_1+$ $c_1c_3c_3'x_2y_1+c_2c_3c_3'x_2y_1+c_3c_3'x_2y_1+c_3c_1'c_3'x_2y_1+c_1c_2'c_3'x_2y_1+c_1c_2c_2'c_3'x_2y_1+c_1c_3c_2'c_3'x_2y_1+c_3c_2'c_3'x_2y_1+c_1c_1'c_2'c_3'x_2y_1+$ $c_2c_1'c_2'c_3'x_2y_1+c_3c_1'c_2'c_3'x_2y_1+c_1c_2c_3x_3y_1+c_2c_3x_3y_1+c_1c_2c_1'x_3y_1+c_1c_2c_3c_1'x_3y_1+c_2c_3c_1'x_3y_1+c_1c_2c_2'x_3y_1+c_1c_3c_2'x_3y_1+$ $c_3c_2'x_3y_1+c_1c_2c_1'c_3'x_3y_1+c_2c_1'c_3'x_3y_1+c_2c_3c_1'c_3'x_3y_1+c_1c_3c_2'c_3'x_3y_1+c_2c_3c_2'c_3'x_3y_1+c_3c_2'c_3'x_3y_1+\frac{1}{2}c_2c_3c_1'y_1(y_1+1)+$ $\frac{1}{2}c_2c_3c_2'y_1(y_1+1)+\frac{1}{2}c_2c_3'y_1(y_1+1)+\frac{1}{2}c_1c_2c_3'y_1(y_1+1)+\frac{1}{2}c_2c_3c_3'y_1(y_1+1)+y_1(y_1+1)-c_2c_3(x_1+y_1)+c_2c_3c_1'c_2'y_2+$ $c_1c_2c_2'c_3'y_2+c_1c_3x_2y_2+c_2c_3c_1'x_2y_2+c_1c_3c_2'x_2y_2+c_3c_1'c_2'x_2y_2+c_1c_3c_3'x_2y_2+c_1c_1'c_3'x_2y_2+c_2c_1'c_3'x_2y_2+c_1c_1'c_2'c_3'x_2y_2+$ $c_2c_1'c_2'c_3'x_2y_2+c_3c_1'c_2'c_3'x_2y_2+c_1c_2c_1'x_3y_2+c_2c_3c_1'x_3y_2+c_1c_3c_2'x_3y_2+c_2c_1'c_2'x_3y_2+c_2c_3c_1'c_2'x_3y_2+c_1c_3c_3'x_3y_2+c_1c_2c_1'c_3'x_3y_2+$

$c_2c'_1c'_3x_3y_2 + c_3c'_1c'_3x_3y_2 + c_1c_2c'_2c'_3x_3y_2 + c_2c'_1c'_2c'_3x_3y_2 + c_3c'_1c'_2c'_3x_3y_2 + c_1c_3y_1y_2 + c_2c_3c'_1y_1y_2 + c_1c_3c'_2y_1y_2 + c_2c_3c'_2y_1y_2 +$
$c_3c'_2y_1y_2 + c_1c_2c'_3y_1y_2 + c_2c_3c'_3y_1y_2 + c_3c'_3y_1y_2 + c_1c'_1c'_3y_1y_2 + c_1c_2c'_1c'_3y_1y_2 + c_1c_3c'_1c'_3y_1y_2 + c_1c'_2c'_3y_1y_2 + c_1c_2c'_2c'_3y_1y_2 +$
$c_1c_3c'_2c'_3y_1y_2 + \frac{1}{2}c_1c_3y_2(y_2+1) + \frac{1}{2}c_2c_3c'_1y_2(y_2+1) + \frac{1}{2}c_1c_3c'_2y_2(y_2+1) + \frac{1}{2}c_1c_3c'_3y_2(y_2+1) + c_2c_3c'_1c'_3y_3 + c_1c_3c'_2c'_3y_3 +$
$c_2c_3c'_1x_2y_3 + c_1c_3c'_2x_2y_3 + c_1c_2c'_1c'_2x_2y_3 + c_2c'_1c'_2x_2y_3 + c_3c'_1c'_2x_2y_3 + c_1c_2c'_1c'_3x_2y_3 + c_2c'_1c'_3x_2y_3 + c_3c'_1c'_3x_2y_3 + c_1c_2c'_1x_3y_3 +$
$c_2c_3c'_1x_3y_3+c_1c_3c'_2x_3y_3+c_2c'_1c'_2x_3y_3+c_2c_3c'_1c'_2x_3y_3+c_1c_2c'_1c'_3x_3y_3+c_2c_3c'_1c'_3x_3y_3+c_2c'_1c'_2c'_3x_3y_3+c_3c'_1c'_2c'_3x_3y_3+c_1c_2c_3y_1y_3+$
$c_2c_3y_1y_3+c_1c_2c'_1y_1y_3+c_1c_2c_3c'_1y_1y_3+c_2c_3c'_1y_1y_3+c_1c_2c'_2y_1y_3+c_1c_3c'_2y_1y_3+c_3c'_2y_1y_3+c_1c_2c'_3y_1y_3+c_2c_3c'_3y_1y_3+c_1c_2c'_1y_2y_3+$
$c_2c_3c'_1y_2y_3 + c_1c_3c'_2y_2y_3 + c_2c'_1c'_2y_2y_3 + c_2c_3c'_1c'_2y_2y_3 + c_2c'_1c'_3y_2y_3 + c_2c_3c'_1c'_3y_2y_3 + \frac{1}{2}c_2c_3c'_1y_3(y_3+1) + \frac{1}{2}c_1c_3c'_2y_3(y_3+1) +$
$c_1c'_1c'_3x_2z_1 + c_2c'_1c'_3x_2z_1 + c_1c'_2c'_3x_2z_1 + c_2c'_2c'_3x_2z_1 + c_2c'_1c'_3y_1z_1 + c_1c'_2c'_3y_1z_1 + c_2c'_2c'_3y_1z_1 + c_1c'_1c'_3y_2z_1 + c_2c'_1c'_3y_2z_1 +$
$c_1c'_2c'_3y_2z_1 + c_2c'_2c'_3y_2z_1 - c_2c'_3(x_1+z_1) - c'_2c'_3(x_1+z_1) + c_3c'_1c'_2x_2z_2 + c_1c'_1c'_3x_2z_2 + c_3c'_1c'_3x_2z_2 + c'_1c'_3x_2z_2 + c_1c'_2c'_3x_2z_2 +$
$c_1c'_1c'_2c'_3x_2z_2 + c_2c'_1c'_2c'_3x_2z_2 + c_3c'_1c'_2c'_3x_2z_2 + c_1c_2c'_1c'_2x_3z_2 + c_2c'_1c'_2x_3z_2 + c_2c_3c'_1c'_2x_3z_2 + c_1c_2c'_1c'_3x_3z_2 + c_2c'_1c'_3x_3z_2 +$
$c_2c_3c'_1c'_3x_3z_2 + c_1c_3y_1z_2 + c_1c_3c'_1y_1z_2 + c_1c_3c'_2y_1z_2 + c_1c_3c'_1c'_2y_1z_2 + c_1c_2c'_3y_1z_2 + c_1c_3c'_3y_1z_2 + c_1c_2c'_1c'_3y_1z_2 + c_2c'_1c'_3y_1z_2 +$
$c_1c_3c'_1c'_3y_1z_2+c_1c'_2c'_3y_1z_2+c_2c'_2c'_3y_1z_2+c_1c'_1c'_2c'_3y_1z_2+c_2c'_1c'_2c'_3y_1z_2+c_3c'_1c'_2c'_3y_1z_2+c_2c'_1c'_3y_2z_2+c_1c'_2c'_3y_2z_2+c_1c'_1c'_2c'_3y_2z_2+$
$c_2c'_1c'_2c'_3y_2z_2+c_3c'_1c'_2c'_3y_2z_2+c_1c_2c'_1c'_2y_3z_2+c_2c'_1c'_2y_3z_2+c_2c_3c'_1c'_2y_3z_2+c_1c_2c'_1c'_3y_3z_2+c_2c'_1c'_3y_3z_2+c_2c_3c'_1c'_3y_3z_2+c_1c_2c'_1c'_2z_3+$
$c_1x_2z_3 + c_1c_2x_2z_3 + c_1c'_1x_2z_3 + c_1c_2c'_1x_2z_3 + c_2c_3c'_1x_2z_3 + c_3c'_1x_2z_3 + c_1c'_2x_2z_3 + c_1c_2c'_2x_2z_3 + c_1c'_3x_2z_3 + c_1c_2c'_3x_2z_3 +$
$c_1c_2c'_1c'_3x_2z_3+c_2c'_1c'_3x_2z_3+c_2c_3c'_1c'_3x_2z_3+c_1c'_2c'_3x_2z_3+c_1c_2c'_2c'_3x_2z_3+c_2c'_1c'_2c'_3x_2z_3+c_3c'_1c'_2c'_3x_2z_3+c_1c'_1c'_2x_3z_3+c_2c'_1c'_2x_3z_3+$
$c_2c_3c'_1c'_2x_3z_3 + c_1c_2c'_1c'_3x_3z_3 + c_2c'_1c'_3x_3z_3 + c_2c_3c'_1c'_3x_3z_3 + c_1c'_2c'_3x_3z_3 + c_2c'_1c'_2c'_3x_3z_3 + c_3c'_1c'_2c'_3x_3z_3 + c_1c_2y_1z_3 + c_2c_3y_1z_3 +$
$c_1c_2c'_1y_1z_3+c_2c'_1y_1z_3+c_2c_3c'_1y_1z_3+c_1c_3c'_2y_1z_3+c_2c_3c'_2y_1z_3+c_3c'_2y_1z_3+c_1c_2c'_3y_1z_3+c_2c_3c'_3y_1z_3+c_1c_2c'_1c'_3y_1z_3+c_2c'_1c'_3y_1z_3+$
$c_2c_3c'_1c'_3y_1z_3 + c_1c_3c'_2c'_3y_1z_3 + c_2c_3c'_2c'_3y_1z_3 + c_3c'_2c'_3y_1z_3 + c_1c_2y_2z_3 + c_1c_2c'_1y_2z_3 + c_2c'_1y_2z_3 + c_2c_3c'_1y_2z_3 + c_1c_2c'_2y_2z_3 +$
$c_2c'_1c'_2y_2z_3+c_3c'_1c'_2y_2z_3+c_1c_2c'_3y_2z_3+c_1c_2c'_1c'_3y_2z_3+c_2c'_1c'_3y_2z_3+c_2c_3c'_1c'_3y_2z_3+c_1c_2c'_2c'_3y_2z_3+c_2c'_1c'_2c'_3y_2z_3+c_3c'_1c'_2c'_3y_2z_3+$
$c_2c_3c'_1c'_2y_3z_3 + c_3c'_1c'_2y_3z_3 + c_1c_2c'_1c'_3y_3z_3 + c_2c'_1c'_3y_3z_3 + c_2c_3c'_1c'_3y_3z_3 + c_2c'_1c'_2c'_3y_3z_3 + c_3c'_1c'_2c'_3y_3z_3 + \frac{1}{2}c_1c'_1c'_2z_3(z_3+1),$

and

$C_\gamma(g_1,g_2,g_3)|_{F222} = c_2c'_3x_1 + c_3c'_1x_1x_2 + c_3c'_1c'_2x_1x_2 + c_2c'_1c'_3x_1x_2 + c_3c'_1c'_3x_1x_2 + \frac{1}{2}c_3c'_2x_1(1+x_1)x_2 + \frac{1}{2}c'_3x_1(1+$
$x_1)x_2 + \frac{1}{2}c_1c'_3x_1(1+x_1)x_2 + \frac{1}{2}c_2c'_3x_1(1+x_1)x_2 + \frac{1}{2}c_3c'_3x_1(1+x_1)x_2 + \frac{1}{2}c'_1c'_3x_1(1+x_1)x_2 + c_2c_3c'_1x_1x_3 + c_2c'_2x_1x_3 +$
$c_2c_3c'_2x_1x_3+c'_1c'_2x_1x_3+c_2c'_1c'_2x_1x_3+c_2c_3c'_3x_1x_3+c'_2c'_3x_1x_3+c_2c'_2c'_3x_1x_3+c'_1c'_2c'_3x_1x_3+\frac{1}{2}c_1c'_2x_1(1+x_1)x_3+\frac{1}{2}c_3c'_2x_1(1+$
$x_1)x_3+\frac{1}{2}c'_1c'_2x_1(1+x_1)x_3+\frac{1}{2}c_2c'_3x_1(1+x_1)x_3+c_3c'_1x_2x_3+c_1c'_2x_2x_3+c_1c_2c'_2x_2x_3+c_1c'_3x_2x_3+c_1c'_1x_1x_2x_3+c_3c'_1x_1x_2x_3+$
$c_1c_3c'_1x_1x_2x_3+c_2c_3c'_1x_1x_2x_3+c'_2x_1x_2x_3+c_1c'_2x_1x_2x_3+c_2c'_2x_1x_2x_3+c_1c_3c'_2x_1x_2x_3+c_2c'_1c'_2x_1x_2x_3+c'_3x_1x_2x_3+c_3c'_3x_1x_2x_3+$
$c'_1c'_3x_1x_2x_3 + c_1c'_1c'_3x_1x_2x_3 + c_1c'_2c'_3x_1x_2x_3 + c'_1c'_2c'_3x_1x_2x_3 + c_1c_3c'_2y_1 + c_1c_2c'_3y_1 + c_2c_3c'_3y_1 + c_1c_3c'_2x_1y_1 + c_2c'_3x_1y_1 +$
$c_1c_2c'_3x_1y_1+c_2c_3c'_3x_1y_1+c_3c'_1x_2y_1+c_1c_3c'_2x_2y_1+c_3c'_1c'_2x_2y_1+c_1c'_3x_2y_1+c_2c'_3x_2y_1+c_1c_2c'_3x_2y_1+c_2c_3c'_3x_2y_1+c_1c'_1c'_3x_2y_1+$
$c_2c'_1c'_3x_2y_1+c_3c'_1c'_3x_2y_1+c_3c'_1x_1x_2y_1+c_1c_3c'_2x_1x_2y_1+c_3c'_1c'_2x_1x_2y_1+c'_3x_1x_2y_1+c_1c'_3x_1x_2y_1+c_1c_2c'_3x_1x_2y_1+c_2c'_1c'_3x_1x_2y_1+$
$c_2c'_1x_3y_1 + c_1c_2c'_1x_3y_1 + c_2c_3c'_1x_3y_1 + c_2c'_2x_3y_1 + c_1c_3c'_2x_3y_1 + c_2c_3c'_2x_3y_1 + c_2c'_3x_3y_1 + c_2c_3c'_1x_1x_3y_1 + c_1c'_2x_1x_3y_1 +$
$c_2c'_2x_1x_3y_1 + c_1c_2c'_2x_1x_3y_1 + c_1c_3c'_2x_1x_3y_1 + c_2c_3c'_2x_1x_3y_1 + c'_1c'_2x_1x_3y_1 + c_2c'_1c'_2x_1x_3y_1 + c_2c'_3x_1x_3y_1 + c_1c_2c'_3x_1x_3y_1 +$
$c'_2c'_3x_1x_3y_1+c_2c'_2c'_3x_1x_3y_1+c'_1c'_2c'_3x_1x_3y_1+c_2c'_1x_2x_3y_1+c_3c'_1x_2x_3y_1+c_2c_3c'_2x_2x_3y_1+c_2c'_1c'_2x_2x_3y_1+c'_3x_2x_3y_1+c_1c'_3x_2x_3y_1+$
$c_2c'_1c'_3x_2x_3y_1+c_2c'_2c'_3x_2x_3y_1+c'_1c'_2c'_3x_2x_3y_1+\frac{1}{2}c_3c'_2x_2y_1(1+y_1)+\frac{1}{2}c_1c'_3x_2y_1(1+y_1)+\frac{1}{2}c_2c'_3x_2y_1(1+y_1)+\frac{1}{2}c_3c'_3x_2y_1(1+y_1)+$
$\frac{1}{2}c_1c'_2x_3y_1(1+y_1)+\frac{1}{2}c_2c'_2x_3y_1(1+y_1)+\frac{1}{2}c_3c'_2x_3y_1(1+y_1)+c_1c'_3x_1y_2+c_2c'_3x_1y_2+c_2c_3c'_3x_1y_2+\frac{1}{2}c'_3x_1(1+x_1)y_2+c_3c'_1x_1x_2y_2+$
$c_3c'_2x_1x_2y_2+c_1c_3c'_2x_1x_2y_2+c_3c'_1c'_2x_1x_2y_2+c'_3x_1x_2y_2+c_1c'_3x_1x_2y_2+c_2c'_3x_1x_2y_2+c_1c_2c'_3x_1x_2y_2+c_3c'_3x_1x_2y_2+c_1c'_1c'_3x_1x_2y_2+$
$c_2c'_1c'_3x_1x_2y_2+c_2c'_1x_3y_2+c_1c_2c'_1x_3y_2+c_3c'_1x_3y_2+c_1c_3c'_1x_3y_2+c_1c_2c'_2x_3y_2+c_1c_3c'_2x_3y_2+c_1c'_1c'_3x_3y_2+c_1c'_2c'_3x_3y_2+c'_1x_1x_3y_2+$
$c_1c'_1x_1x_3y_2+c_1c_3c'_1x_1x_3y_2+c_2c_3c'_1x_1x_3y_2+c'_2x_1x_3y_2+c_2c'_2x_1x_3y_2+c_1c_2c'_2x_1x_3y_2+c_3c'_2x_1x_3y_2+c_2c_3c'_2x_1x_3y_2+c_1c'_3x_1x_3y_2+$
$c_2c'_3x_1x_3y_2 + c_1c_2c'_3x_1x_3y_2 + c_3c'_3x_1x_3y_2 + c_1c'_1c'_3x_1x_3y_2 + c_2c'_1c'_3x_1x_3y_2 + c_1c'_2c'_3x_1x_3y_2 + c_2c'_2c'_3x_1x_3y_2 + c_1c'_1x_2x_3y_2 +$
$c_1c_2c'_1x_2x_3y_2 + c_3c'_1x_2x_3y_2 + c_1c_3c'_1x_2x_3y_2 + c_1c'_2x_2x_3y_2 + c_1c_3c'_2x_2x_3y_2 + c_1c'_3x_2x_3y_2 + c_1c'_1c'_3x_2x_3y_2 + c_1c'_2c'_3x_2x_3y_2 +$
$c_1c_3c'_2y_1y_2+c_1c'_3y_1y_2+c_1c_2c'_3y_1y_2+c_1c_3c'_2x_1y_1y_2+c_1c'_3x_1y_1y_2+c_1c_2c'_3x_1y_1y_2+c_1c_3c'_2x_2y_1y_2+c_1c'_3x_2y_1y_2+c_1c_2c'_3x_2y_1y_2+$
$c_1c'_2x_3y_1y_2+c_1c_2c'_2x_3y_1y_2+c_1c_3c'_2x_3y_1y_2+c_2c'_3x_3y_1y_2+c_1c_2c'_3x_3y_1y_2+c_3c'_3x_3y_1y_2+c_2c'_1x_1y_3+c_2c_3c'_1x_1y_3+\frac{1}{2}c_2c'_2x_1(1+$
$x_1)y_3 + \frac{1}{2}c_2c'_3x_1(1+x_1)y_3 + c_1c'_1x_1x_2y_3 + c_2c'_1x_1x_2y_3 + c_1c_3c'_1x_1x_2y_3 + c_2c_3c'_1x_1x_2y_3 + c_1c_2c'_2x_1x_2y_3 + c_3c'_2x_1x_2y_3 +$
$c_2c'_1c'_2x_1x_2y_3 + c_1c_2c'_3x_1x_2y_3 + c_3c'_3x_1x_2y_3 + c_2c'_1c'_3x_1x_2y_3 + c_1c'_2x_1x_3y_3 + c_2c'_2x_1x_3y_3 + c_1c_2c'_2x_1x_3y_3 + c_3c'_2x_1x_3y_3 +$
$c_1c_3c'_2x_1x_3y_3 + c_2c_3c'_2x_1x_3y_3 + c_2c'_1c'_2x_1x_3y_3 + c_1c_2c'_3x_1x_3y_3 + c_2c'_1c'_3x_1x_3y_3 + c_1c'_1x_2x_3y_3 + c_2c'_1x_2x_3y_3 + c_3c'_1x_2x_3y_3 +$
$c_1c_2c'_2x_2x_3y_3 + c_1c_2c'_1y_1y_3 + c_2c_3c'_1y_1y_3 + c_1c'_2y_1y_3 + c_1c_3c'_2y_1y_3 + c_2c'_1c'_2y_1y_3 + c_2c'_3y_1y_3 + c_2c_3c'_3y_1y_3 + c_2c'_1c'_3y_1y_3 +$
$c_2c'_1x_1y_1y_3 + c_2c_3c'_1x_1y_1y_3 + c_1c'_2x_1y_1y_3 + c_1c_2c'_2x_1y_1y_3 + c_1c_3c'_2x_1y_1y_3 + c_2c_3c'_2x_1y_1y_3 + c_2c'_1c'_2x_1y_1y_3 + c_2c'_3x_1y_1y_3 +$
$c_1c_2c'_3x_1y_1y_3 + c_2c'_1c'_3x_1y_1y_3 + c_2c'_1x_2y_1y_3 + c_1c'_2x_2y_1y_3 + c_2c'_2x_2y_1y_3 + c_1c_2c'_2x_2y_1y_3 + c_1c_3c'_2x_2y_1y_3 + c_2c_3c'_2x_2y_1y_3 +$
$c_2c'_1c'_2x_2y_1y_3 + c_1c_2c'_3x_2y_1y_3 + c_2c'_1c'_3x_2y_1y_3 + c_2c'_1x_3y_1y_3 + c_1c'_2x_3y_1y_3 + c_2c'_2x_3y_1y_3 + c_1c_2c'_2x_3y_1y_3 + c_1c_3c'_2x_3y_1y_3 +$
$c_2c_3c'_2x_3y_1y_3 + c_2c'_1c'_2x_3y_1y_3 + c_1c_2c'_3x_3y_1y_3 + c_2c'_1c'_3x_3y_1y_3 + c_1c'_1y_2y_3 + c_1c_2c'_1y_2y_3 + c_1c_3c'_1y_2y_3 + c_1c'_2y_2y_3 + c_1c_3c'_2y_2y_3 +$
$c_1c'_1x_1y_2y_3+c_2c'_1x_1y_2y_3+c_1c_3c'_1x_1y_2y_3+c_2c_3c'_1x_1y_2y_3+c_1c'_2x_1y_2y_3+c_2c'_2x_1y_2y_3+c_1c_3c'_2x_1y_2y_3+c_2c_3c'_2x_1y_2y_3+c_1c'_1x_2y_2y_3+$
$c_1c_2c'_1x_2y_2y_3+c_1c_3c'_1x_2y_2y_3+c_1c'_2x_2y_2y_3+c_1c_3c'_2x_2y_2y_3+c_1c_2c'_2x_3y_2y_3+c_3c'_2x_1z_1+c_1c_3c'_2x_1z_1+c_1c_2c'_3x_1z_1+c_2c_3c'_3x_1z_1+$
$c_3c'_1x_2z_1+c_3c'_1c'_2x_2z_1+c'_3x_2z_1+c_2c'_3x_2z_1+c_3c'_3x_2z_1+c_2c_3c'_3x_2z_1+c_1c'_1c'_3x_2z_1+c_2c'_1c'_3x_2z_1+c_3c'_2x_1x_2z_1+c_1c_3c'_2x_1x_2z_1+$
$c_3c'_1c'_2x_1x_2z_1+c_1c'_3x_1x_2z_1+c_2c'_3x_1x_2z_1+c_1c_2c'_3x_1x_2z_1+c_3c'_3x_1x_2z_1+c'_1c'_3x_1x_2z_1+c_2c'_1c'_3x_1x_2z_1+c_2c'_1x_3z_1+c_1c_2c'_1x_3z_1+$
$c_2c_3c'_1x_3z_1 + c_1c'_2x_3z_1 + c_2c'_2x_3z_1 + c_1c_2c'_2x_3z_1 + c_2c_3c'_2x_3z_1 + c_1c_2c'_3x_3z_1 + c_2c_3c'_3x_3z_1 + c_2c_3c'_1x_1x_3z_1 + c_2c'_2x_1x_3z_1 +$
$c_1c_2c'_2x_1x_3z_1 + c_1c_3c'_2x_1x_3z_1 + c_2c_3c'_2x_1x_3z_1 + c'_1c'_2x_1x_3z_1 + c_1c_2c'_3x_1x_3z_1 + c_2c'_1c'_3x_1x_3z_1 + c_2c'_2c'_3x_1x_3z_1 + c_3c'_1x_2x_3z_1 +$
$c_1c_3c'_1x_2x_3z_1 + c_2c_3c'_1x_2x_3z_1 + c_3c'_2x_2x_3z_1 + c_1c_3c'_2x_2x_3z_1 + c_2c_3c'_2x_2x_3z_1 + c_1c'_3x_2x_3z_1 + c_2c'_3x_2x_3z_1 + c_3c'_3x_2x_3z_1 +$
$c_1c'_1c'_3x_2x_3z_1+c_2c'_1c'_3x_2x_3z_1+c_1c'_2c'_3x_2x_3z_1+c_2c'_2c'_3x_2x_3z_1+c_1c_3c'_2y_1z_1+c_2c'_3y_1z_1+c_1c_2c'_3y_1z_1+c_2c_3c'_3y_1z_1+c_3c'_2x_1y_1z_1+$
$c_2c'_3x_1y_1z_1+c_3c'_1x_2y_1z_1+c_1c_3c'_2x_2y_1z_1+c_3c'_1c'_2x_2y_1z_1+c'_3x_2y_1z_1+c_1c_2c'_3x_2y_1z_1+c_3c'_3x_2y_1z_1+c_2c'_1c'_3x_2y_1z_1+c_2c_3c'_1x_3y_1z_1+$
$c'_2x_3y_1z_1+c_1c_2c'_2x_3y_1z_1+c_1c_3c'_2x_3y_1z_1+c_2c_3c'_2x_3y_1z_1+c_2c'_3x_3y_1z_1+c_1c_2c'_3x_3y_1z_1+c_2c'_1c'_3x_3y_1z_1+c_2c'_2c'_3x_3y_1z_1+c_1c'_3y_2z_1+$

$c_2c_3'y_2z_1 + c_2c_3c_3'y_2z_1 + c_3c_1'x_1y_2z_1 + c_3c_2'x_1y_2z_1 + c_1c_3c_2'x_1y_2z_1 + c_3'x_1y_2z_1 + c_2c_3'x_1y_2z_1 + c_1c_2c_3'x_1y_2z_1 + c_3c_1'x_2y_2z_1 +$
$c_3c_1'c_2'x_2y_2z_1 + c_3'x_2y_2z_1 + c_1c_3'x_2y_2z_1 + c_3c_3'x_2y_2z_1 + c_1c_1'c_3'x_2y_2z_1 + c_2c_1'c_3'x_2y_2z_1 + c_1'x_3y_2z_1 + c_1c_1'x_3y_2z_1 + c_1c_3c_1'x_3y_2z_1 +$
$c_2c_3c_1'x_3y_2z_1 + c_2'x_3y_2z_1 + c_1c_3c_2'x_3y_2z_1 + c_2c_3c_2'x_3y_2z_1 + c_1c_3'x_3y_2z_1 + c_2c_3'x_3y_2z_1 + c_1c_1'c_3'x_3y_2z_1 + c_2c_1'c_3'x_3y_2z_1 + c_1c_2'c_3'x_3y_2z_1 +$
$c_2c_2'c_3'x_3y_2z_1 + c_1c_3c_2'y_1y_2z_1 + c_1c_3'y_1y_2z_1 + c_1c_2c_3'y_1y_2z_1 + c_1c_2c_1'y_3z_1 + c_2c_3c_1'y_3z_1 + c_1c_2c_2'y_3z_1 + c_2c_3c_2'y_3z_1 + c_1c_2c_3'y_3z_1 +$
$c_2c_3c_3'y_3z_1 + c_2c_1'x_1y_3z_1 + c_2c_3c_1'x_1y_3z_1 + c_1c_2'x_1y_3z_1 + c_1c_2c_2'x_1y_3z_1 + c_1c_3c_2'x_1y_3z_1 + c_2c_3c_2'x_1y_3z_1 + c_2c_3'x_1y_3z_1 + c_1c_2c_3'x_1y_3z_1 +$
$c_2c_1'x_2y_3z_1 + c_3c_1'x_2y_3z_1 + c_1c_3c_1'x_2y_3z_1 + c_2c_3c_1'x_2y_3z_1 + c_1c_2'x_2y_3z_1 + c_2c_2'x_2y_3z_1 + c_3c_2'x_2y_3z_1 + c_1c_3c_2'x_2y_3z_1 + c_2c_3c_2'x_2y_3z_1 +$
$c_3c_3'x_2y_3z_1 + c_2c_1'x_3y_3z_1 + c_1c_2'x_3y_3z_1 + c_2c_2'x_3y_3z_1 + c_2c_1'y_1y_3z_1 + c_2c_3c_1'y_1y_3z_1 + c_1c_2'y_1y_3z_1 + c_2c_2'y_1y_3z_1 + c_1c_2c_2'y_1y_3z_1 +$
$c_1c_3c_2'y_1y_3z_1 + c_2c_3c_2'y_1y_3z_1 + c_1c_2c_3'y_1y_3z_1 + c_1c_1'y_2y_3z_1 + c_2c_1'y_2y_3z_1 + c_1c_3c_1'y_2y_3z_1 + c_2c_3c_1'y_2y_3z_1 + c_1c_2'y_2y_3z_1 + c_2c_2'y_2y_3z_1 +$
$c_1c_3c_2'y_2y_3z_1 + c_2c_3c_2'y_2y_3z_1 + \frac{1}{2}c_1'c_3'x_2z_1(1+z_1) + \frac{1}{2}c_2'x_3z_1(1+z_1) + \frac{1}{2}c_2c_2'x_3z_1(1+z_1) + \frac{1}{2}c_1'c_2'x_3z_1(1+z_1) + \frac{1}{2}c_2c_3'x_3z_1(1+$
$z_1) + \frac{1}{2}c_3'y_2z_1(1+z_1) + \frac{1}{2}c_2c_2'y_3z_1(1+z_1) + \frac{1}{2}c_2c_3'y_3z_1(1+z_1) + c_3c_2'x_1z_2 + c_1c_3c_2'x_1z_2 + c_1c_3'x_1z_2 + c_1c_2c_3'x_1z_2 + c_2c_3c_3'x_1z_2 +$
$c_1'c_3'x_1z_2 + c_1c_1'c_3'x_1z_2 + c_3c_1'c_3'x_1z_2 + \frac{1}{2}c_3c_2'x_1(1+x_1)z_2 + \frac{1}{2}c_3'x_1(1+x_1)z_2 + \frac{1}{2}c_1c_3'x_1(1+x_1)z_2 + \frac{1}{2}c_2c_3'x_1(1+x_1)z_2 + \frac{1}{2}c_3c_3'x_1(1+$
$x_1)z_2 + \frac{1}{2}c_1'c_3'x_1(1+x_1)z_2 + c_3c_2'x_1x_2z_2 + c_1c_3c_2'x_1x_2z_2 + c_3c_1'c_2'x_1x_2z_2 + c_2c_3'x_1x_2z_2 + c_1c_2c_3'x_1x_2z_2 + c_3c_3'x_1x_2z_2 + c_1'c_3'x_1x_2z_2 +$
$c_1c_1'c_3'x_1x_2z_2 + c_2c_1'c_3'x_1x_2z_2 + c_1c_1'x_3z_2 + c_2c_1'x_3z_2 + c_3c_1'x_3z_2 + c_3c_1'x_1x_3z_2 + c_1c_2'x_1x_3z_2 + c_2c_2'x_1x_3z_2 + c_1c_2c_2'x_1x_3z_2 +$
$c_1c_3c_2'x_1x_3z_2 + c_2c_3c_2'x_1x_3z_2 + c_1'c_2'x_1x_3z_2 + c_2c_1'c_2'x_1x_3z_2 + c_1c_3'x_1x_3z_2 + c_1c_2c_3'x_1x_3z_2 + c_2c_1'c_3'x_1x_3z_2 + c_2c_2'c_3'x_1x_3z_2 +$
$c_1'c_2'c_3'x_1x_3z_2 + c_2c_1'x_2x_3z_2 + c_1c_2c_1'x_2x_3z_2 + c_1c_3c_1'x_2x_3z_2 + c_1c_3c_2'x_2x_3z_2 + c_1c_3'x_2x_3z_2 + c_1'c_3'x_2x_3z_2 + c_1c_1'c_3'x_2x_3z_2 +$
$c_1c_2'c_3'x_2x_3z_2 + c_3c_2'y_1z_2 + c_1c_3c_2'y_1z_2 + c_3c_1'c_2'y_1z_2 + c_1c_2c_3'y_1z_2 + c_3c_3'y_1z_2 + c_2c_3c_3'y_1z_2 + c_1c_1'c_3'y_1z_2 + c_2c_1'c_3'y_1z_2 + c_3c_1'c_3'y_1z_2 +$
$c_3c_2'x_1y_1z_2 + c_3c_1'c_2'x_1y_1z_2 + c_2c_3'x_1y_1z_2 + c_1'c_3'x_1y_1z_2 + c_2c_1'c_3'x_1y_1z_2 + c_3c_2'x_2y_1z_2 + c_1c_3c_2'x_2y_1z_2 + c_1c_3'x_2y_1z_2 + c_2c_3'x_2y_1z_2 +$
$c_1c_2c_3'x_2y_1z_2 + c_1'c_3'x_2y_1z_2 + c_1'x_3y_1z_2 + c_2c_1'x_3y_1z_2 + c_3c_1'x_3y_1z_2 + c_1c_2'x_3y_1z_2 + c_2c_2'x_3y_1z_2 + c_1c_2c_2'x_3y_1z_2 + c_3c_2'x_3y_1z_2 +$
$c_1c_3c_2'x_3y_1z_2 + c_2c_3c_2'x_3y_1z_2 + c_1'c_2'x_3y_1z_2 + c_2c_1'c_2'x_3y_1z_2 + c_3'x_3y_1z_2 + c_1c_3'x_3y_1z_2 + c_1c_2c_3'x_3y_1z_2 + c_3c_3'x_3y_1z_2 + c_1'c_3'x_3y_1z_2 +$
$c_2c_1'c_3'x_3y_1z_2 + c_2c_2'c_3'x_3y_1z_2 + c_1'c_2'c_3'x_3y_1z_2 + \frac{1}{2}c_3c_2'y_1(1+y_1)z_2 + \frac{1}{2}c_1c_3'y_1(1+y_1)z_2 + \frac{1}{2}c_2c_3'y_1(1+y_1)z_2 + \frac{1}{2}c_3c_3'y_1(1+y_1)z_2 +$
$c_1c_3c_2'x_1y_2z_2 + c_3c_1'c_2'x_1y_2z_2 + c_3'x_1y_2z_2 + c_1c_2c_3'x_1y_2z_2 + c_3c_3'x_1y_2z_2 + c_1c_1'c_3'x_1y_2z_2 + c_2c_1'c_3'x_1y_2z_2 + c_1c_3'x_2y_2z_2 + c_1c_1'x_3y_2z_2 +$
$c_1c_2c_1'x_3y_2z_2 + c_3c_1'x_3y_2z_2 + c_1c_3c_1'x_3y_2z_2 + c_1c_3c_2'x_3y_2z_2 + c_1c_3'x_3y_2z_2 + c_1c_1'c_3'x_3y_2z_2 + c_1c_2'c_3'x_3y_2z_2 + c_1c_3c_2'y_1y_2z_2 +$
$c_1c_3'y_1y_2z_2 + c_1c_2c_3'y_1y_2z_2 + c_1c_2'x_1y_3z_2 + c_2c_2'x_1y_3z_2 + c_1c_2c_2'x_1y_3z_2 + c_3c_2'x_1y_3z_2 + c_1c_3c_2'x_1y_3z_2 + c_2c_3c_2'x_1y_3z_2 + c_2c_1'c_2'x_1y_3z_2 +$
$c_1c_2c_3'x_1y_3z_2 + c_3c_3'x_1y_3z_2 + c_2c_1'c_3'x_1y_3z_2 + c_2c_1'x_2y_3z_2 + c_1c_2c_1'x_2y_3z_2 + c_3c_1'x_2y_3z_2 + c_1c_3c_1'x_2y_3z_2 + c_1c_2'x_2y_3z_2 + c_1c_3c_2'x_2y_3z_2 +$
$c_1c_1'x_3y_3z_2 + c_2c_1'x_3y_3z_2 + c_3c_1'x_3y_3z_2 + c_1c_2'x_3y_3z_2 + c_2c_1'y_1y_3z_2 + c_1c_2'y_1y_3z_2 + c_2c_2'y_1y_3z_2 + c_1c_2c_2'y_1y_3z_2 + c_1c_3c_2'y_1y_3z_2 +$
$c_2c_3c_2'y_1y_3z_2 + c_2c_1'c_2'y_1y_3z_2 + c_1c_2c_3'y_1y_3z_2 + c_2c_1'c_3'y_1y_3z_2 + c_1c_1'y_2y_3z_2 + c_1c_2c_1'y_2y_3z_2 + c_1c_3c_1'y_2y_3z_2 + c_1c_2'y_2y_3z_2 +$
$c_1c_3c_2'y_2y_3z_2 + c_3c_2'x_1z_1z_2 + c_3c_1'c_2'x_1z_1z_2 + c_1c_3'x_1z_1z_2 + c_2c_3'x_1z_1z_2 + c_3c_3'x_1z_1z_2 + c_1'c_3'x_1z_1z_2 + c_2c_1'c_3'x_1z_1z_2 + c_3c_2'x_2z_1z_2 +$
$c_3c_1'c_2'x_2z_1z_2 + c_2c_3'x_2z_1z_2 + c_3c_3'x_2z_1z_2 + c_1c_1'c_3'x_2z_1z_2 + c_2c_1'c_3'x_2z_1z_2 + c_1'x_3z_1z_2 + c_1c_1'x_3z_1z_2 + c_3c_1'x_3z_1z_2 + c_2'x_3z_1z_2 +$
$c_3c_2'x_3z_1z_2 + c_3c_3'x_3z_1z_2 + c_3c_2'y_1z_1z_2 + c_3c_1'c_2'y_1z_1z_2 + c_3'y_1z_1z_2 + c_1c_3'y_1z_1z_2 + c_2c_3'y_1z_1z_2 + c_3c_3'y_1z_1z_2 + c_2c_1'c_3'y_1z_1z_2 +$
$c_3c_2'y_2z_1z_2 + c_3c_1'c_2'y_2z_1z_2 + c_3'y_2z_1z_2 + c_2c_3'y_2z_1z_2 + c_3c_3'y_2z_1z_2 + c_1c_1'c_3'y_2z_1z_2 + c_2c_1'c_3'y_2z_1z_2 + c_1c_1'y_3z_1z_2 + c_3c_1'y_3z_1z_2 +$
$c_3c_2'y_3z_1z_2 + c_3c_3'y_3z_1z_2 + \frac{1}{2}c_1'c_3'z_1(1 + z_1)z_2 + c_2'x_1z_3 + c_1c_2'x_1z_3 + c_2c_2'x_1z_3 + c_1c_2c_2'x_1z_3 + c_3c_2'x_1z_3 + c_1c_3c_2'x_1z_3 +$
$c_2c_3'x_1z_3 + c_1c_2c_3'x_1z_3 + c_2c_3c_3'x_1z_3 + c_2c_1'c_3'x_1z_3 + c_2c_2'c_3'x_1z_3 + c_1'c_2'c_3'x_1z_3 + \frac{1}{2}c_2'x_1(1 + x_1)z_3 + \frac{1}{2}c_1c_2'x_1(1 + x_1)z_3 +$
$\frac{1}{2}c_2c_2'x_1(1+x_1)z_3 + \frac{1}{2}c_3c_2'x_1(1+x_1)z_3 + \frac{1}{2}c_1'c_2'x_1(1+x_1)z_3 + c_1'c_3'x_2z_3 + c_1c_1'c_3'x_2z_3 + c_1c_2'c_3'x_2z_3 + c_1'c_2'c_3'x_2z_3 + c_1c_1'x_1x_2z_3 +$
$c_2c_2'x_1x_2z_3 + c_1c_2c_2'x_1x_2z_3 + c_3c_2'x_1x_2z_3 + c_1c_3c_2'x_1x_2z_3 + c_1'c_2'x_1x_2z_3 + c_3'x_1x_2z_3 + c_1c_2c_3'x_1x_2z_3 + c_2c_1'c_3'x_1x_2z_3 + c_2'x_1x_3z_3 +$
$c_1c_2'x_1x_3z_3 + c_1c_2c_2'x_1x_3z_3 + c_1c_3c_2'x_1x_3z_3 + c_2c_3c_2'x_1x_3z_3 + c_1'c_2'x_1x_3z_3 + c_2c_1'c_2'x_1x_3z_3 + c_2c_3'x_1x_3z_3 + c_1c_2c_3'x_1x_3z_3 +$
$c_2c_1'c_3'x_1x_3z_3 + c_2'c_3'x_1x_3z_3 + c_1c_2'x_2x_3z_3 + c_1c_2c_2'x_2x_3z_3 + c_1c_3'x_2x_3z_3 + c_1'c_3'x_2x_3z_3 + c_1c_2c_2'y_1z_3 + c_1c_3c_2'y_1z_3 + c_1c_2c_3'y_1z_3 +$
$c_2c_3c_3'y_1z_3 + c_2'x_1y_1z_3 + c_2c_3'x_1y_1z_3 + c_2c_1'c_3'x_1y_1z_3 + c_2c_2'c_3'x_1y_1z_3 + c_1'c_2'c_3'x_1y_1z_3 + c_1'x_2y_1z_3 + c_1c_2'x_2y_1z_3 + c_1c_2c_2'x_2y_1z_3 +$
$c_1c_3c_2'x_2y_1z_3 + c_1'c_2'x_2y_1z_3 + c_2c_3'x_2y_1z_3 + c_1c_2c_3'x_2y_1z_3 + c_1c_1'c_3'x_2y_1z_3 + c_2'c_3'x_2y_1z_3 + c_1c_2'c_3'x_2y_1z_3 + c_2c_2'c_3'x_2y_1z_3 +$
$c_2c_1'x_3y_1z_3 + c_2'x_3y_1z_3 + c_1c_2'x_3y_1z_3 + c_1c_2c_2'x_3y_1z_3 + c_3c_2'x_3y_1z_3 + c_1c_3c_2'x_3y_1z_3 + c_2c_3c_2'x_3y_1z_3 + c_1'c_2'x_3y_1z_3 + c_2c_1'c_2'x_3y_1z_3 +$
$c_2c_3'x_3y_1z_3 + c_1c_2c_3'x_3y_1z_3 + c_2c_1'c_3'x_3y_1z_3 + c_2'c_3'x_3y_1z_3 + \frac{1}{2}c_1c_2'y_1(1 + y_1)z_3 + \frac{1}{2}c_2c_2'y_1(1 + y_1)z_3 + \frac{1}{2}c_3c_2'y_1(1 + y_1)z_3 +$
$c_1'x_1y_2z_3 + c_1c_1'x_1y_2z_3 + c_2'x_1y_2z_3 + c_2c_2'x_1y_2z_3 + c_1c_2c_2'x_1y_2z_3 + c_3c_2'x_1y_2z_3 + c_1c_3c_2'x_1y_2z_3 + c_3'x_1y_2z_3 + c_1c_3'x_1y_2z_3 +$
$c_1c_2c_3'x_1y_2z_3 + c_3c_3'x_1y_2z_3 + c_1'c_3'x_1y_2z_3 + c_2c_1'c_3'x_1y_2z_3 + c_2'c_3'x_1y_2z_3 + c_2c_2'c_3'x_1y_2z_3 + c_1c_2'x_2y_2z_3 + c_1'c_3'x_2y_2z_3 + c_1c_1'c_3'x_2y_2z_3 +$
$c_1c_2'c_3'x_2y_2z_3 + c_1'c_2'c_3'x_2y_2z_3 + c_1'x_3y_2z_3 + c_1c_2'x_3y_2z_3 + c_1c_2c_2'x_3y_2z_3 + c_1c_3'x_3y_2z_3 + c_1'c_3'x_3y_2z_3 + c_1c_2'y_1y_2z_3 + c_1c_2c_2'y_1y_2z_3 +$
$c_1c_3c_2'y_1y_2z_3 + c_1c_3'y_1y_2z_3 + c_2c_3'y_1y_2z_3 + c_1c_2c_3'y_1y_2z_3 + c_3c_3'y_1y_2z_3 + c_1c_1'c_3'y_1y_2z_3 + c_1c_2'c_3'y_1y_2z_3 + c_1c_2'x_1y_3z_3 + c_2c_2'x_1y_3z_3 +$
$c_1c_2c_2'x_1y_3z_3 + c_3c_2'x_1y_3z_3 + c_1c_3c_2'x_1y_3z_3 + c_2c_3c_2'x_1y_3z_3 + c_2c_1'c_2'x_1y_3z_3 + c_1c_2c_3'x_1y_3z_3 + c_2c_1'c_3'x_1y_3z_3 + c_1c_2c_2'x_2y_3z_3 +$
$c_1c_2'x_3y_3z_3 + c_2c_1'y_1y_3z_3 + c_1c_2'y_1y_3z_3 + c_2c_2'y_1y_3z_3 + c_1c_2c_2'y_1y_3z_3 + c_1c_3c_2'y_1y_3z_3 + c_2c_3c_2'y_1y_3z_3 + c_2c_1'c_2'y_1y_3z_3 + c_1c_2c_3'y_1y_3z_3 +$
$c_2c_1'c_3'y_1y_3z_3 + c_1c_2c_2'y_2y_3z_3 + c_2'x_1z_1z_3 + c_1c_2'x_1z_1z_3 + c_1'c_2'x_1z_1z_3 + c_2c_1'c_3'x_1z_1z_3 + c_2c_2'c_3'x_1z_1z_3 + c_1'x_2z_1z_3 + c_1c_1'x_2z_1z_3 +$
$c_2'x_2z_1z_3 + c_1c_1'c_3'x_2z_1z_3 + c_2c_1'c_3'x_2z_1z_3 + c_1c_2'c_3'x_2z_1z_3 + c_2c_2'c_3'x_2z_1z_3 + c_2c_1'x_3z_1z_3 + c_1c_2'x_3z_1z_3 + c_2c_2'x_3z_1z_3 + c_2'y_1z_1z_3 +$
$c_1c_2'y_1z_1z_3 + c_2c_1'c_3'y_1z_1z_3 + c_2c_2'c_3'y_1z_1z_3 + c_1'y_2z_1z_3 + c_1c_1'y_2z_1z_3 + c_2'y_2z_1z_3 + c_1c_1'c_3'y_2z_1z_3 + c_2c_1'c_3'y_2z_1z_3 + c_1c_2'c_3'y_2z_1z_3 +$
$c_2c_2'c_3'y_2z_1z_3 + c_2c_1'y_3z_1z_3 + c_1c_2'y_3z_1z_3 + c_2c_2'y_3z_1z_3 + \frac{1}{2}c_1'c_2'z_1(1+z_1)z_3 + c_2'x_1z_2z_3 + c_2c_2'x_1z_2z_3 + c_3c_2'x_1z_2z_3 + c_1c_3'x_1z_2z_3 +$
$c_2c_3'x_1z_2z_3 + c_3c_3'x_1z_2z_3 + c_1'c_3'x_1z_2z_3 + c_2c_2'c_3'x_1z_2z_3 + c_1'c_3'x_2z_2z_3 + c_1c_1'c_3'x_2z_2z_3 + c_1c_2'c_3'x_2z_2z_3 + c_1'c_2'c_3'x_2z_2z_3 + c_1'x_3z_2z_3 +$
$c_1c_2'x_3z_2z_3 + c_2c_2'y_1z_2z_3 + c_3c_2'y_1z_2z_3 + c_3'y_1z_2z_3 + c_1c_3'y_1z_2z_3 + c_2c_3'y_1z_2z_3 + c_3c_3'y_1z_2z_3 + c_2c_2'c_3'y_1z_2z_3 + c_1c_1'c_3'y_2z_2z_3 +$
$c_1c_2'c_3'y_2z_2z_3 + c_1'c_2'c_3'y_2z_2z_3 + c_1c_2'y_3z_2z_3 + (c_1x_1x_2 + x_1y_2 + c_1x_1y_2 + c_1x_2y_2)(x_3 + z_3).$

18 $H^*(I4_1/amd(No.141), \mathbb{Z}_2)$

The standard spectral sequence at E_2 page is

$$\begin{array}{c|ccccccccccc}
q=3 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 \\
q=2 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 \\
q=1 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 \\
q=0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10
\end{array} \tag{62}$$

$$\begin{array}{c|ccccccccccc}
q=3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q=2 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 \\
q=1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q=0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 \\
\hline
E_3^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10
\end{array} \tag{63}$$

Final result is $H^{1,\dots,10}(I4_1/amd,\mathbb{Z}_2)=\mathbb{Z}_2^{3,6,10,14,18,22,26,30,34,38}$,
Point group is generated by $\langle C_2, C_2', M, P\rangle = Dih_4$ which has cohomology $\mathbb{F}_2[e,\sigma_1,\sigma_2]/(e\sigma_1)$,

```
T1:=[[1, 0, 0, 0], [0, 1, 0, 1/2], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T2:=[[1, 0, 0, 1/2], [0, 1, 0, 0], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T3:=[[1, 0, 0, 1/2], [0, 1, 0, 1/2], [0, 0, 1, 0], [0, 0, 0, 1]];
C3:=[[0, 0, 1, 0], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 0, 1]];
P:=[[-1, 0, 0, 0], [0, -1, 0, 0], [0, 0, -1, 0], [0, 0, 0, 1]];
C2:=[[-1, 0, 0, 1/4], [0, -1, 0, 1/4], [0, 0, 1, 0], [0, 0, 0, 1]];
C2p:=[[-1, 0, 0, 1/4], [0, 1, 0, 0], [0, 0, -1, 1/4], [0, 0, 0, 1]];
M:=[[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]];
```

```
G:=Group(T1,T2,T3,C2,C2p,M,P);
Gp:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G))));
R:=ResolutionAlmostCrystalGroup(Image(Gp),6);
```

Use `R!.elts;` to get all the elements recorded; `List([1..6],x->R!.dimension(x));` to get the dimension of resolution entries; `List([1..18],x->R!.boundary(2,x));` and so on to get the boundary maps.

19 $H^*(R\bar{3}m(No.166),\mathbb{Z}_2)$

The spectral sequence collapses at E_2:

$$\begin{array}{c|ccccccccccc}
q=3 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
q=2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
q=1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
q=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10
\end{array} \tag{64}$$

$E_\infty^{p,q}=E_2^{p,q}=\mathbb{Z}_2^{p+1}$. Note that $E_2^{0,q}=\mathbb{Z}_2$ for $q=0,1,2,3$, where $q=1$ is $\langle\chi_1+\chi_2+\chi_3\rangle$ and $q=2$ is $\langle\beta_1+\beta_2+\beta_3\rangle$ (See Eq. (129) and (131)).

20 $H^*(F23(No.196), \mathbb{Z}_2)$

$F23 = \langle T_1, T_2, T_3, C_2, C_2', C_3\rangle$, where $C_2' = C_3^{-1}C_2C_3$ so can be omitted. Now we have

$$C_3T_iC_3^{-1} = T_{i+1}, \quad C_3C_2C_3^{-1} = C_2C_2', \quad C_3C_2'C_3^{-1} = C_2,$$

so we have

$$C_3 \colon (\chi_1, \chi_2, \chi_3, \chi_c, \chi_{c'}) \mapsto (\chi_2, \chi_3, \chi_1, \chi_{c'}, \chi_c + \chi_{c'}),$$

where χ_c and $\chi_{c'}$ are the characteris for C_2 and C_2', respectively, which we denoted by x and y respectively in the above. GAP gives

$$H^{1:15}(F23, \mathbb{Z}_2) = (0, 3, 6, 2, 6, 10, 6, 10, 14, 10, 14, 18, 14, 18, 22). \tag{65}$$

Denote $K_q = H^q(F222, \mathbb{Z}_2)$. Note that $(a_1a_2 = \chi_1\chi_2 + \chi_2\chi_3 + \chi_1\chi_3)$

$$\begin{aligned}
K_1^{C_3} &= \langle \chi_1 + \chi_2, \chi_1 + \chi_3, x, y\rangle^{C_3} = \{0\}, \\
K_2^{C_3} &= \langle (a_1a_2, (\chi_1 + \chi_2)x, x^2, (\chi_1 + \chi_2)y, (\chi_1 + \chi_3)y, xy, y^2\rangle^{C_3} \\
&= \langle (a_1a_2, (\chi_1 + \chi_2)x + (\chi_2 + \chi_3)y, x^2 + xy + y^2\rangle \\
&= \langle a_1a_2, (a_1 + a_2)x + a_1y, x^2 + xy + y^2)\rangle \\
&\cong \mathbb{Z}_2^3, \\
K_3^{C_3} &= \langle c_1, c_2, a_1x^2, a_1xy, a_1y^2, a_2y^2, x^3, x^2y, xy^2, y^3\rangle^{C_3} \\
&= \langle c_1, x^3 + xy^2 + y^3, x^2y + xy^2, a_1y^2 + a_2(x^2 + y^2)\rangle
\end{aligned} \tag{66}$$

We see that the spectral sequence has $E_2^{p\geq 1,q} = H^{p\geq 1}(\langle C_3\rangle, H^q(F23, \mathbb{Z}_2)) = H^{p\geq 1}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}_2^{4n-2}) = 0$ since there is no map from $\mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}_2^{4n-2}$. Therefore the spectral sequence collapse at E_2 and the only nonzero places are the first column ($p = 0$). We just need to compute $E_2^{p=0,q} = H^q(F23, \mathbb{Z}_2)^{C_3} = K_q^{C_3}$ as already calculated above.

So our task is to calculate the number of C_3-invariant terms (in the mod-2 sense). Note that at each degree n, we have four types, according to the degree of $\chi_{1,2,3}$ in them: (1) $c_1 \cdot f_{n-3}(x, y)$, (2) $c_2 \cdot f_{n-2}^{no}(x, y)$, (3) $a_1h_{n-1}^1(y) + a_2h_{n-1}^2(x, y)$, and (4) $f_n(x, y)$. Note that the second type cannot be invariant under C_3, so should be discarded.

First, let us calculate the number of invariants $f_n(x, y)$ for $n = 1, 2, ...$ Note that the number must be 2^m since all these invariant polynomials form a group under addition. Call this number $m(n)$. Writing some short codes in Mathematica

```
Table[MatrixRank[DeleteDuplicates[Mod[Table[CoefficientList[(x^(n-i)y^i+y^(n-i)(x+y)^i+(x+y)^(n-i)x^i)/.y->1,
```

we get table 3:

Table 3: Here the number $m(n)$, which is the number of generators in the additive group of C_3-invariant polynomials $\{C_3.f(x, y) = f(x, y)|f(x, y) = \sum_{j=0}^{n} p_jx^{n-j}y^j\}$ with $\mathbb{Z}_2$ coefficients $\{p_j\}_{j=0}^{n}$, is also the dimension of mod-2 cohomology for the alternating group A_4, i.e. $H^n(A_4, \mathbb{Z}_2) = \mathbb{Z}_2^{m(n)}$. This has been checked up to the end of list, $n \leq 20$. $M(n)$ denotes the terms of of the $m(n)$ terms that further preserves the M symmetry. The $*$ terms are what is defined. For a conjectured expression for $m(n)$ (which we think is correct), see Eq. (91).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$m(n)$	1*	0	1	2	1	2	3	2	3	4	3	4	5	4	5	6	5	6	7	6	7
$m(n-3)$	-	-	-	1*	0	1	2	1	2	3	2	3	4	3	4	5	4	5	6	5	6
$m(n) + m(n-3)$	-	-	-	3	1	3	5	3	5	7	5	7	9	7	9	11	9	11	13	11	13
$H^n(F23, \mathbb{Z}_2)$	1	0	3	6	2	6	10	6	10	14	10	14	18	14	18	22					
$M(n)$	1*	0	1	1	1	1	2	1	3												
$M(n-3)$	-	-	-	1*	0	1	1	1	1	2	1	3									
$M(n) + M(n-3)$	-	-	-	2	1	2	3	2	4												

Now we treat part (3). Note that $a_1 = \chi_1 + \chi_2 \xrightarrow{C_3} \chi_2 + \chi_3 = a_1 + a_2$, while $a_2 = \chi_1 + \chi_3 \xrightarrow{C_3} \chi_2 + \chi_1 = a_1$. So we have $a_1h_{n-1}^1(y) + a_2h_{n-1}^2(x, y) \xrightarrow{C_3} (a_1 + a_2)h_{n-1}^1(x + y) + a_1h_{n-1}^2(y, x + y) = a_1(h_{n-1}^1(x + y) + h_{n-1}^2(y, x + y)) + a_2h_{n-1}^1(x + y)$, we see that $h_{n-1}^1(y) = h_{n-1}^1(x+y) + h_{n-1}^2(y, x+y)$ and $h_{n-1}^1(x+y) = h_{n-1}^2(x, y)$, so that $h_{n-1}^1(y) = h_{n-1}^1(x+y) + h_{n-1}^1(x)$. but we know $h_{n-1}^1(y) = y^{n-1}$, so this condition shows that only when either $n = 2$ or when n is odd number this term is allowed.

We conjecture that

$$H^{n\geq 3}(F23,\mathbb{Z}_2)=\{c_1f_{n-3}(x,y),(c_2+c_3)f_{n-3}(x,y),f_n(x,y),a_1g_{n-1}(x,y)+(a_1+a_2)g_{n-1}(y,x+y)+(a_2g_{n-1}(x+y,x)\},$$

where $c_2=\chi_1\chi_2x$ and $c_3=\chi_1\chi_3y$. We prove that there is a one-to-one correspondence between the last two terms: note that we have $(x,y)\to(y,x+y)$, while $(a_2,a_1)\to(a_1,a_1+a_2)$. In $a_1g_{n-1}(x,y)+(a_1+a_2)g_{n-1}(y,x+y)+a_2g_{n-1}(x+y,x)$, making substitution $(a_1,a_2)\to(y,x)$ we get a term of the form $f_n(x,y)$; On the other hand, any term $f_n(x,y)$ is of the form $h(x,y)+h(y,x+y)+h(x+y,x)$, making substitution $h(x,y)\to a_1p(x,y), h(y,x+y)\to(a_1+a_2)p(y,x+y), h(x+y,x)\to a_2p(x+y,x)$, we get a term of the form $a_1g_{n-1}(x,y)+(a_1+a_2)g_{n-1}(y,x+y)+a_2g_{n-1}(x+y,x)$.

We have

$$\begin{aligned}
&n=1\colon \{0\},\\
&n=2\colon \langle a_1a_2,a_1y,x^2+xy+y^2\rangle\cong\mathbb{Z}_2^3,\\
&n=3\colon \langle c_1,c_2,x^3+x^2y+y^3,x^3+xy^2+y^3,a_2x^2+a_1(x^2+y^2),a_2(x^2+y^2)+a_1y^2\rangle\cong\mathbb{Z}_2^6,
\end{aligned}\tag{67}$$

$$H^*(F23,\mathbb{Z}_2)=\mathbb{F}_2[x,y,a_1,a_2,c_1,c_2]^{C_3}/(a_1^2=a_1y,a_2^2=a_2x,a_1x+a_2y=a_2x+a_1y=0,c_1^2=c_2^2=0),$$

where we have in mind that (but this is technically illegal) $a_1=\chi_1+\chi_2$, $a_2=\chi_1+\chi_3$, $c_1=\chi_1\chi_2\chi_3$, $c_2=\chi_1\chi_2x+\chi_1\chi_3y$. The above shows that the spectral sequence is only nonzero in the first column ($p=0$), with

$$E_2^{p=0,q}=\mathbb{Z}_2^{2(m(q)+m(q-3))}\quad\text{for } q\geq 4,\quad E^{0,0:3}=\mathbb{Z}^{(1,0,3,6)}.$$

About relation: at degree two we have a_1,a_2,x,y but eventually the $n=2$ cohomology dimension is 3, so we must write 7 relations. First of all, there are two relations among x and y so that the only term appearing at $n=2$ is x^2+xy+y^2. The relation one should write is $x^2=y^2=xy$.

in $F23$, since $(a_1,a_2,x,y)\to(a_1+a_2,a_1,y,x+y)$, so under C_3 $a_1x\to(a_1+a_2)y\to a_2(x+y)$ so the invariant term is $a_1(x+y)+a_2x$; $a_1y\to(a_1+a_2)(x+y)\to a_2x$ which gives the term a_1x+a_2y; $a_2x\to a_1y\to(a_1+a_2)(x+y)$ gives a_1x+a_2y; and $a_2y\to a_1(x+y)\to(a_1+a_2)x$ gives $a_2x+(a_1+a_2)y$. So since they must reduce to the same term, we have $a_2x+a_1(x+y)=0$. We conjecture that $a_1^2=a_1y$ and $a_2^2=a_2x$, and $a_2y=a_1x$ holds. Now we are still missing one relation.

In F222, the term b_1 is mapped to $a_1a_2-b_1$ under C_2', so the term a_1a_2 is invariant. Then, in $F23$, $a_1a_2\to(a_1+a_2)a_1\to a_2(a_1+a_2)$ so the invariant quantity is $a_1a_2+a_1^2+a_2^2$, and $n=2\colon(a_1a_2+a_1^2+a_2^2,x^2+xy+y^2,a_1x+a_2y)$. This means there are two relation among a_1,a_2 when $x=y=0$, and we easily see that these are $a_1^2=a_2^2=0$. So there must be one more relation among x,y,a_1,a_2 that we are missing. under $M\colon(a_1,a_2,x,y)\to(a_1,a_1+a_2,x+y,y)$ so we see that $a_1a_2+a_1^2+a_2^2$ and x^2+y^2+xy are invariant, $a_1x+a_2y\to a_1(x+y)+(a_1+a_2)y$ is also invariant. Finally, under P, $a_1^2+a_1a_2+a_2^2\to(a_1+y)^2+(a_1a_2+x^2+xy+y^2)+(a_2+x)^2=a_1^2+a_2^2+a_1a_2+xy$, so the term $a_1^2+a_1a_2+a_2^2$ is not invariant; $a_1x+a_2y\to(a_1+y)x+(a_2+x)y$ is invariant.

Now we have dimensions: $(x^2,xy,y^2)=1,(a_1^2,a_1y)=1,(a_2^2,a_2x)=0,(a_1a_2)=1,(a_1x,a_2y)=1$, and we now must make $((a_1^2,a_1y),(a_2^2,a_2x),(a_1a_2))=1$, which we know is true when $x=y=0$, but apparently this holds also when x,y nonzero. We claim that the only possibility is that $(a_1^2,a_1y)=(a_2^2,a_2x)$, meaning that we have the relation $a_1^2=a_1y=a_2^2=a_2x$. Note again we clarify what this means: this means that the coefficient of these four basis vectors must equal! It does not mean that if one basis vector vanish all the other four must vanish. To summarize: the most general elements are

$$p_1x^2+p_2xy+p_3y^2+p_4a_1^2+p_5a_1y+p_6a_2^2+p_7a_2x+p_8a_1a_2+p_9a_1x+p_{10}a_2y,$$

and we claim that

$$p_1=p_2=p_3,p_4=p_5+1=p_6=p_7+1=p_8,p_9=p_{10}+1.$$

Note that when going to $Fd\bar{3}m$, $x=y=0$, and instead we have a new generator corresponding to P, which we call $\chi_p=z$. So we start with

$$a_1^2+a_2^2+a_1a_2,$$

we know from our general experience that, the order-two nature of PT_i (sice $PT_iP=T_i^{-1}$, not $PT_iP=T_i$) we must have $\chi_i^2=\chi_p\chi_i$, or $a_1^2=za_1$, $a_2^2=za_2$. a_1,a_2,z are three generators, and we need 4 relations to knock them down to dimension 2. We have just found two relations. We have one relation $a_1^2=a_2^2$ inherited from above, which still must hold true now: $a_1^2=a_2^2$. finally, we have

20.1 Calculation by definition

Elemennts of $g \in F23$ can be written in standard form

$$g = T_1^x T_2^y T_3^z C_2^c C_2^{\prime c'} C_3^r, \tag{68}$$

where $c, c' = 0, 1$ and $r = 0, 1, 2$, and $x, y, z \in \mathbb{Z}$.

We have using $C_3 T_i = T_{i+1} C_3$, $C_3 C_2 = C_2 C_2' C_3$, and $C_3 C_2' = C_2 C_3$, so we have $C_3 C_2^c C_2^{\prime c'} = C_2^c C_2^{\prime c} C_3 C_2^{\prime c'} = C_2^c C_2^{\prime c} C_2^{c'} C_3 = C_2^{c+c'} C_2^{\prime c} C_3$; and $C_3^2 C_2^c C_2^{\prime c'} = C_3 C_2^{c+c'} C_2^{\prime c} C_3 = C_2^{c'} C_2^{\prime c+c'} C_3^2$, so that $C_3^{t_1} C_2^c C_2^{\prime c'} = C_2^{(c,c',c+c')_{2t_1+1}} C_2^{\prime (c,c',c+c')_{2t_1+2}} C_3^{t_1}$,

and $C_3 T_1^{x_2} T_2^{y_2} T_3^{z_2} = T_2^{x_2} T_3^{y_2} T_1^{z_2} C_3 = T_1^{z_2} T_2^{x_2} T_3^{y_2} C_3$, and $C_3^2 T_1^{x_2} T_2^{y_2} T_3^{z_2} = T_1^{y_2} T_2^{z_2} T_3^{x_2} C_3^2$, so we see that $C_3^{t_1} T_1^{x_2} T_2^{y_2} T_3^{z_2} = T_1^{(r_2)_{2t_1+1}} T_2^{(r_2)_{2t_1+2}} T_3^{(r_2)_{2t_1+3}} C_3^{t_1}$,

so if we define

$$r_i = (x_i, y_i, z_i), \qquad s_i = (c_i, c_i', c_i + c_i'), \quad c_i, c_i' \in \mathbb{Z}_2,$$

and using

$$\begin{aligned}
g_1 g_2 &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} C_2^{\prime c_1'} T_1^{x_2} T_2^{y_2} T_3^{z_2} C_2^{c_2} C_2^{\prime c_2'} \\
&= T_1^{x_1+(1-c_1)((1-c_1')x_2+c_1'z_2)+c_1(-c_1'x_2-c_1'y_2+(1-c_1')y_2-c_1'z_2)} \\
&\quad \cdot T_2^{y_1+c_1((1-c_1')x_2+c_1'z_2)+(1-c_1)(-c_1'x_2-c_1'y_2+(1-c_1')y_2-c_1'z_2)} \\
&\quad \cdot T_3^{z_1-c_1((1-c_1')x_2+c_1'z_2)-c_1(-c_1'x_2-c_1'y_2+(1-c_1')y_2-c_1'z_2)+(1-2c_1)(c_1'x_2+(1-c_1')z_2)} \\
&\quad \cdot C_2^{c_1+c_2} C_2^{\prime c_1'+c_2'},
\end{aligned} \tag{69}$$

$$(x_2, y_2, z_2) \to ((r_2)_{2t_1+1}, (r_2)_{2t_1+2}, (r_2)_{2t_1+3}), \qquad (c_2, c_2') \to ((s_2)_{2t_1+1}, (s_2)_{2t_1+2}),$$

$$\begin{aligned}
g_1 g_2 &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} C_2^{\prime c_1'} C_3^{t_1} T_1^{x_2} T_2^{y_2} T_3^{z_2} C_2^{c_2} C_2^{\prime c_2'} C_3^{t_2} \\
&= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} C_2^{\prime c_1'} T_1^{(r_2)_{2t_1+1}} T_2^{(r_2)_{2t_1+2}} T_3^{(r_2)_{2t_1+3}} C_2^{(s_2)_{2t_1+1}} C_2^{\prime (s_2)_{2t_1+2}} C_3^{t_1+t_2} \\
&= T_1^{x_1+(1-c_1)((1-c_1')(r_2)_{2t_1+1}+c_1'(r_2)_{2t_1+3})+c_1(-c_1'(r_2)_{2t_1+1}-c_1'(r_2)_{2t_1+2}+(1-c_1')(r_2)_{2t_1+2}-c_1'(r_2)_{2t_1+3})} \\
&\quad \cdot T_2^{y_1+c_1((1-c_1')(r_2)_{2t_1+1}+c_1'(r_2)_{2t_1+3})+(1-c_1)(-c_1'(r_2)_{2t_1+1}-c_1'(r_2)_{2t_1+2}+(1-c_1')(r_2)_{2t_1+2}-c_1'(r_2)_{2t_1+3})} \\
&\quad \cdot T_3^{z_1-c_1((1-c_1')(r_2)_{2t_1+1}+c_1'(r_2)_{2t_1+3})-c_1(-c_1'(r_2)_{2t_1+1}-c_1'(r_2)_{2t_1+2}+(1-c_1')(r_2)_{2t_1+2}-c_1'(r_2)_{2t_1+3})+(1-2c_1)(c_1'(r_2)_{2t_1+1}+(1-c_1')(r_2)_{2t_1+3})} \\
&\quad \cdot C_2^{c_1+(s_2)_{2t_1+1}} C_2^{\prime c_1'+(s_2)_{2t_1+2}} C_3^{t_1+t_2}.
\end{aligned} \tag{70}$$

Then we have

$$C_3^{-1} g C_3 = T_1^y T_2^z T_3^x C_2^{c'} C_2^{\prime c+c'},$$

In other words, $(x, y, z, c, c') \mapsto (y, z, x, c', c + c')$ ($c + c'$ is in the mod 2 sense). So we see that a_1, a_2, x, y are not invariant under C_3 action.

Remember that in F222, in $H^2(F222, \mathbb{Z}_2)$ we have

$$H^2(F222, \mathbb{Z}_2) = \langle a_1a_2, a_1x, a_1y, a_2x, x^2, xy, y^2 \rangle \cong \mathbb{Z}_2^7, \quad a_1^2 = a_1y, \quad a_2^2 = a_2x, \quad a_1x = a_2y. \tag{71}$$

For $a_1a_2(g_1, g_2) = (x_1 + y_1)(x_2 + z_2)$, we check that

$$a_1a_2(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) - a_1a_2(g_1, g_2) = (a_1y)(g_1, g_2) + \mu(g_1) + \mu(g_2) - \mu(g_1g_2),$$

where $\mu(g) = \frac{1}{2}(y + z)(y + z + 1) + \frac{1}{2}(x + z)(x + z + 1) + x + (x + y)c'$.

Let's also mention in passing that $a_1x = xa_1$, $a_2x = xa_2$, $a_1y = ya_1$, $a_2y = ya_2$, up to coboundary $\mu(g) = (x + y)c, (x + z)c, (x + y)c', (x + z)c'$, respectively. Then taking $\mu(g) = (x + y)c + (x + z)c' + x$ or $(x + y)c + (x + z)c' + y$ or $(x + y)c + (x + z)c' + z$ we see that $a_2y = a_1x$. This means that

$$\begin{aligned}
a_1x(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) - a_1x(g_1, g_2) &= (a_1x + a_1y + a_2y)(g_1, g_2) = a_1y(g_1, g_2), \\
a_1y(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) - a_1y(g_1, g_2) &= (a_1x + a_1y + a_2y)(g_1, g_2) = a_1y(g_1, g_2), \\
a_2x(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) - a_2x(g_1, g_2) &= (a_2x + a_1y)(g_1, g_2),
\end{aligned}$$

We also have $(a_2y(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) - a_2y(g_1, g_2) = (a_1x + a_1y + a_2y)(g_1, g_2) = a_1y(g_1, g_2))$ but since $a_2y = a_1x$ this is redundant. We see that

$$C_3.(a_1x + a_1y) = a_1x + a_1y,$$

$$C_3.(a_1a_2 + a_1y) = a_1a_2 + a_1y,$$

$$C_3.(a_1a_2 + a_1x) = a_1a_2 + a_1x,$$

and of course only two out them are independent. We have $(C_3.x^2 - x^2)(g_1, g_2) = c_1'c_2' + c_1c_2 = (x^2 + y^2)(g_1, g_2)$, $(C_3.xy - xy)(g_1, g_2) = c_1c_2' + c_1'(c_2 + c_2') = y^2(g_1, g_2)$ (note that using $\mu(g) = cc'$ we can show that $c_1c_2' + c_2c_1'$ is a coboundary), and $(C_3.y^2 - y^2)(g_1, g_2) = c_1'c_2' + (c_1 + c_1')(c_2 + c_2') = c_1c_2 = x^2(g_1, g_2)$, so $C_3.(x^2 + xy + y^2) = x^2 + xy + y^2$.

This way we have found all the three for $E_2^{0,2}$: $E_2^{0,2} = \langle a_1(x+y), a_1(a_2+y), x^2+xy+y^2)\rangle$. Since we know the spectral sequence collapses at E_2 with nonzero element only on $E_2^{0,q}$, we have, from $(p, q) = (0, 2)$,

$$H^2(F23, \mathbb{Z}_2) = \langle \beta_1, \beta_2, \beta_3 \rangle, \tag{72}$$

where

$$F23 \to F222, \quad \beta_1 \mapsto a_1a_2 + a_1x, \quad \beta_2 \mapsto a_1a_2 + a_1y, \quad \beta_3 \mapsto x^2 + xy + y^2. \tag{73}$$

The spectral sequence is useful in that it allows us to know what elements of $H^*(F222, \mathbb{Z}_2)$ the elements in $H^*(F23, \mathbb{Z}_2)$ reduces to. This is useful because if we find a representative cochain ω that satisfy the cocycle condition, if we can find the corresponding element in $H^*(F222, \mathbb{Z}_2)$, this means that ω is a true cocycle, not a coboundary. But the form of ω (as a representative cochain) can be extremely hard to find.

Using brutal force computation in Mathematica, we find one representative cochain for the cocycle $x^2 + xy + y^2$ after adding the C_3 element. It is:

$$\beta_3(g_1, g_2) = \begin{cases} c_1c_2 + c_1c_2' + c_1'c_2', & (t_1, t_2) = (0, 0) \\ c_1c_2 + c_1c_2' + c_1'c_2', & (t_1, t_2) = (0, 1) \\ c_1c_2 + c_1c_2' + c_1'c_2', & (t_1, t_2) = (0, 2) \\ c_2 + c_2c_1' + c_1c_2', & (t_1, t_2) = (1, 0) \\ 1 + c_2 + c_2c_1' + c_1c_2', & (t_1, t_2) = (1, 1) \\ c_2 + c_2c_1' + c_1c_2', & (t_1, t_2) = (1, 2) \\ c_1c_2 + c_2c_1' + c_2' + c_1'c_2', & (t_1, t_2) = (2, 0) \\ c_1c_2 + c_2c_1' + c_2' + c_1'c_2', & (t_1, t_2) = (2, 1) \\ 1 + c_1c_2 + c_2c_1' + c_2' + c_1'c_2', & (t_1, t_2) = (0, 0) \end{cases} \tag{74}$$

where know that this is the only cocycle in $H^2(A_4, \mathbb{Z}_2) = \langle \beta_3 \rangle \cong \mathbb{Z}_2$.

Next we want to find the other two cocycles β_1, β_2.

Goal: find β_1 and β_2.

By examining Weicheng's result on wallpaper groups with a C_3 element, we see that if $\beta(g_1, g_2)$ is a 2-cocycle, then it can be made independent of t_2; then depending on the value of $t_1 = 0, 1, 2$, at least two of them always involve things like $x_2(x_2 + 1)/2$ and so on that comes g_2 (not g_1). Also, we notice that the terms are at most linear in c_i and c_i'. With all these in mind, we can start conjecturing the form of functions $f(x_1, y_1, z_1, c_1, c_1', x_2, y_2, z_2, c_2, c_2')$. Using Mathematica, we successfully find the following cocycle:

$$\beta_1(g_1, g_2)$$
$$= \begin{cases} (x_1 + y_1)(x_2 + z_2 + c_2), & t_1 = 0, \\ \frac{1}{2}x_2(1+x_2) + \frac{1}{2}z_2(1+z_2) + c_2x_1 + c_2'x_1 + c_2'x_2 + c_2y_1 + c_2'y_1 + c_2y_2 + x_1y_2 + x_2y_2 + y_1y_2 + c_2z_2 + c_2'z_2 + x_1z_2 + y_1z_2 + y_2z_2, & t_1 = 1, \\ \frac{1}{2}x_2(1+x_2) + \frac{1}{2}y_2(1+y_2) + c_2'x_1 + c_2x_2 + x_1x_2 + c_2'y_1 + x_2y_1 + c_2y_2 + c_2'y_2 + x_1y_2 + y_1y_2 + c_2'z_2 + x_2z_2 + y_2z_2, & t_1 = 2. \end{cases} \tag{75}$$

$$
\begin{aligned}
&\beta_2(g_1,g_2)\\
&=\begin{cases}
(c_2+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_1'm_2+c_2'm_2+c_2'm_1m_2)x_1\\
\quad +(c_2'+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_2'm_1m_2)y_1\\
\quad +(c_2+c_2'+m_2+c_1'm_2+c_2'm_2)z_1, & (t_1,t_2)=(0,0)\\
(1+c_2+c_2m_1+c_2'm_1+m_2+c_1m_2+c_2m_2+c_1'm_2+c_2'm_2+m_1m_2+c_2'm_1m_2)x_1\\
\quad +(1+c_2'+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+m_1m_2+c_2'm_1m_2)y_1\\
\quad +(c_2+c_2'+c_1'm_2+c_2'm_2)z_1, & (t_1,t_2)=(0,1)\\
(1+c_2+m_1+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_1'm_2+c_2'm_2+m_1m_2+c_2'm_1m_2)x_1\\
\quad +(c_2'+m_1+c_2m_1+c_2'm_1+m_2+c_1m_2+c_2m_2+m_1m_2+c_2'm_1m_2)y_1\\
\quad +(1+c_2+c_2'+c_1'm_2+c_2'm_2)z_1, & (t_1,t_2)=(0,2)\\
(c_2+c_2'+m_2+c_1m_2+c_1'm_2+c_2'm_2)x_1\\
\quad +(c_2+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_2'm_2+c_2'm_1m_2)y_1\\
\quad +(c_2'+c_2m_1+c_2'm_1+c_2m_2+c_1'm_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(1,0)\\
(c_2+c_2'+c_1m_2+c_1'm_2+c_2'm_2)x_1\\
\quad +(1+c_2+c_2m_1+c_2'm_1+m_2+c_1m_2+c_2m_2+c_2'm_2+m_1m_2+c_2'm_1m_2)y_1\\
\quad +(1+c_2'+c_2m_1+c_2'm_1+c_2m_2+c_1'm_2+m_1m_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(1,1)\\
(1+c_2+c_2'+c_1m_2+c_1'm_2+c_2'm_2)x_1\\
\quad +(1+c_2+m_1+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_2'm_2+m_1m_2+c_2'm_1m_2)y_1\\
\quad +(c_2'+m_1+c_2m_1+c_2'm_1+m_2+c_2m_2+c_1'm_2+m_1m_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(1,2)\\
(c_2'+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_1'm_2+c_2'm_1m_2)x_1\\
\quad +(c_2+c_2'+m_2+c_1m_2+c_2'm_2)y_1\\
\quad +(c_2+c_2m_1+c_2'm_1+c_2m_2+c_1'm_2+c_2'm_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(2,0)\\
(1+c_2'+c_2m_1+c_2'm_1+c_1m_2+c_2m_2+c_1'm_2+m_1m_2+c_2'm_1m_2)x_1\\
\quad +(c_2+c_2'+c_1m_2+c_2'm_2)y_1\\
\quad +(1+c_2+c_2m_1+c_2'm_1+m_2+c_2m_2+c_1'm_2+c_2'm_2+m_1m_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(2,1)\\
(c_2'+m_1+c_2m_1+c_2'm_1+m_2+c_1m_2+c_2m_2+c_1'm_2+m_1m_2+c_2'm_1m_2)x_1\\
\quad +(1+c_2+c_2'+c_1m_2+c_2'm_2)y_1\\
\quad +(1+c_2+m_1+c_2m_1+c_2'm_1+c_2m_2+c_1'm_2+c_2'm_2+m_1m_2+c_2'm_1m_2)z_1, & (t_1,t_2)=(2,2)
\end{cases}
\end{aligned}
\tag{76}
$$

20.2 Conjecture – now proved! E_2 collapses

For the E_2 page (which collapses to E_∞) of the group $F23$ we have

$q=4$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$q=3$	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**4**	**3**	**4**	**5**	**4**
$q=2$	**1**	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**4**	**3**	**4**	**5**
$q=1$	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**4**	**3**	**4**	**5**	**4**	**5**
$q=1$	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**4**	**3**	**4**	**5**	**4**
$E_2^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$	$p=12$	$p=13$
H_{tot}^{p-1}	–	1	0	3	6	2	6	10	6	10	14	10	14	18

(77)

i.e. the mod-2 cohomology dimensions are: $E_2^{p,q=0}=m(p)$, $E_2^{p,q=1}=m(p+1)$, $E_2^{p,q=2}=\delta_{p=0}+\delta_{p>0}m(p-1)$, $E_2^{p,q=3}=m(p)$. And E_2 collapses.

21 $H^*(F\bar{4}3m(No.216),\mathbb{Z}_2)$

$F\bar{4}3m=\langle T_1,T_2,T_3,C_3,C_2,C_2',M\rangle$ is the index-2 extension of $F23$, where M, according to Eq. (29), is just mirror that switches x and y coordinates. We have $MT_1M=T_2$, $MT_2M=T_1$, $MT_3M=T_3$, $MC_3M=C_3^{-1}$, $MC_2M=C_2$, $MC_2'M=C_2C_2'$.

Elements of $g\in F\bar{4}3m$ can be written in standard form

$$g=T_1^xT_2^yT_3^zC_2^cC_2'^{c'}C_3^tM^m, \tag{78}$$

where $m,c,c'=0,1$ and $r=0,1,2$, and $x,y,z\in\mathbb{Z}$.

Using the group multiplication if $F23$

$$\begin{aligned} g_1 g_2 &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} {C'}_2^{c'_1} C_3^{t_1} T_1^{x_2} T_2^{y_2} T_3^{z_2} C_2^{c_2} {C'}_2^{c'_2} C_3^{t_2} \\ &= T_1^{x_1+(1-c_1)((1-c'_1)(r_2)_{2t_1+1}+c'_1(r_2)_{2t_1+3})+c_1(-c'_1(r_2)_{2t_1+1}-c'_1(r_2)_{2t_1+2}+(1-c'_1)(r_2)_{2t_1+2}-c'_1(r_2)_{2t_1+3})} \\ &\quad \cdot T_2^{y_1+c_1((1-c'_1)(r_2)_{2t_1+1}+c'_1(r_2)_{2t_1+3})+(1-c_1)(-c'_1(r_2)_{2t_1+1}-c'_1(r_2)_{2t_1+2}+(1-c'_1)(r_2)_{2t_1+2}-c'_1(r_2)_{2t_1+3})} \\ &\quad \cdot T_3^{z_1-c_1((1-c'_1)(r_2)_{2t_1+1}+c'_1(r_2)_{2t_1+3})-c_1(-c'_1(r_2)_{2t_1+1}-c'_1(r_2)_{2t_1+2}+(1-c'_1)(r_2)_{2t_1+2}-c'_1(r_2)_{2t_1+3})+(1-2c_1)(c'_1(r_2)_{2t_1+1}+(1-c'_1)(r_2)_{2t_1+3})} \\ &\quad \cdot C_2^{c_1+(s_2)_{2t_1+1}} {C'}_2^{c'_1+(s_2)_{2t_1+2}} C_3^{t_1+t_2}, \end{aligned} \tag{79}$$

where $r_i = (x_i, y_i, z_i),\ s_i = (c_i, c'_i, c_i + c'_i),\ c_i, c'_i \in \mathbb{Z}_2$,

We have

$$\begin{aligned} g_1 g_2 &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} {C'}_2^{c'_1} C_3^{t_1} M^{m_1} T_1^{x_2} T_2^{y_2} T_3^{z_2} C_2^{c_2} {C'}_2^{c'_2} C_3^{t_2} M^{m_2} \\ &= T_1^{x_1} T_2^{y_1} T_3^{z_1} C_2^{c_1} {C'}_2^{c'_1} C_3^{t_1} T_1^{x_2+m_1(y_2-x_2)} T_2^{y_2+m_1(x_2-y_2)} T_3^{z_2} C_2^{c_2+m_1c'_2} {C'}_2^{c'_2} C_3^{(1+m_1)t_2} M^{m_1+m_2} \\ &= T_1^{x_1+(1-c_1)((1-c'_1)(r_2[m_1])_{2t_1+1}+c'_1(r_2[m_1])_{2t_1+3})+c_1(-c'_1(r_2[m_1])_{2t_1+1}-c'_1(r_2[m_1])_{2t_1+2}+(1-c'_1)(r_2[m_1])_{2t_1+2}-c'_1(r_2[m_1])_{2t_1+3})} \\ &\quad \cdot T_2^{y_1+c_1((1-c'_1)(r_2[m_1])_{2t_1+1}+c'_1(r_2[m_1])_{2t_1+3})+(1-c_1)(-c'_1(r_2[m_1])_{2t_1+1}-c'_1(r_2[m_1])_{2t_1+2}+(1-c'_1)(r_2[m_1])_{2t_1+2}-c'_1(r_2[m_1])_{2t_1+3})} \\ &\quad \cdot T_3^{z_1-c_1((1-c'_1)(r_2[m_1])_{2t_1+1}+c'_1(r_2[m_1])_{2t_1+3})-c_1(-c'_1(r_2[m_1])_{2t_1+1}-c'_1(r_2[m_1])_{2t_1+2}+(1-c'_1)(r_2[m_1])_{2t_1+2}-c'_1(r_2[m_1])_{2t_1+3})+(1-2c_1)(c'_1(r_2[m_1])_{2t}} \\ &\quad \cdot C_2^{c_1+(s_2[m_1])_{2t_1+1}} {C'}_2^{c'_1+(s_2[m_1])_{2t_1+2}} C_3^{t_1+(1+m_1)t_2} M^{m_1+m_2}, \end{aligned} \tag{80}$$

where

$$r_i[m] = (x_i + m(y_i - x_i), y_i + m(x_i - y_i), z_i), \quad s_i[m] = (c_i + mc'_i, c'_i, c_i + (1+m)c'_i), \quad c_i, c'_i, m \in \mathbb{Z}_2.$$

It is easy to check that $m(g) = m$ satisfies the cocycle condition. Since we know it cannot be a coboundary, it is a cocycle. It is the only nontrivial cocycle in $H^1(F\bar{4}3m, \mathbb{Z}_2)$. Also, $\omega(g_1, g_2) = m_1 m_2$ is a cocycle in $H^2(F\bar{4}3m, \mathbb{Z}_2)$.

We checked explicitly in Mathematica that the cocycles in Eqs. (74) and (75) given in the last section is invariant: for $g_1, g_2 \in F23$, we have

$$\beta_1(Mg_1M, Mg_2M) - \beta_1(g_1, g_2) = \mu_{\beta_1}(g_1) + \mu_{\beta_1}(g_2) - \mu_{\beta_1}(g_1 g_2), \tag{81}$$

where $\mu_{\beta_1}(g) = \frac{1}{2}(x+y)(x+y+1) + (x+y)c'$.

$$\beta_3(Mg_1M, Mg_2M) - \beta_3(g_1, g_2) = \mu_{\beta_3}(g_1) + \mu_{\beta_3}(g_2) - \mu_{\beta_3}(g_1 g_2), \tag{82}$$

where $\mu_{\beta_2}(g) = c(c' - 1)$.

Setting $t_i = 0$, we know that both $ym(g_1, g_2) = c'_1 m_2$ and $e_2(g_1, g_2) = c_1c_2 + c_1c'_2 + c'_1c'_2 + m_1(c_2 + c_1c'_2 + c_2c'_2)$ are cocycles. We can also easily calculate (by brutal force using Mathematica) that

$$\beta_3^{F\bar{4}3m}(g_1, g_2) = \begin{cases} c_1c_2 + c_1c'_2 + c'_1c'_2 + m_1(c_2 + c_1c'_2 + c_2c'_2), & (t_1, t_2) = (0,0) \\ c_1c_2 + c_1c'_2 + c'_1c'_2 + m_1(c_2 + c_1c'_2 + c_2c'_2), & (t_1, t_2) = (0,1) \\ c_1c_2 + c_1c'_2 + c'_1c'_2 + m_1(c_2 + c_1c'_2 + c_2c'_2), & (t_1, t_2) = (0,2) \\ c_2 + c_2c'_1 + c_1c'_2 + m_1(c_2 + c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (1,0) \\ 1 + c_2 + c_2c'_1 + c_1c'_2 + m_1(1 + c_2 + c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (1,1) \\ c_2 + c_2c'_1 + c_1c'_2 + m_1(1 + c_2 + c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (1,2) \\ c_1c_2 + c_2c'_1 + c'_2 + c'_1c'_2 + m_1(c_2 + c_1c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (2,0) \\ c_1c_2 + c_2c'_1 + c'_2 + c'_1c'_2 + m_1(1 + c_2 + c_1c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (2,1) \\ 1 + c_1c_2 + c_2c'_1 + c'_2 + c'_1c'_2 + m_1(1 + c_2 + c_1c'_2 + c_2c'_2 + c'_1c'_2), & (t_1, t_2) = (0,0) \end{cases} \tag{83}$$

which reduces to the β_3 of $F23$ (see Eq. (74))) when setting $m_i = 0$. This means β_3 is a real cocycle, not a coboundary.

On the other hand, from Adem–Milgram (P185) we know it should reduce to another order-two element of the group

$V_1 = \langle (12), (34) \rangle = \langle M, MC_2 \rangle$, so let's see this. This is to set $c_1' = c_2' = 0$, and we have

$$\beta_3^{F\bar{4}3m}(g_1, g_2)|_{c'_{1,2}=0} = \begin{cases} c_1c_2 + m_1c_2, & (t_1, t_2) = (0,0) \\ c_1c_2 + m_1c_2, & (t_1, t_2) = (0,1) \\ c_1c_2 + m_1c_2, & (t_1, t_2) = (0,2) \\ c_2 + m_1c_2, & (t_1, t_2) = (1,0) \\ (1+m_1)(1+c_2), & (t_1, t_2) = (1,1) \\ c_2 + m_1(1+c_2), & (t_1, t_2) = (1,2) \\ c_1c_2 + m_1c_2, & (t_1, t_2) = (2,0) \\ c_1c_2 + m_1(1+c_2), & (t_1, t_2) = (2,1) \\ 1 + c_1c_2 + m_1(1+c_2), & (t_1, t_2) = (0,0) \end{cases} \tag{84}$$

we have verified in Mathematica that when this, restricting to V_1, gives a true cocycle.

Next we want to find the cocycle of $F\bar{4}3m$ that reduces to β_1 in $F23$. To do this, we first find the expression at $t_1 = t_2 = 0$:

$$\begin{aligned}
\beta_1(g_1, g_2)|_{t_1=0} &= (x_1+y_1)(x_2+z_2+c_2) + m_2\left[\frac{1}{2}(x_1+y_1)(x_1+y_1+1) + (x_1+y_1)(x_2+y_2+c_1'+c_2')\right], \\
\beta_1(g_1, g_2)|_{t_1=1} &= c_2m_1m_2x_2 + c_2m_1m_2y_2 + c_2m_1x_1 + c_2m_1x_2 + c_2m_1y_1 + c_2m_1z_2 + c_2m_2x_1 + c_2m_2x_2 + c_2m_2y_1 \\
&\quad + c_2m_2z_2 + c_2x_1 + c_2y_1 + c_2y_2 + c_2z_2 + c_1'm_2x_1 + c_1'm_2y_1 + c_2'm_1m_2x_1 + c_2'm_1m_2y_1 + c_2'm_1m_2y_2 \\
&\quad + c_2'm_1m_2z_2 + c_2'm_1x_2 + c_2'm_1y_2 + c_2'm_2x_2 + c_2'm_2y_2 + c_2'x_1 + c_2'x_2 + c_2'y_1 + c_2'z_2 + m_1m_2x_1x_2 \\
&\quad + m_1m_2x_1y_2 + m_1m_2x_2y_1 + m_1m_2x_2z_2 + \frac{1}{2}m_1m_2x_2(x_2+1) + m_1m_2y_1y_2 + m_1m_2y_2z_2 \\
&\quad + \frac{1}{2}m_1m_2y_2(y_2+1) + m_1x_1x_2 + m_1x_1z_2 + m_1x_2y_1 + m_1x_2y_2 + m_1x_2z_2 + m_1y_1z_2 + \frac{1}{2}m_1y_2(y_2+1) \\
&\quad + \frac{1}{2}m_1z_2(z_2+1) + m_2x_1x_2 + m_2x_1y_1 + m_2x_1z_2 + \frac{1}{2}m_2(x_1+1)x_1 + m_2x_2y_1 + m_2x_2y_2 + m_2x_2z_2 + m_2y_1z_2 \\
&\quad + \frac{1}{2}m_2y_1(y_1+1) + \frac{1}{2}m_2y_2(y_2+1) + \frac{1}{2}m_2z_2(z_2+1) \\
&\quad + x_1y_2 + x_1z_2 + x_2y_2 + \frac{1}{2}x_2(x_2+1) + y_1y_2 + y_1z_2 + y_2z_2 + \frac{1}{2}z_2(z_2+1), \\
\beta_1(g_1, g_2)|_{t_1=2} &= c_2m_1m_2x_2 + c_2m_1m_2y_2 + c_2m_1x_1 + c_2m_1x_2 + c_2m_1y_1 + c_2m_1z_2 + c_2m_2x_1 + c_2m_2y_1 + c_2m_2y_2 + c_2m_2z_2 \\
&\quad + c_2x_2 + c_2y_2 + c_1'm_2x_1 + c_1'm_2y_1 + c_2'm_1m_2x_1 + c_2'm_1m_2y_1 + c_2'm_1m_2y_2 + c_2'm_1m_2z_2 + c_2'm_1x_2 + c_2'm_1y_2 \\
&\quad + c_2'm_2x_1 + c_2'm_2x_2 + c_2'm_2y_1 + c_2'm_2z_2 + c_2'x_1 + c_2'y_1 + c_2'y_2 + c_2'z_2 + m_1m_2x_1x_2 + m_1m_2x_1y_2 + m_1m_2x_2y_1 \\
&\quad + m_1m_2x_2z_2 + \frac{1}{2}m_1m_2x_2(x_2+1) + m_1m_2y_1y_2 + m_1m_2y_2z_2 + \frac{1}{2}m_1m_2y_2(y_2+1) + m_1x_1x_2 + m_1x_1z_2 \\
&\quad + m_1x_2y_1 + m_1x_2y_2 + m_1x_2z_2 + m_1y_1z_2 + \frac{1}{2}m_1y_2(y_2+1) + \frac{1}{2}m_1z_2(z_2+1) + m_2x_1y_1 + m_2x_1y_2 + m_2x_1z_2 \\
&\quad + \frac{1}{2}m_2(x_1+1)x_1 + m_2x_2y_2 + \frac{1}{2}m_2x_2(x_2+1) + m_2y_1y_2 + m_2y_1z_2 + \frac{1}{2}m_2y_1(y_1+1) + m_2y_2z_2 + \frac{1}{2}m_2z_2(z_2+1) \\
&\quad + x_1x_2 + x_1y_2 + x_2y_1 + x_2z_2 + \frac{1}{2}x_2(x_2+1) + y_1y_2 + y_2z_2 + \frac{1}{2}y_2(y_2+1), \\
&\text{where} \quad m_i, c_i, c_i' = 0, 1, \quad t_i = 0, 1, 2, \quad x_i, y_i, z_i \in \mathbb{Z}.
\end{aligned} \tag{85}$$

Note that it is important to remember that m_i, c_i, c_i' can only take $\mathbb{Z}_2$ values. Had we take the wrong value $m_1 = 2$, say, we will find that β_1 does not "satisfy" the cocycle condition.

We see that

$$M\colon (\chi_1, \chi_2, \chi_3, \chi_c, \chi_{c'}) \mapsto (\chi_2, \chi_1, \chi_3, \chi_c + \chi_{c'}, \chi_{c'})$$

which induces

$$M\colon (a_1, a_2, x, y) \mapsto (a_1, a_1 + a_2, x + y, y),$$

Note that M permute $x^3 + x^2y + y^3$ and $x^3 + xy^2 + y^3$; M also permute $a_2x^2 + a_1(x^2 + y^2)$ and $a_2(x^2 + y^2) + a_1y^2$, and this already reduces the $n = 3$ dimension down to $\mathbb{Z}_2^4$. We see that M must stabilize a_1a_2(?), a_1y(?), $x^2 + xy + y^2$(checked), and c_1, and c_2. M stablizes $x^2y + xy^2$ and $a_2y^2 + a_1x^2$.

We further see that $M.H^1(F23, \mathbb{Z}_2) - H^1(F23, \mathbb{Z}_2) = \langle x^2y + xy^2, a_2y^2 + a_1x^2 \rangle$ which has dimension 2, therefore the $q = 3$ line in the spectral sequence is also determined. Further, we see that the $q = 4$ is also obtained.

We have, using GAP command `R:=ResolutionSpaceGroup(SpaceGroupIT(3,216),16);`,

$$H^{0:15}(F\bar{4}3m, \mathbb{Z}_2) = (1, 1, 4, 8, 8, 12, 16, 16, 20, 24, 24, 28, 32, 32, 36, 40), \tag{86}$$

the GAP computation by constructing resolution up to $n \geq 7$ exists memory limit, but we tested that `G:=SpaceGroupBBNWZ("F-43m")` `SpaceGroupOnRightBBNWZ( 3, 7, 4, 2, 1 ); GroupCohomology(G,2,2);` actually is a better way to compute it. Note that one changes `GroupCohomology(G,2,2);` to `GroupCohomology(G,3,2);` and so on.

Using $H^{1:5}(F23, \mathbb{Z}_2) = (0, 3, 6, 2, 6)$, we can deduce the spectral sequence

$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\iddots$
$q=5$	$\mathbf{4^{\leq 6}}$	**2**	**2**	**2**	**2**	**2**	$\cdots$
$q=4$	$\mathbf{2^{\leq 2}}$	**2**	**2**	**2**	**2**	**2**	$\cdots$
$q=3$	$\mathbf{4^{\leq 6}}$	**2**	**2**	**2**	**2**	**2**	$\cdots$
$q=2$	$\mathbf{3^{\leq 3}}$	**3**	**3**	**3**	**3**	**3**	$\cdots$
$q=1$	$\mathbf{0^{\leq 0}}$	**0**	**0**	**0**	**0**	**0**	$\cdots$
$q=0$	**1**	**1**	**1**	**1**	**1**	**1**	$\cdots$
$E_2^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$\cdots\cdots$

(87)

where a bold place means this place is stabilized at the E_2 page. We see that all the d_2 maps in the region tabulated vanish.

$$n_1\colon \langle m\rangle \cong \mathbb{Z}_2;$$
$$n_2\colon \langle a_1^2 + a_1a_2 + a_2^2, x^2 + xy + y^2, a_2x; m^2\rangle \cong \mathbb{Z}_2^4,$$
$$n_3\colon \langle c_1, c_2, x^2y + xy^2, a_2y^2 + a_1x^2; m(a_1^2 + a_1a_2 + a_2^2), m(x^2 + xy + y^2), ma_2x; m^3\rangle \cong \mathbb{Z}_2^8,$$

with $m(x^2y + xy^2) = m(a_1y^2 + a_1x^2) = 0$,

21.1 Conjecture – now proved! E_2 collapses

If we use the translation spectral sequence, we conjecture that the spectral sequence collapses at E_2, and we have the form for the E_2 page of $F\bar{4}3m$

$q=4$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$q=3$	**1**	**1**	**2**	**3**	**3**	**4**	**5**	**5**	**6**	**7**	**7**	**8**	**9**	**9**
$q=2$	**1**	**2**	**2**	**3**	**4**	**4**	**5**	**6**	**6**	**7**	**8**	**8**	**9**	**10**
$q=1$	**0**	**1**	**2**	**2**	**3**	**4**	**4**	**5**	**6**	**6**	**7**	**8**	**8**	**9**
$q=0$	**1**	**1**	**2**	**3**	**3**	**4**	**5**	**5**	**6**	**7**	**7**	**8**	**9**	**9**
$E_2^{p,q}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$	$p=12$	$p=13$
H_{tot}^{p-1}	0	1	1	4	8	8	12	16	16	20	24	24	28	32

(88)

the entries with small p of course have been verified. The H_{tot}^p stands form $H^n(F\bar{4}3m, \mathbb{Z}_2)$ and has been calculated up to $n = 9$.

We can compare Eq. (88) with the result for F23 in (77). To do this, consider the SES $0 \to A_4 \to S_4 \to \mathbb{Z}_2 \to 0$. The spectral sequence writes $E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(A_4, \mathbb{Z}_2)) \Rightarrow H^{p+q}(S_4, \mathbb{Z}_2)$. In Adem Milgram P208 there is a theorem:

$$0 \to H^*(S_4, \mathbb{Z}_2)/(\sigma_1) \xrightarrow{res} H^*(A_4, \mathbb{Z}_2) \xrightarrow{tr} \mathrm{Ann}(\sigma_1) \to 0,$$

where $H^*(S_4, \mathbb{Z}_2) = \mathbb{F}_2[\sigma_1, \sigma_2, c_3]/(\sigma_1 c_3)$ where σ_1, σ_2, c_3 have degrees $1, 2, 3$ respectively, and $\mathrm{Ann}(\sigma_1)$ is the elements in the ring that annihilates σ_1, which, in our case, are $\{c_3^i\}_{i=1,2,3,...}$. Let's see how to count $H^*(S_4, \mathbb{Z}_2)$. As we mentioned later, this is equal to the number of solutions for $a + 2b + 3c = n$, with the constraint that when $a \geq 1$ then $c = 0$. Note that when $a = 0$, the number of solutions is $[(n-3)/6] + 1 - \delta_{n \neq 4 \bmod 6}$; when $a \geq 1$, then we must have $c = 0$ and the number of solutions are $[n/2] + 1$. Now, $H^*(S_4, \mathbb{Z}_2)/(\sigma_1)$ means that only taking one of the $[(n-3)/6] + 1 - \delta_{n \neq 4 \bmod 6}$ solutions for degree n while the j-th solution for $0 \leq j \leq [n/2]$ which now has degree $2j$. This is then "product with" c_3^i to get a class in $H^m(A_4)$ which needs $m = 2j + 3i$, and one notices the number of solutions is exactly $[(m-3)/6] + 1 - \delta_{m \neq 4 \bmod 6}$. But we see there is some over counting. In fact, the problem is very simple: the above extension tells us that the dimension of $H^m(A_4, \mathbb{Z}_2)$ equals the nonnegative integer solution of $2b + 3(c + d) = m$. Numerate: when $m = 6p$, then $c + d = 0, 2, 4, ..., 2p$, so there are $1 + 3 + 5 + \cdots + 2p + 1 = (p + 1)^2$ solutions; when $m = 6p + 1$, then $c + d = 1, 3, ..., 2p - 1$, so there are $2 + 4 + \cdots + 2p = (1 + p)p$ solutions; when $m = 6p + 2$, then $c + d = 0, 2, ..., 2p$ so $(p + 1)^2$ solutions; when

$m = 6p+3$, then $c+d = 1, 3, ..., 2p+1$, so there are $2+4+\cdots+2p+2 = (p+2)(p+1)$ solutions; when $m = 6p+4$, there are $(p+1)^2$ solutions; and finally when $m = 6p+5$, there are $(p+2)(p+1)$ solutions. But again this seems to be the wrong number.

By inspection we find that it seems we have

$$m(n) = f(n) - [(n+2)/3], \tag{89}$$

where $m(n)$ is the cohomology dimension for A_4 and $f(n)$ is that for S_4. Using $f(n) = [n/2] + [(n-3)/6] + 2 - \delta_{n\equiv 4 \text{ mod } 6}$, we should have

$$m(n) = [n/2] + [(n-3)/6] - [(n+2)/3] + 2 - \delta_{n\equiv 4 \text{ mod } 6}, \tag{90}$$

then we have

$$m(n = 0:30) = (1, 0, 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, 5, 4, 5, 6, 5, 6, 7, 6, 7, 8, 7, 8, 9, 8, 9, 10, 9, 10, 11), \tag{91}$$

which agrees with those listed in Table 3.

Conjecture: we have

$$H^n(F\bar{4}3m, \mathbb{Z}_2) = \mathbb{Z}_2^{4f(n)-4}, \quad n \geq 2, \quad H^{0,1}(F\bar{4}3m, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{92}$$

Prove: note that the $q = 1$ and $q = 2$ are calculating $H^*(S_4, \mathcal{A}_q)$, which have been worked out by GAP (see later). So this essentially proves that the E_2 page collapses.

Now the interesting question is how to obtain the ring structure. Since we know the ring structure for $q = 0, 3$, we only need the ring structure for $q = 1$ and $q = 2$.

22 $H^*(Fd\bar{3}m(No.227), \mathbb{Z}_2)$

We have $Fd\bar{3}m = \langle T_1, T_2, T_3, C_2, C_2', C_3, M, P\rangle$, where P is inversion. We have $PMP = M$, $PC_3P = C_3$, $PC_2P = T_3^{-1}C_2$, $PC_2'P = T_2^{-1}C_2'$, $PC_2C_2'P = T_3^{-1}C_2T_2^{-1}C_2' = T_1^{-1}C_2C_2'$, $PT_iP = T_i^{-1}$, $i = 1, 2, 3$. Recall that C_2' is two-fold rotation about the axis $(1/8, 0, 1/8) + \hat{y}$, while C_2 is two-fold rotation about the axis $(1/8, 1/8, 0) + \hat{z}$. Point group wise, C_2 is a permutation $(12)(34)$, while C_2' is a permutation $(13)(24)$, and C_3 is (123); $M$is mirror that switches x and y coordinates so M is (12).

Note: in Adem-Milgram the cohomology ring $\mathbb{F}_2[x, y, c]/(xc)$, where x, y, c have degree 1,2,3, respectively: in his convention, x lives in $V_1 = \langle(12), (34)\rangle$ and c lives in $V_2 = \langle(12)(34), (13)(24)\rangle$, and y reduces to both the degree-2 cohomology element of V_1 and V_2 when restricting to them. Corresponding to our notation, $V_1 = \langle M, MC_2\rangle$ while $V_2 = \langle C_2, C_2'\rangle$.

Recall that his y correspond to our β_3, so β_3 should also reduce reduces to an expression of purely c, c' (see Eq. (74)), hence our β_3 a

Elements of $g \in Fd\bar{3}m$ can be written in standard form

$$g = T_1^xT_2^yT_3^zC_2^cC_2'^{c'}C_3^rM^mP^p, \tag{93}$$

where $p, m, c, c' = 0, 1$ and $r = 0, 1, 2$, and $x, y, z \in \mathbb{Z}$.

$$\begin{aligned}
g_1g_2 &= T_1^{x_1}T_2^{y_1}T_3^{z_1}C_2^{c_1}C_2'^{c_1'}C_3^{t_1}M^{m_1}P^{p_1}T_1^{x_2}T_2^{y_2}T_3^{z_2}C_2^{c_2}C_2'^{c_2'}C_3^{t_2}M^{m_2}P^{p_2} \\
&= T_1^{x_1}T_2^{y_1}T_3^{z_1}C_2^{c_1}C_2'^{c_1'}C_3^{t_1}M^{m_1}T_1^{(1-2p_1)x_2-p_1c_2c_2'}T_2^{(1-2p_1)y_2-p_1c_2'(1-c_2)}T_3^{(1-2p_1)z_2-p_1c_2(1-c_2')}C_2^{c_2}C_2'^{c_2'}C_3^{t_2}M^{m_2}P^{p_1+p_2} \\
&= T_1^{x_1+(1-c_1)((1-c_1')(r_2[m_1,p_1])_{2t_1+1}+c_1'(r_2[m_1,p_1])_{2t_1+3})+c_1(-c_1'(r_2[m_1,p_1])_{2t_1+1}-c_1'(r_2[m_1,p_1])_{2t_1+2}+(1-c_1')(r_2[m_1,p_1])_{2t_1+2}-c_1'(r_2[m_1,p_1])_{2t_1+3})} \\
&\quad \cdot T_2^{y_1+c_1((1-c_1')(r_2[m_1,p_1])_{2t_1+1}+c_1'(r_2[m_1,p_1])_{2t_1+3})+(1-c_1)(-c_1'(r_2[m_1,p_1])_{2t_1+1}-c_1'(r_2[m_1,p_1])_{2t_1+2}+(1-c_1')(r_2[m_1,p_1])_{2t_1+2}-c_1'(r_2[m_1,p_1])_{2t_1+3})} \\
&\quad \cdot T_3^{z_1-c_1((1-c_1')(r_2[m_1,p_1])_{2t_1+1}+c_1'(r_2[m_1,p_1])_{2t_1+3})-c_1(-c_1'(r_2[m_1,p_1])_{2t_1+1}-c_1'(r_2[m_1,p_1])_{2t_1+2}+(1-c_1')(r_2[m_1,p_1])_{2t_1+2}-c_1'(r_2[m_1,p_1])_{2t_1+3})+(1-} \\
&\quad \cdot C_2^{c_1+(s_2[m_1])_{2t_1+1}}C_2'^{c_1'+(s_2[m_1])_{2t_1+2}}C_3^{t_1+(1+m_1)t_2}M^{m_1+m_2}P^{p_1+p_2},
\end{aligned} \tag{94}$$

where based on the previous definition $r_i[m] = (x_i + m(y_i - x_i), y_i + m(x_i - y_i), z_i)$, we know have

$$r_i[m,p] = ((1-2p)x_i-pc_ic_i'+m((1-2p)y_i-p_1c_i'(1-c_i)-(1-2p)x_i+pc_ic_i'), (1-2p)y_i-pc_i'(1-c_i)+m((1-2p)x_i-pc_ic_i'-(1-2p)y_i+p_1c_i'(1-$$

$$s_i[m] = (c_i + mc_i', c_i', c_i + (1+m)c_i'), \quad c_i, c_i', m, p \in \mathbb{Z}_2.$$

For $g = (x, y, z, c, c', t) \in F\bar{4}3m$, we have

$$PgP = P(x,y,z,c,c',t)P = (-cc'-x, -(1-c)c'-y, -c(1-c')-z, c, c', t, m),$$

using this rule, we can actually check that, the cocycle of $F\bar{4}3m$, β_3, is stabilized by P:

$$\beta_3^{F\bar{4}3m}(Pg_1P, Pg_2P) - \beta_3^{F\bar{4}3m}(g_1,g_2) = 0. \tag{95}$$

Also, we checked that for the cocycle β_1 of $F\bar{4}3m$ in Eq. (85), we have

$$\beta_3^{F\bar{4}3m}(Pg_1P, Pg_2P) - \beta_3^{F\bar{4}3m}(g_1,g_2) \overset{??}{=} \mu_7(g_1) + \mu_7(g_2) - \mu_7(g_1g_2), \tag{96}$$

It might be tempting to restrict to $t = m = 0$ and conclude that $\mu_7(g)|_{t=m=0} = (x+y)c+x$; similarly we can restrict to $t = x = y = z = 0$ and obtain $\mu_7(g)|_{t=x=y=z=0} = c'(1-m)+c$, but we notice that these two are not compatible. In fact the former will leave no solution of the latter. So the situation does seem a bit tricky.

We do a more detailed analysis. First, notice that

$$(\beta_3^{F\bar{4}3m}(Pg_1P, Pg_2P) - \beta_3^{F\bar{4}3m}(g_1,g_2))|_{t_1=t_2=0} = (c_2+m_2+c_2m_2)(x_1+y_1) + cp1(x_2+m_2(1+x_1+x_2+y_1+y_2)+z_2),$$

restricting to a $\mu(g)$ that contains up to linear x, y, z, we find the solution is

$$\mu_6(g) = c + c'm + cx + mx + c'mx + cy + my + c'my + C_1m + C_2c' + C_3y + (1-C_3)x$$

such that

$$(\beta_3^{F\bar{4}3m}(Pg_1P, Pg_2P) - \beta_3^{F\bar{4}3m}(g_1,g_2)P|_{t=0} = \mu_6(g_1) + \mu_6(g_2) - \mu_6(g_1g_2). \tag{97}$$

The complete coboundary: in turns out that we find: setting

$$\mu_8(g) = c + c' + c'm + x + cx + mx + c'mx + cy + my + c'my, \tag{98}$$

then we have

$$(\beta_3^{F\bar{4}3m}(Pg_1P, Pg_2P) - \beta_3^{F\bar{4}3m}(g_1,g_2)P = \mu_8(g_1) + \mu_8(g_2) - \mu_8(g_1g_2), \quad \forall g_1, g_2 \in F\bar{4}3m, \tag{99}$$

and this shows unambiguously that β_3 is inherited by $H^2(Fd\bar{3}m, \mathbb{Z}_2)$.

Therefore

$$P\colon (\chi_1,\chi_2,\chi_3,\chi_c,\chi_{c'},\chi_m) \mapsto (\chi_1, \chi_2+\chi_{c'}, \chi_3+\chi_c, \chi_c, \chi_{c'}, \chi_m).$$

GAP gives

$$H^{1:5}(Fd\bar{3}m, \mathbb{Z}_2) = (2,5,9,11,15).$$

Update: new version of GAP, combined with Polymake, gives

$$H^{0:15}(Fd\bar{3}m, \mathbb{Z}_2) = (1,2,5,9,11,15,19,21,25,29,31,35,39,41,45,49), \tag{100}$$

the command is

```
R:=ResolutionSpaceGroup(SpaceGroupIT(3,227),15);
for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15] do
Print("H^",n,"=",Cohomology(HomToIntegersModP(R,2),n),"\n");
od;
```

Compared with $H^*(F\bar{4}3m, \mathbb{Z}_2)$, we see that the difference is 1, 1, 1, 3, 3, 3, 5, 5, 5, 7, 7, 7....

So $P\colon (a_1, a_2) \to (a_1+y, a_2+x)$. We have $a_1^2 + a_1a_2 + a_2^2 \to a_1^2 + a_1a_2 + a_2^2 + x^2 + xy + y^2$, (we used the relation $a_1x + a_2y = 0$). Therefore

$$n = 1\colon \langle m; p\rangle \cong \mathbb{Z}_2^2,$$

$$n = 2\colon \langle x_1^2 + xy + y^2, a_2x, m^2; mp; p^2\rangle \cong \mathbb{Z}_2^5,$$

note $a_1^2 + a_1a_2 + a_2^2$ is modded out in the line of $q = 2$.

Now we look at the $(p,q) = (0,3)$ place: the stabilizer of P is $(m^3, m(x^2+xy+y^2), x^2y + xy^2, c_1, c_2)$ more???

$$
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\le 8 & & & & & & & \cdots \\
\le 8 & & & & & & & \cdots \\
\mathbf{3}^{\le \mathbf{4}} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \cdots \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots
\end{array}
$$

$$H^*(Fd\bar{3}m,\mathbb{Z}_2)=\mathbb{F}_2[a_1,a_2,x,y,m,p,(c_1,c_2)]/R,$$
$$R=(a_1^2=a_1p=a_2^2=a_2p,a_1x=a_2y,a_1m=a_2m=xm=ym=0,(\text{third or higher degree relations}))$$
$$H^1(Fd\bar{3}m,\mathbb{Z}_2)=\langle m;p\rangle,$$
$$H^2(Fd\bar{3}m,\mathbb{Z}_2)=\langle x^2+xy+y^2,a_2x,m^2;mp,p^2\rangle$$

$$H^*(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{F}_2[x_1,y_2,c_3]/(x_1c_3)\otimes\mathbb{F}_2[x_2]=\mathbb{F}_2[x_1,x_2,y_2,c_3]/(x_1c_3),\tag{101}$$

so we have $y_2\mapsto x^2+xy+y^2$, $x_1\mapsto m$, $x_2\mapsto p$,

We have

$$H^{0:9}(S_4,\mathbb{Z}_2)=(1,1,2,3,3,4,5,5,6,7),\tag{102}$$

We see that the cohomology dimension for $H^n(S_4,\mathbb{Z}_2)$ is equal to $a+2b+3c=n$ with the constraint that when $c\ge 1$, then $a=0$. If $c=0$, then there are $[n/2]+1$ solutions, where [] means taking the integer equal or below. When $c\ge 1$, then $a=0$, so we need to find the number of solutions of $2b+3c'=n-3$. If $n-3=6m$, then total number of solutions is $m+1$; if $n-3=6m+1$, then m; if $n-3=6m+2$, then $m+1$; if $n-3=6m+3$, then $m+1$; if $n-3=6m+4$, then $m+1$; if $n-3=6m+5$, then $m+1$. So the total number of solution for $2b+3c'=n-3$ is $[(n-3)/6]+1-\delta_{n\equiv 4 \bmod 6}$. So we have

$$\dim(H^n(S_4,\mathbb{Z}_2))=f(n)\equiv[n/2]+[(n-3)/6]+2-\delta_{n\equiv 4 \bmod 6},\quad n=0,1,2,....\tag{103}$$

Then using Kunneth formula, we see that

$$\dim(H^n(S_4\times\mathbb{Z}_2,\mathbb{Z}_2))=\sum_{i=0}^{n}f(i).\tag{104}$$

This agrees with the GAP calculation

$$H^{0:3}(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)=(1,2,4,7).\tag{105}$$

using our formula, we have

$$H^{0:10}(S_4\times\mathbb{Z}_2,\mathbb{Z}_2)=(1,2,4,7,10,14,19,24,30,37,44).\tag{106}$$

At degree two the four generators are x_1^2,x_2^2,x_1x_2,y_2.

Next step: directly solve for the four H^2 element of $H^2(F\bar{4}3m,\mathbb{Z}_2)$ and see which one is killed by the inversion P of $Fd\bar{3}m$. Write the standard group multiplication rule for 216 and 227.

22.1 The explicit 1-, 2-, and 3-cocycles and ring structure

There are two 1-cocycles, which are

$$\alpha_1^{Fd\bar{3}m}(g)=m,\qquad \alpha_2^{Fd\bar{3}m}(g)=p,\tag{107}$$

note that this agrees with Adem-Milgram: the claim is that $H^*(S_4,\mathbb{Z}_2)=\mathbb{F}_2[x,y,c]/(xc)$, where x,y,c have degree 1,2,3, and that x is (i.e. reduces to) from $V_1=\langle(12),(34)\rangle$ and c is from $V_2=\langle(12)(34),(13)(24)\rangle$, while y reduces both the degree-2 elements of V_1 and V_2 when restricting to V_1 and V_2 respectively. Note they match with what we have: recall that C_2 and C_2' are two fold rotation along $(1/8,1/8,0)+\hat{z}$ and $(1/8,0,1/8)+\hat{y}$, respectively, and correspond to the permutation (12)(34) and (13)(24), respectively, while the mirror M switches x and y and correspond to (12). The 1-cocycle $\alpha_1^{Fd\bar{3}m}=m$ is exactly x, and the $\beta_3^{Fd\bar{3}m}(g_1,g_2)$ introduced below correspond to y. It's an interesting question to look for c, which we will address below.

Note that α_1 is x; we will call α_2 x_1; $y=\beta_3^{Fd\bar{3}m}$ and c are defiend as above; Further, we will call the translation 2-cocycle $\beta=\beta_1^{Fd\bar{3}m}$. We have

$$n=1\colon\langle x,x_1\rangle\tag{108}$$
$$n=2\colon\langle\beta,y,x^2,xx_1,x_1^2\rangle\tag{109}$$

$$n=3\colon \langle x\beta, x_1\beta, xy, x^3, x^2x_1, x_1^2x, x_1^3, c, \tau\rangle, \qquad x_1y=0. \tag{110}$$

We conjecture that τ is the only element in $E_2^{2,1}$ that survives the d_2 map, and that $E_\infty^{0,3}=0$. To be verified.

Below we give invariants for the 3-cocycles:

$$\sigma^3\colon \qquad \Omega(M,M,M)=-1, \tag{111a}$$

$$\iota^3\colon \qquad \Omega(P,P,P)=-1, \tag{111b}$$

$$\sigma^3,\sigma^2\iota,\sigma\iota^2,\iota^3\colon \qquad \Omega(MP,MP,MP)=-1, \tag{111c}$$

$$\sigma^3,\sigma\iota^2\colon \qquad \Omega(M,MP,MP)\Omega(MP,M,MP)\Omega(MP,MP,M)=-1, \tag{111d}$$

$$\sigma\delta\colon \qquad \Omega(M,C_2,C_2)\Omega(C_2,M,C_2)\Omega(C_2,C_2,M)=-1, \tag{111e}$$

$$\psi\colon \qquad \Omega(C_2,C_2',C_2')\Omega(C_2',C_2,C_2')\Omega(C_2',C_2',C_2)=-1, \tag{111f}$$

$$\iota^3,\beta\iota\colon \qquad \Omega(T_1T_2T_3P,T_1T_2T_3P,T_1T_2T_3P)=-1, \tag{111g}$$

$$\begin{aligned}\beta\sigma\colon \qquad & \Omega(M,T_3,T_1)\Omega(M,T_1T_2^{-1}T_3,T_1)\Omega(M,T_1,T_3)\Omega(M,T_1,T_1T_2^{-1}T_3)\cdot\\ & \Omega(T_3,T_3M,T_1)\Omega(T_3,T_1,T_3M)\Omega(T_3,T_1M,T_3)\Omega(T_3,T_1M,T_1T_2^{-1}T_3)\cdot\\ & \Omega(T_1T_2^{-1}T_3,T_1^{-1}T_2T_3M,T_1)\Omega(T_1T_2^{-1}T_3,T_2M,T_3)\cdot\\ & \Omega(T_1T_2^{-1}T_3,T_2M,T_3)\Omega(T_1T_2^{-1}T_3,T_1,T_1^{-1}T_2T_3M)\cdot\\ & \Omega(T_1,M,T_3)\Omega(T_1,M,T_1T_2^{-1}T_3)\Omega(T_1,T_3,T_3M)\Omega(T_1,T_1T_2^{-1}T_3,T_1^{-1}T_2T_3M)=-1,\end{aligned} \tag{111h}$$

$$\tau\colon \qquad \Omega(T_3C_2,T_2C_2',T_2C_2')\Omega(T_2C_2',T_3C_2,T_2C_2')\Omega(T_2C_2',T_2C_2',T_3C_2)=-1. \tag{111i}$$

$$
\begin{aligned}
\beta_1^{Fd\bar{3}m}(g_1,g_2)|_{t_1=0} =&(x_1+y_1)(c_2+x_2+z_2)+m_2\left((x_1+y_1)(c_1'+c_2'+x_2+y_2)+\frac{1}{2}(x_1+y_1)(x_1+y_1+1)\right)+\\
&+p_1(c_2x_1+c_2x_2+c_2y_1+c_2y_2+c_2+c_2'm_2x_1+c_2'm_2x_2+c_2'm_2y_1+c_2'm_2y_2+\\
&+c_2'm_2+c_2'+m_2x_2+m_2y_2+x_2)\\
\beta_1^{Fd\bar{3}m}(g_1,g_2)|_{t_1=1} =&c_2m_1m_2x_2+c_2m_1m_2y_2+c_2m_1x_1+c_2m_1x_2+c_2m_1y_1+c_2m_1z_2+c_2m_2x_1+c_2m_2x_2+\\
&+c_2m_2y_1+c_2m_2z_2+c_2x_1+c_2y_1+c_2y_2+c_2z_2+c_1'm_2x_1+c_1'm_2y_1+c_2'm_1m_2x_1+\\
&+c_2'm_1m_2y_1+c_2'm_1m_2y_2+c_2'm_1m_2z_2+c_2'm_1x_2+c_2'm_1y_2+c_2'm_2x_2+c_2'm_2y_2+\\
&+c_2'x_1+c_2'x_2+c_2'y_1+c_2'z_2+m_1m_2x_1x_2+m_1m_2x_1y_2+m_1m_2x_2y_1+m_1m_2x_2z_2+\\
&+\frac{1}{2}m_1m_2x_2(x_2+1)+m_1m_2y_1y_2+m_1m_2y_2z_2+\frac{1}{2}m_1m_2y_2(y_2+1)+m_1x_1x_2+m_1x_1z_2+\\
&+m_1x_2y_1+m_1x_2y_2+m_1x_2z_2+m_1y_1z_2+\frac{1}{2}m_1y_2(y_2+1)+\frac{1}{2}m_1z_2(z_2+1)+m_2x_1x_2+\\
&+m_2x_1y_1+m_2x_1z_2+\frac{1}{2}m_2(x_1+1)x_1+m_2x_2y_1+m_2x_2y_2+m_2x_2z_2+m_2y_1z_2+\frac{1}{2}m_2y_1(y_1+1)+\\
&+\frac{1}{2}m_2y_2(y_2+1)+\frac{1}{2}m_2z_2(z_2+1)+x_1y_2+x_1z_2+x_2y_2+\frac{1}{2}x_2(x_2+1)+y_1y_2+y_1z_2+y_2z_2+\\
&+\frac{1}{2}z_2(z_2+1)+p_1(c_2m_1m_2x_2+c_2m_1m_2y_2+c_2m_1x_1+c_2m_1x_2+c_2m_1y_1+c_2m_1z_2+c_2m_1+\\
&+c_2m_2x_1+c_2m_2x_2+c_2m_2y_1+c_2m_2z_2+c_2m_2+c_2x_1+c_2x_2+c_2y_1+c_2z_2+c_2'm_1m_2x_1+c_2'm_1m_2y_1+\\
&+c_2'm_1m_2y_2+c_2'm_1m_2z_2+c_2'm_1m_2+c_2'm_1x_2+c_2'm_1y_2+c_2'm_1+c_2'x_1+c_2'x_2+c_2'y_1+c_2'z_2+c_2'+\\
&+m_1m_2x_2+m_1m_2y_2+m_1y_2+m_1z_2+m_2x_2+m_2z_2+z_2)\\
\beta_1^{Fd\bar{3}m}(g_1,g_2)|_{t_1=2} =&c_2m_1m_2x_2+c_2m_1m_2y_2+c_2m_1x_1+c_2m_1x_2+c_2m_1y_1+c_2m_1z_2+c_2m_2x_1+c_2m_2y_1+\\
&+c_2m_2y_2+c_2m_2z_2+c_2x_2+c_2y_2+c_1'm_2x_1+c_1'm_2y_1+c_2'm_1m_2x_1+c_2'm_1m_2y_1+c_2'm_1m_2y_2+\\
&+c_2'm_1m_2z_2+c_2'm_1x_2+c_2'm_1y_2+c_2'm_2x_1+c_2'm_2x_2+c_2'm_2y_1+c_2'm_2z_2+c_2'x_1+c_2'y_1+c_2'y_2+\\
&+c_2'z_2+m_1m_2x_1x_2+m_1m_2x_1y_2+m_1m_2x_2y_1+m_1m_2x_2z_2+\frac{1}{2}m_1m_2x_2(x_2+1)+m_1m_2y_1y_2+\\
&+m_1m_2y_2z_2+\frac{1}{2}m_1m_2y_2(y_2+1)+m_1x_1x_2+m_1x_1z_2+m_1x_2y_1+m_1x_2y_2+m_1x_2z_2+m_1y_1z_2+\\
&+\frac{1}{2}m_1y_2(y_2+1)+\frac{1}{2}m_1z_2(z_2+1)+m_2x_1y_1+m_2x_1y_2+m_2x_1z_2+\frac{1}{2}m_2(x_1+1)x_1+m_2x_2y_2+\\
&+\frac{1}{2}m_2x_2(x_2+1)+m_2y_1y_2+m_2y_1z_2+\frac{1}{2}m_2y_1(y_1+1)+m_2y_2z_2+\frac{1}{2}m_2z_2(z_2+1)+x_1x_2+\\
&+x_1y_2+x_2y_1+x_2z_2+\frac{1}{2}x_2(x_2+1)+y_1y_2+y_2z_2+\frac{1}{2}y_2(y_2+1)+p_1(c_2m_1m_2x_2+\\
&+c_2m_1m_2y_2+c_2m_1x_1+c_2m_1x_2+c_2m_1y_1+c_2m_1z_2+c_2m_1+c_2m_2x_1+c_2m_2y_1+c_2m_2y_2+c_2m_2z_2+\\
&+c_2m_2+c_2+c_2'm_1m_2x_1+c_2'm_1m_2y_1+c_2'm_1m_2y_2+c_2'm_1m_2z_2+c_2'm_1m_2+c_2'm_1x_2+c_2'm_1y_2+c_2'm_1+\\
&+c_2'm_2x_1+c_2'm_2y_1+c_2'm_2y_2+c_2'm_2z_2+c_2'm_2+c_2'x_1+c_2'y_1+c_2'y_2+c_2'z_2+m_1m_2x_2+m_1m_2y_2+\\
&+m_1y_2+m_1z_2+m_2y_2+m_2z_2+y_2),
\end{aligned}
\tag{112}
$$

$$
\beta_3^{Fd\bar{3}m}(g_1,g_2)=\begin{cases}
c_1c_2+c_1c_2'+c_1'c_2'+m_1(c_2+c_1c_2'+c_2c_2'), & (t_1,t_2)=(0,0)\\
c_1c_2+c_1c_2'+c_1'c_2'+m_1(c_2+c_1c_2'+c_2c_2'), & (t_1,t_2)=(0,1)\\
c_1c_2+c_1c_2'+c_1'c_2'+m_1(c_2+c_1c_2'+c_2c_2'), & (t_1,t_2)=(0,2)\\
c_2+c_2c_1'+c_1c_2'+m_1(c_2+c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(1,0)\\
1+c_2+c_2c_1'+c_1c_2'+m_1(1+c_2+c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(1,1)\\
c_2+c_2c_1'+c_1c_2'+m_1(1+c_2+c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(1,2)\\
c_1c_2+c_2c_1'+c_2'+c_1'c_2'+m_1(c_2+c_1c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(2,0)\\
c_1c_2+c_2c_1'+c_2'+c_1'c_2'+m_1(1+c_2+c_1c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(2,1)\\
1+c_1c_2+c_2c_1'+c_2'+c_1'c_2'+m_1(1+c_2+c_1c_2'+c_2c_2'+c_1'c_2'), & (t_1,t_2)=(0,0)
\end{cases}
\tag{113}
$$

$$\beta_2^{Fd\bar{3}m}(g_1,g_2)=m_1m_2 \tag{114}$$

$$\beta_4^{Fd\bar{3}m}(g_1,g_2)=m_1p_2 \tag{115}$$

$$\beta_5^{Fd\bar{3}m}(g_1,g_2) = p_1p_2 \tag{116}$$

The 3-cocycles: Corresponding to the cohomology of O_h, which is $\mathbb{F}_2[x,x_1,y,c]/(xc)$, at degree-3 there are 7 elements, which are $x^3, x^2x_1, xx_1^2, x_1^3, xy, x_1y, c$; out of them, x_1y is killed by the d_2 differential, so we are left with $x^3, x^2x_1, xx_1^2, x_1^3, xy, c$, which we will label by $\gamma_{1,2,3,4,5,6}$. we have

$$\gamma_1^{Fd\bar{3}m}(g_1,g_2,g_3) = m_1m_2m_3, \tag{117}$$

$$\gamma_2^{Fd\bar{3}m}(g_1,g_2,g_3) = m_1m_2p_3, \tag{118}$$

$$\gamma_3^{Fd\bar{3}m}(g_1,g_2,g_3) = m_1p_2p_3, \tag{119}$$

$$\gamma_4^{Fd\bar{3}m}(g_1,g_2,g_3) = p_1p_2p_3, \tag{120}$$

$$\gamma_5^{Fd\bar{3}m}(g_1,g_2,g_3) = m_1\beta_3^{Fd\bar{3}m}(g_2,g_3), \tag{121}$$

Note using the resolution in GAP we find that the following two are also genuine cocycles:

$$\gamma_6^{Fd\bar{3}m}(g_1,g_2,g_3) = m_1\beta_1^{Fd\bar{3}m}(g_2,g_3), \tag{122}$$

$$\gamma_7^{Fd\bar{3}m}(g_1,g_2,g_3) = p_1\beta_1^{Fd\bar{3}m}(g_2,g_3), \tag{123}$$

where recall that $\beta_1^{Fd\bar{3}m}(g_2,g_3)$ is the one that reduces to the translation 2-cocycle.

The second-last one, corresponding to c, is

$$\begin{aligned}
\gamma_8^{Fd\bar{3}m}(g_1,g_2,g_3) =& \delta_{(t_1,t_2,t_3)=(0,0,0)}\big(c_1c_3c_2' + c_1c_2'c_3' + c_3c_1'c_2'm_1 + c_1'c_2'c_3'm_1 + c_1c_3m_2 + c_3c_1'm_2 + c_3c_1'c_2'm_2 + c_1c_3'm_2 + c_1'c_3'm_2 + c_1' \\
&+ \delta_{(t_1,t_2,t_3)=(0,0,1)}\big(c_1c_2 + c_2c_1' + c_1c_3c_2' + c_1'c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_3c_1'c_2' + c_2m_1 + c_1c_2m_1 + c_2c_3m_1 + c_1c_2c_1'm_1 \\
&+ \delta_{(t_1,t_2,t_3)=(0,0,2)}\big(c_1c_2 + c_2c_1' + c_1c_2' + c_1c_2c_2' + c_1'c_2' + c_1c_2c_3' + c_1c_2'c_3' + c_2c_1'm_1 + c_1'c_2'm_1 + c_2c_1'c_2'm_1 + c_2c_1'c_3' \\
&+ \delta_{(t_1,t_2,t_3)=(0,1,0)}\big(c_1c_3c_2' + c_3c_1'c_2' + c_1c_2'c_3' + c_1'c_2'c_3' + c_2c_3m_1 + c_2c_3c_1'm_1 + c_1c_3c_2'm_1 + c_2c_3c_2'm_1 + c_2c_3'm_1 + c_2 \\
&+ \delta_{(t_1,t_2,t_3)=(0,1,1)}\big(c_1 + c_1c_3 + c_1c_2c_3 + c_1' + c_1c_1' + c_2c_1' + c_3c_1' + c_1c_3c_1' + c_2c_3c_1' + c_1c_2c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_2 \\
&+ \delta_{(t_1,t_2,t_3)=(0,1,2)}\big(c_1c_2 + c_1c_1' + c_1c_2' + c_1'c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_1c_1'c_3' + c_2c_1'c_3' + c_1m_1 + c_1c_2m_1 + c_1'm_1 + c_1c_1' \\
&+ \delta_{(t_1,t_2,t_3)=(0,2,0)}\big(c_1c_3 + c_1c_2c_3 + c_3c_1' + c_1c_3c_1' + c_2c_3c_1' + c_1c_3' + c_1c_2c_3' + c_1'c_3' + c_1c_1'c_3' + c_2c_1'c_3' + c_2c_3c_1'm_1 + \\
&+ \delta_{(t_1,t_2,t_3)=(0,2,1)}\big(c_1 + c_1c_2 + c_1' + c_1c_1' + c_1c_2c_2' + c_1c_3c_2' + c_1'c_2' + c_2c_1'm_1 + c_2c_1'c_2'm_1 + c_3c_1'c_2'm_1 + c_1m_2 + c_1 \\
&+ \delta_{(t_1,t_2,t_3)=(0,2,2)}\big(c_1c_1' + c_2c_1' + c_1c_2c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_1c_3' + c_1'c_3' + c_1c_2'c_3' + c_1'c_2'c_3' + c_1m_1 + c_1c_2m_1 + c_1' \\
&+ \delta_{(t_1,t_2,t_3)=(1,0,0)}\big(c_1c_2c_3 + c_2c_3c_1' + c_3c_2' + c_2c_3c_2' + c_1c_2c_3' + c_2c_1'c_3' + c_2'c_3' + c_2c_2'c_3' + c_1c_2c_3m_1 + c_2c_3c_1'm_1 + c_1 \\
&+ \delta_{(t_1,t_2,t_3)=(1,0,1)}\big(c_2 + c_1c_2c_3 + c_2c_1' + c_2c_3c_1' + c_2c_2' + c_1c_2c_2' + c_3c_2' + c_1c_3c_2' + c_2c_3c_2' + c_1'c_2' + c_2c_1'c_2' + c_3c_1'c_2' + \\
&+ \delta_{(t_1,t_2,t_3)=(1,0,2)}\big(c_2 + c_1c_2c_1' + c_2' + c_1c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_2c_3' + c_2c_1'c_3' + c_2'c_3' + c_1'c_2'c_3' + c_1c_2m_1 + c_1c_2c_1'm \\
&+ \delta_{(t_1,t_2,t_3)=(1,1,0)}\big(c_1c_2c_3 + c_2c_3c_1' + c_3c_2' + c_1c_3c_2' + c_2c_3c_2' + c_3c_1'c_2' + c_1c_2c_3' + c_2c_1'c_3' + c_2'c_3' + c_1c_2'c_3' + c_2c_2'c_3' + \\
&+ \delta_{(t_1,t_2,t_3)=(1,1,1)}\big(1 + c_1c_2 + c_3 + c_2c_3 + c_1c_2c_3 + c_1c_1' + c_1c_2c_1' + c_1c_3c_1' + c_1c_2' + c_2c_2' + c_1c_2c_2' + c_1c_1'c_2' + c_1c_2 \\
&+ \delta_{(t_1,t_2,t_3)=(1,1,2)}\big(c_1 + c_2 + c_1' + c_2c_1' + c_2' + c_1c_2c_2' + c_1'c_2' + c_2c_1'c_2' + c_1c_3' + c_1c_2c_3' + c_1'c_3' + c_2c_1'c_3' + m_1 + c_2m_1 \\
&+ \delta_{(t_1,t_2,t_3)=(1,2,0)}\big(c_3 + c_2c_3 + c_1c_2c_3 + c_1c_3c_1' + c_3' + c_2c_3' + c_1c_2c_3' + c_1c_1'c_3' + c_1c_2c_3m_1 + c_2c_3c_1'm_1 + c_1c_2c_3'm_1 \\
&+ \delta_{(t_1,t_2,t_3)=(1,2,1)}\big(1 + c_2 + c_1c_2 + c_1c_2c_3 + c_1c_1' + c_2c_1' + c_1c_2c_1' + c_2c_3c_1' + c_1c_2' + c_3c_2' + c_2c_3c_2' + c_1'c_2' + c_1c_1'c_2' + \\
&+ \delta_{(t_1,t_2,t_3)=(1,2,2)}\big(c_1 + c_1' + c_2c_1' + c_1c_2c_1' + c_1c_2' + c_1c_2c_2' + c_1'c_2' + c_1c_1'c_2' + c_3' + c_1c_3' + c_1'c_3' + c_1c_1'c_3' + c_2'c_3' + c_1 \\
&+ \delta_{(t_1,t_2,t_3)=(2,0,0)}\big(c_3c_2' + c_1c_3c_2' + c_3c_1'c_2' + c_2'c_3' + c_1c_2'c_3' + c_1'c_2'c_3' + c_3c_2'm_1 + c_1c_3c_2'm_1 + c_2'c_3'm_1 + c_1c_2'c_3'm_1 + \\
&+ \delta_{(t_1,t_2,t_3)=(2,0,1)}\big(c_2 + c_1c_2 + c_2c_3 + c_1c_2c_3 + c_1c_2c_1' + c_1c_2' + c_1c_2c_2' + c_1c_3c_2' + c_2c_3c_2' + c_1c_1'c_2' + c_1c_2m_1 + c_2c_3 \\
&+ \delta_{(t_1,t_2,t_3)=(2,0,2)}\big(c_2 + c_1c_2 + c_2c_1' + c_1c_2' + c_1c_2c_2' + c_2c_1'c_2' + c_2c_3' + c_1c_2c_3' + c_2c_1'c_3' + c_2'c_3' + c_1c_2'c_3' + c_1'c_2'c_3' + \\
&+ \delta_{(t_1,t_2,t_3)=(2,1,0)}\big(c_2c_3 + c_1c_2c_3 + c_1c_3c_2' + c_2c_3c_2' + c_2c_3' + c_1c_2c_3' + c_1c_2'c_3' + c_2c_2'c_3' + c_2c_3m_1 + c_3c_2'm_1 + c_2c_3'm \\
&+ \delta_{(t_1,t_2,t_3)=(2,1,1)}\big(1 + c_1 + c_3 + c_1c_3 + c_2c_3 + c_1c_2c_3 + c_1' + c_2c_1' + c_1c_2c_1' + c_3c_1' + c_2c_3c_1' + c_2' + c_2c_2' + c_1c_3c_2' + \\
&+ \delta_{(t_1,t_2,t_3)=(2,1,2)}\big(1 + c_1 + c_1' + c_1c_1' + c_1c_2c_1' + c_2' + c_1c_2c_2' + c_1'c_2' + c_1c_1'c_2' + c_3' + c_1c_3' + c_2c_3' + c_1c_2c_3' + c_1'c_3' + \\
&+ \delta_{(t_1,t_2,t_3)=(2,2,0)}\big(c_3 + c_1c_3 + c_2c_3 + c_1c_2c_3 + c_3c_1' + c_2c_3c_1' + c_1c_3c_2' + c_2c_3c_2' + c_3c_1'c_2' + c_3' + c_1c_3' + c_2c_3' + c_1c_2 \\
&+ \delta_{(t_1,t_2,t_3)=(2,2,1)}\big(1 + c_1 + c_2 + c_1c_2 + c_1' + c_2c_1' + c_2' + c_1c_2' + c_2c_2' + c_1c_2c_2' + c_3c_2' + c_1c_3c_2' + c_2c_1'c_2' + c_3c_1'c_2' + \\
&+ \delta_{(t_1,t_2,t_3)=(2,2,2)}\big(1 + c_1 + c_2 + c_1c_2 + c_1' + c_1c_1' + c_2c_1' + c_1c_2c_1' + c_2' + c_1c_2' + c_1'c_2' + c_1c_1'c_2' + c_2c_1'c_2' + c_1c_1'c_3' +
\end{aligned} \tag{124}$$

We define

$$C_{\psi}(g_1,g_2,g_3)|_{T_d} = C_{\psi'}(g_1,g_2,g_3)|_{T_d} + C_{s\psi}(g_1,g_2,g_3)|_{T_d}, \tag{125}$$

where

$C_{\psi'}(g_1,g_2,g_3)|_{T_d} = \delta_{(t_1,t_2,t_3)=(0,0,0)}\big(c_1c_2c_3 + c_1c_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,0,1)}\big(c_1c_2c_3 + c_2c'_1 + c'_1c'_2 + c_1c'_1c'_2 + c_1c_2c'_3 + c_2c'_1c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,0,2)}\big(c_1c_2 + c_1c_2c_3 + c_1c'_2 + c_1c'_1c'_2 + c_1c_2c'_3 + c_2c'_1c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,1,0)}\big(c_3c'_1 + c_1c_3c'_1 + c_3c'_1c'_2 + c_1c'_3 + c_1c_2c'_3 + c'_1c'_3 + c_1c'_1c'_3 + c_2c'_1c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,1,1)}\big(c_1c_2 + c_1c'_1 + c_3c'_1 + c_1c_3c'_1 + c'_1c'_2 + c_3c'_1c'_2 + c_1c_2c'_3 + c'_1c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,1,2)}\big(c'_1 + c_2c'_1 + c'_1c'_2 + c_1c'_1c'_2 + c_3c'_1c'_2 + c_1c_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,2,0)}\big(c_1c_2c_3 + c_2c_3c'_1 + c_1c_3c'_2 + c_3c'_1c'_2 + c_2c'_1c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,2,1)}\big(c_1 + c_1c_2 + c_1c_2c_3 + c_1c_3c'_1 + c_2c_3c'_1 + c_1c'_2 + c_1c_3c'_2 + c_1c'_1c'_2 + c_3c'_1c'_2 + c_1c'_3 + c'_1c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(0,2,2)}\big(c_1 + c_1c_2 + c_1c_3 + c_1c_2c_3 + c'_1 + c_1c'_1 + c_2c_3c'_1 + c_1c_3c'_2 + c'_1c'_2 + c_3c'_1c'_2 + c'_1c'_3 + c_1c'_1c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,0,0)}\big(c_2c_3 + c_2c_3c'_1 + c_2c'_3 + c_2c'_1c'_3 + c'_2c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,0,1)}\big(c_2 + c_1c_2 + c_2c_3 + c_2c'_1 + c_2c_3c'_1 + c_1c'_1c'_2 + c_1c_2c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,0,2)}\big(c_2 + c_2c_3 + c_2c'_1 + c_2c_3c'_1 + c_1c'_2 + c_1c'_1c'_2 + c_1c_2c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,1,0)}\big(c_1c_3c'_1 + c_3c'_2 + c_1c_3c'_2 + c_3c'_1c'_2 + c'_3 + c_1c_2c'_3 + c'_1c'_3 + c_1c'_1c'_3 + c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,1,1)}\big(1 + c_1 + c_2 + c'_1 + c_1c'_1 + c_2c'_1 + c_1c_3c'_1 + c'_2 + c_1c'_2 + c_3c'_2 + c_1c_3c'_2 + c'_1c'_2 + c_3c'_1c'_2 + c'_3 + c_1c'_3 + c_2c'_3 + c'_1c'_3 + c_2c'_1c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,1,2)}\big(1 + c_1 + c_2 + c_1c_2 + c'_1 + c_2c'_1 + c_3c'_2 + c_1c_3c'_2 + c_1c'_1c'_2 + c_3c'_1c'_2 + c_2c'_3 + c_2c'_1c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,2,0)}\big(c_1c_2c_3 + c_1c_3c'_2 + c_2c'_3 + c_1c_2c'_3 + c_2c'_1c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,2,1)}\big(1 + c_2 + c_3 + c_1c_3 + c_1c_2c_3 + c'_1 + c_2c'_1 + c_3c'_1 + c_1c_3c'_1 + c_1c'_2 + c_1c_3c'_2 + c_1c'_1c'_2 + c_1c'_3 + c'_2c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(1,2,2)}\big(1 + c_2 + c_3 + c_1c_2c_3 + c'_1 + c_1c'_1 + c_2c'_1 + c_3c'_1 + c'_2 + c_1c'_2 + c_1c_3c'_2 + c'_1c'_2 + c_1c'_1c'_3 + c'_2c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,0,0)}\big(c_1c_2c_3 + c_2c_3c'_1 + c_1c_2c'_3 + c_2c'_1c'_3 + c'_2c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,0,1)}\big(c_2 + c_1c_2 + c_1c_2c_3 + c_2c_3c'_1 + c'_2 + c_1c'_2 + c'_1c'_2 + c_1c'_1c'_2 + c_2c'_3 + c_2c'_1c'_3 + c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,0,2)}\big(c_1c_2 + c_1c_2c_3 + c_2c'_1 + c_2c_3c'_1 + c_1c'_2 + c_1c'_1c'_2 + c_2c'_3 + c_2c'_1c'_3 + c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,1,0)}\big(c_3 + c_1c_3 + c_3c'_1 + c_1c_3c'_1 + c_3c'_2 + c_1c_3c'_2 + c'_3 + c_2c'_3 + c_1c'_1c'_3 + c_2c'_1c'_3 + c_1c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,1,1)}\big(c_1c_2 + c_3 + c_1c_3 + c'_1 + c_1c'_1 + c_2c'_1 + c_3c'_1 + c_1c_3c'_1 + c'_2 + c_1c'_2 + c_3c'_2 + c_1c_3c'_2 + c'_3 + c_1c'_3 + c_1c_2c'_3 + c_2c'_1c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,1,2)}\big(1 + c_1 + c_2 + c_1c_2 + c'_2 + c_1c'_2 + c_3c'_2 + c_1c_3c'_2 + c'_1c'_2 + c_1c'_1c'_2 + c_1c_2c'_3 + c_2c'_1c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,2,0)}\big(c_2c_3 + c_2c_3c'_1 + c_3c'_2 + c_3c'_1c'_2 + c_2c'_3 + c_1c_2c'_3 + c'_2c'_3 + c'_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,2,1)}\big(c_1 + c_1c_2 + c_2c_3 + c'_1 + c_2c'_1 + c_3c'_1 + c_1c_3c'_1 + c_2c_3c'_1 + c_1c'_2 + c_3c'_2 + c_1c'_1c'_2 + c_3c'_1c'_2 + c'_3 + c'_1c'_3 + c'_2c'_3 + c_1c'_2c'_3\big) + \delta_{(t_1,t_2,t_3)=(2,2,2)}\big(1 + c_1c_2 + c_1c_3 + c_2c_3 + c_1c'_1 + c_2c'_1 + c_3c'_1 + c_2c_3c'_1 + c'_2 + c_1c'_2 + c_3c'_2 + c_3c'_1c'_2 + c'_3 + c_1c'_3 + c'_1c'_3 + c_1c'_1c'_3 + c'_2c'_3 + c_1c'_2c'_3\big)$,

and

$$C_{s\psi}(g_1,g_2,g_3)|_{T_d} = \gamma_8^{Fd\bar{3}m}(g_1,g_2,g_3).$$

22.2 Twisted coefficients $H^3_\rho(G_{227},\mathbb{Z})$

we know that there are in total four actions we can define, depending on whether $\overline{C}_6$ and S complex conjugate U(1) i.e. whether they orientation-reverse $\mathbb{Z}$. Note that this is equivalent to asking whether inversion P and mirror M orientation-reverse $\mathbb{Z}$. We have give in the above the case where both do not, and both reverse, corresponding to a trivial action on $\mathbb{Z}$ and the action $\mathbb{Z}^{\text{or}}$. One can check the mixed case where only M reverse or only P reverse.

To see that there are in total four actions determined by the sign of $\overline{C}_6$ and S: note that using the standard generator given in our LSM paper, which are $T_1, T_2, T_3, C_2, C'_2, C_3, M, P$, we have

$$S = C_2MP,$$

and

$$C'_2 = T_2\overline{C}_6S\overline{C}_6S, \quad C_2 = \overline{C}_6^2S\overline{C}_6S\overline{C}_6^{-1}.$$

Now, the code to check the twisted cohomology with $\rho_{01}(\overline{C}_6) = +1$, $\rho_{01}(S) = -1$ is:

```
LoadPackage("cryst");
LoadPackage("hap");
LoadPackage("polycyclic");

T1227 := [[1,0,0,1],[0,1,0,0],[0,0,1,0],[0,0,0,1]];
T2227 := [[1,0,0,0],[0,1,0,1],[0,0,1,0],[0,0,0,1]];
T3227 := [[1,0,0,0],[0,1,0,0],[0,0,1,1],[0,0,0,1]];
C2227 := [[0,1,0,0],[1,0,0,0],[-1,-1,-1,1/2],[0,0,0,1]];
C2p227 := [[0,0,1,0],[-1,-1,-1,1/2],[1,0,0,0],[0,0,0,1]];
C3227 := [[0,0,1,0],[1,0,0,0],[0,1,0,0],[0,0,0,1]];
```

```
M227   := [[0,1,0,0],[1,0,0,0],[0,0,1,0],[0,0,0,1]];
P227   := [[-1,0,0,0],[0,-1,0,0],[0,0,-1,0],[0,0,0,1]];

G227raw := Group(T1227,T2227,T3227,C2227,C2p227,C3227,M227,P227);

# 1. Convert to GAP right-action affine crystallographic group
A227 := AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G227raw)));
gensA := GeneratorsOfGroup(A227);
isoA := IsomorphismPcpGroup(A227);
PcpA := Image(isoA);
R227 := ResolutionAlmostCrystalGroup(PcpA,5);

SignGrp := Group([[-1]]);
plus  := One(SignGrp);
minus := GeneratorsOfGroup(SignGrp)[1];

gensPcp := List(gensA, g -> Image(isoA,g));
rhoPcp := GroupHomomorphismByImages(
PcpA, SignGrp,
gensPcp,
[ plus, plus, plus, plus, plus, plus, minus, plus ]
);
if rhoPcp = fail then
Error("The requested sign assignment is not compatible with the relations.");
fi;
Crho := HomToIntegralModule(R227, rhoPcp);
Print("H^2_rho(G,Z) = ", Cohomology(Crho,2), "\n");
Print("H^3_rho(G,Z) = ", Cohomology(Crho,3), "\n");
```

The result we get are

```
H^2_rho(G,Z) = [ 6 ]
```

and

```
H^3_rho(G,Z) = [ 2, 2, 2, 4 ]
```

Let me remind that the above is action ρ_{01}, namely $\overline{C}_6$ has trivial action while S has nontrivial action.

Let us also give the other three actions:

```
rhoPcp := GroupHomomorphismByImages(
PcpA, SignGrp,
gensPcp,
[ plus, plus, plus, plus, plus, plus, minus, minus ]
);
```

gives

```
H^2_rho(G,Z) = [ 2 ]
H^3_rho(G,Z) = [ 2, 2, 2, 2, 0 ]
```

Let me remind that the above is action ρ_{10} (same as orientation reversing), namely $\overline{C}_6$ has nontrivial action while S has trivial action.

and `[ plus, plus, plus, plus, plus, plus, plus, minus ]` gives

```
H^2_rho(G,Z) = [ 2 ]
```

```
H^3_rho(G,Z) = [ 2, 2, 2, 2 ]
```

which is the action ρ_{11}, namely both $\overline{C}_6$ and S have nontrivial action.

while `[ plus, plus, plus, plus, plus, plus, plus, plus ]` gives

```
H^2_rho(G,Z) = [ 2, 2 ]
```

```
H^3_rho(G,Z) = [ 2, 2, 2 ]
```

Note: previous we had some wrong code, by defining the action to be the determinant sign of the entry-wise absolute value of the rotation matrix obtained from the upper left 3×3 block of the matrices:

```
Z10:=GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)*Determinant([[AbsInt(x[1,1]),AbsInt(x[1,2]),
AbsInt(x[1,3])],[AbsInt(x[2,1]),AbsInt(x[2,2]),AbsInt(x[2,3])],[AbsInt(x[3,1]),AbsInt(x[3,2]),AbsInt(x[3,3])]
```

and

```
Z01:=GroupHomomorphismByFunction(G,ZZ,x->[[Determinant([[AbsInt(x[1,1]),AbsInt(x[1,2]),
AbsInt(x[1,3])],[AbsInt(x[2,1]),AbsInt(x[2,2]),AbsInt(x[2,3])],[AbsInt(x[3,1]),AbsInt(x[3,2]),AbsInt(x[3,3])]
```

Then we can take

```
C10:=HomToIntegralModule(R,Z10);;
```

and

```
C01:=HomToIntegralModule(R,Z01);;
```

these are wrong actions: the `Z10` defined here has the sign $(-1, -1, +1, +1, -1)$ on the generators (C_2, C_2', C_3, M, P), while `Z01` has $(-1, -1, +1, -1, 1)$. Apply `Cohomology(C01,3);` and `Cohomology(C10,3);` to gives trivial for $H^{1,2,3,4}$, i.e. trivial at degree 1,2,3,4; but in fact it is not clear that they are well defined.

The things is that the action "the action to be the determinant sign of the entry-wise absolute value of the rotation matrix obtained from the upper left 3×3 block of the matrices" is basis dependent, and we really need to use the cartesian basis to achieve this. (Note that the matrix generators defined above are in the affine basis.) So the code below will in fact give the correct result (albeit much slower than the code above with our own generators defined):

```
G227 := SpaceGroupIT(3,227);

ZZ1  := GL(1,Integers);

Z01 := GroupHomomorphismByFunction(G227, ZZ1,x -> [[ Determinant([[ AbsInt(x[1,1]),
AbsInt(x[1,2]), AbsInt(x[1,3]) ],[ AbsInt(x[2,1]), AbsInt(x[2,2]), AbsInt(x[2,3]) ],
[ AbsInt(x[3,1]), AbsInt(x[3,2]), AbsInt(x[3,3]) ]]) ]]);

Print("Is Z01 a homomorphism? ", IsGroupHomomorphism(Z01), "\n");

iso227 := IsomorphismPcpGroup(G227);

P227   := Image(iso227);

R227pcp := ResolutionAlmostCrystalGroup(P227,4);

Z01pcp := GroupHomomorphismByFunction(P227, ZZ1,g -> Image(Z01, PreImage(iso227, g)));

Print("Is Z01pcp a homomorphism? ", IsGroupHomomorphism(Z01pcp), "\n");

C01pcp := HomToIntegralModule(R227pcp, Z01pcp);

Print("H^3_rho(G, Z) on pcp copy = ", Cohomology(C01pcp,3), "\n");
```

23 $Fd\bar{3}m$: the original spectral sequence

With the above sections, we finally have a better, much more correct, understanding of spectral sequence. So let's go back to our original spectral sequence $0 \to N \to G \to Q \to 0$,, where $G = Fd\bar{3}m$, $Q = O_{\mathrm{h}} = S_4 \times \mathbb{Z}_2$, $N = T = \mathbb{Z}^3$, $A = \mathbb{Z}_2$. We have

$$\begin{array}{c|cccccc} \vdots & \vdots & \vdots & \vdots & & \vdots & \iddots \\ q=3 & H^3(N,A)^Q & H^1(Q,H^3(N,A)) & H^2(Q,H^3(N,A)) & H^3(Q,H^3(N,A)) & H^4(Q,H^3(N,A)) & \cdots \\ q=2 & H^2(N,A)^Q & H^1(Q,H^2(N,A)) & H^2(Q,H^2(N,A)) & H^3(Q,H^2(N,A)) & H^4(Q,H^3(N,A)) & \cdots \\ q=1 & H^1(N,A)^Q & H^1(Q,H^1(N,A)) & H^2(Q,H^1(N,A)) & H^3(Q,H^1(N,A)) & H^4(Q,H^3(N,A)) & \cdots \\ q=0 & A & H^1(Q,A^N) & H^2(Q,A^N) & H^3(Q,A^N) & H^4(Q,A^N) & \cdots \\ \hline E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots \end{array} \tag{126}$$

We have

$$\begin{aligned} C_3^{-1}T_1^xT_2^yT_3^zC_3 &= T_3^xT_1^yT_2^z = T_1^yT_2^zT_3^x, \\ C_2T_1^xT_2^yT_3^zC_2 &= (T_3^{-1}T_2)^x(T_3^{-1}T_1)^yT_3^{-z} = T_1^yT_2^xT_3^{-x-y-z}, \\ C_2'T_1^xT_2^yT_3^zC_2' &= (T_2^{-1}T_3)^xT_2^{-y}(T_2^{-1}T_1)^z = T_1^zT_2^{-x-y-z}T_3^x, \\ MT_1^xT_2^yT_3^zM &= T_1^yT_2^xT_3^z, \end{aligned} \tag{127}$$

Now, $H^1(N,A) = \langle\chi_1,\chi_2,\chi_3\rangle$, where $\chi_{1,2,3}$ are the character function for T_1, T_2, T_3. Setting $g = T_1^xT_2^yT_3^z$. We have $\chi_1(C_3^{-1}gC_3) - \chi_1(g) = y - x = \chi_2(g) - \chi_1(g)$, so $C_3.\chi_1 = \chi_2$, $C_3.\chi_2 = \chi_3$, $C_3.\chi_3 = \chi_1$, $\chi_1(C_2gC_2) - \chi_1(g) = y - x = (\chi_2 - \chi_1)(g)$, $\chi_2(C_2gC_2) - \chi_2(g) = x - y = (\chi_1 - \chi_2)(g)$, and $\chi_3(C_2gC_2) - \chi_3(g) = -x-y-z-z = -x-y = (\chi_1+\chi_2)(g)$, so that $C_2.(\chi_1+\chi_2+\chi_3) - (\chi_1+\chi_2+\chi_3) = \chi_1+\chi_2$. Similarly we have $\chi_1(C_2'gC_2') = z = \chi_1(g)$, $\chi_2(C_2'gC_2') = -x-y-z = (\chi_1+\chi_2+\chi_3)(g)$, and $\chi_3(C_2'gC_2') = x = \chi_1$. Therefore, the actions of C_2 and C_3 together shows that $H^1(N,A)^Q = \{0\}$.

Next, we look at $H^2(N,A)^Q$. We have $H^2(N,A) = \langle b_1, b_2, b_3\rangle = \mathbb{Z}_2^3$ where $b_1(g_1,g_2) = x_1y_2$, $b_2(g_1,g_2) = y_1z_2$, and $b_3(g_1,g_2) = x_1z_2$. Now we have $b_1(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) = y_1z_2 = b_2(g_1,g_2)$, $b_2(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) = z_1x_2 = b_3(g_1,g_2)$ up to coboundary, and $b_3(C_3^{-1}g_1C_3, C_3^{-1}g_2C_3) = y_1x_2 = b_1(g_1,g_2)$ up to coboundary. And $b_1(C_2g_1C_2, C_2g_2C_2) = y_1x_2 = b_1(g_1,g_2)$ up to coboundary, $b_2(C_2g_1C_2, C_2g_2C_2) = x_1(x_2+y_2+z_2) = (b_1+b_3)(g_1,g_2)$ up to coboundary, and $b_3(C_2g_1C_2, C_2g_2C_2) = (x_1+y_1+z_1)y_2 = (b_1+b_2)(g_1,g_2)$ up to coboundary. $b_1(C_2'g_1C_2', C_2'g_2C_2') = z_1(x_2+y_2+z_2) = (b_2+b_3)(g_1,g_2)$ up to coboundary, $b_2(C_2'g_1C_2', C_2'g_2C_2') = (x_1+y_1+z_1)x_2 = (b_1+b_3)(g_1,g_2)$ up to coboundary, and $b_3(C_2'g_1C_2', C_2'g_2C_2') = z_1x_2 = b_3(g_1,g_2,g_3)$ up to cobundary, $b_1(Mg_1M, Mg_2M) = b_1(g_1,g_2)$, $b_2(Mg_1M, Mg_2M) = b_3(g_1,g_2)$, $b_3(Mg_1M, Mg_2M) = b_2(g_1,g_2)$. This shows that $Q.(b_1+b_2+b_3) - (b_1+b_2+b_3) = 0$ up to coboundary and that it is the only one that survives in $H^2(N,A)^Q$.

Note that $H^1(N,A) = \mathbb{Z}_2^3$ generated by $\chi_{1,2,3}$ and $Q = O_{\mathrm{h}} = S_4 \times \mathbb{Z}_2$. In order to calculate $H^p(Q, H^1(N,A))$ for $p = 1, 2$, we can again use our Mathematica code. The starting point is to write Q in standard form. This has been done before, from Eq. (94), setting $x_i = y_i = z_i = 0$, we get

$$g_i = C_2^{c_i}C_2^{\prime c_i'}C_3^{t_i}M^{t_i}P^{p_i}, \quad i = 1,2, \qquad g_1g_2 = C_2^{c_1+(s_2[m_1])_{2t_1+1}}C_2^{\prime c_1'+(s_2[m_1])_{2t_1+2}}C_3^{t_1+(1+m_1)t_2}M^{m_1+m_2}P^{p_1+p_2}, \tag{128}$$

where

$$s_i[m] = (c_i + mc_i', c_i', c_i + (1+m)c_i'), \quad c_i, c_i', m, p \in \mathbb{Z}_2, \quad t_i \in \mathbb{Z}_3.$$

The generators have action $C_3.\chi_1 = \chi_2$, $C_3.\chi_2 = \chi_3$, $C_3.\chi_3 = \chi_1$, $C_2.\chi_1 = \chi_2$, $C_2.\chi_2 = \chi_1$, $C_2.\chi_3 = \chi_1+\chi_2+\chi_3$, $P.\chi_i = \chi_i$, $i = 1,2,3$, and $C_2'.\chi_1 = \chi_3$, $C_2'.\chi_2 = \chi_1+\chi_2+\chi_3$, and $C_2'.\chi_3 = \chi_1$. What this really means is that, define $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}_2^3$. Now, these χ_i are actually basis. Transferring the action to coordinates $(\alpha_1,\alpha_2,\alpha_3)$ where $\alpha_{1,2,3} = 0, 1$, we have

$$\begin{aligned} P&\colon (\alpha_1,\alpha_2,\alpha_3) \mapsto (\alpha_1,\alpha_2,\alpha_3), \\ M&\colon (\alpha_1,\alpha_2,\alpha_3) \mapsto (\alpha_2,\alpha_1,\alpha_3), \\ C_3&\colon (\alpha_1,\alpha_2,\alpha_3) \mapsto (\alpha_3,\alpha_1,\alpha_2), \\ C_2'&\colon (\alpha_1,\alpha_2,\alpha_3) \mapsto (\alpha_2+\alpha_3,\alpha_2,\alpha_1+\alpha_2), \\ C_2&\colon (\alpha_1,\alpha_2,\alpha_3) \mapsto (\alpha_2+\alpha_3,\alpha_1+\alpha_3,\alpha_3). \end{aligned} \tag{129}$$

or in matrix notation

$$P = 1_{3\times3}, \quad M = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad C_2' = \begin{pmatrix} & 1 & 1 \\ & 1 & \\ 1 & 1 & \end{pmatrix}, \quad C_2 = \begin{pmatrix} & 1 & 1 \\ 1 & & 1 \\ & & 1 \end{pmatrix}. \tag{130}$$

We can then use the brutal force method in Mathematica to compute $H^1(Q, H^1(N,A))$, equipped with the action (129). Turns out we find $H^1(Q, H^1(N,A)) = \mathbb{Z}_2$. The detail can be found in Mathematica notebook. Furthermore, in

Mathematica we showed that the d_2 map $d_2\colon E_2^{1,1} \to E_2^{3,0}$ is nontrivial, this makes use of the definition of the d_2 map that for $f \in H^1(Q, H^1(N, A))$, we have $(d_2 f)(q_1, q_2, q_3) = f(l(q_3))((l(q_1 q_2))^{-1} l(q_1) l(q_2))$, where $l\colon Q \to G$ is a lifting (see Rotman for the definition of a lifting). This shows that $E_3^{1,1} = 0$.

Similarly, for $E_2^{1,2} = H^1(Q, H^2(N, A))$, we have $C_3.b_1 = b_2$, $C_3.b_2 = b_3$, $C_3.b_3 = b_1$, $C_2.b_1 = b_1$, $C_2.b_2 = b_1 + b_3$, and $C_2.b_3 = b_1 + b_2$, $C_2'.b_1 = b_2 + b_3$, $C_2'.b_2 = b_1 + b_3$, $C_2'.b_3 = b_3$, $M.b_1 = b_1$, $M.b_2 = b_3$, $M.b_3 = b_2$. Again these are action on the basis. Transferring the action to coordinates $(\beta_1, \beta_2, \beta_3)$ where $\beta_{1,2,3} = 0, 1$, we have

$$\begin{aligned}
&P\colon (\beta_1, \beta_2, \beta_3) \mapsto (\beta_1, \beta_2, \beta_3),\\
&M\colon (\beta_1, \beta_2, \beta_3) \mapsto (\beta_1, \beta_3, \beta_2),\\
&C_3\colon (\beta_1, \beta_2, \beta_3) \mapsto (\beta_3, \beta_1, \beta_2),\\
&C_2'\colon (\beta_1, \beta_2, \beta_3) \mapsto (\beta_2, \beta_1, \beta_1 + \beta_2 + \beta_3),\\
&C_2\colon (\beta_1, \beta_2, \beta_3) \mapsto (\beta_1 + \beta_2 + \beta_3, \beta_3, \beta_2).
\end{aligned} \tag{131}$$

or in matrix notation

$$\widetilde{P} = 1_{3\times 3}, \quad \widetilde{M} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \widetilde{C}_3 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad \widetilde{C'}_2 = \begin{pmatrix} & 1 & \\ 1 & & \\ 1 & 1 & 1 \end{pmatrix}, \quad \widetilde{C}_2 = \begin{pmatrix} 1 & 1 & 1 \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{132}$$

Again using the Mathematica code we find that $H^1(Q, H^2(N, A)) = \mathbb{Z}_2^3$.

Then we calculate $H^3(N, A)^Q$: note that $H^3(N, A) = \mathbb{Z}_2 = \langle \gamma \rangle$ with the only nontrivial element $\gamma(g_1, g_2, g_3) = x_1 y_2 z_3$. We have $\gamma(C_3^{-1} g_1 C_3, C_3^{-1} g_2 C_3, C_3^{-1} g_3 C_3) = y_1 z_2 x_3$, $\gamma(C_2 g_1 C_2, C_2 g_2 C_2, C_2 g_3 C_2) = y_1 x_2 (x_3 + y_3 + z_3)$, $\gamma(C_2' g_1 C_2', C_2' g_2 C_2', C_2' g_3 C_2') = z_1 (x_2 + y_2 + z_2) x_3$, and $\gamma(M g_1 M, M g_2 M, M g_3 M) = y_1 x_2 z_3$, which all equal $x_1 y_2 z_3$ up to coboundary. About coboundary: note that the following are all coboundary: $(y_2 z_1 + y_1 z_2) z_3$, $x_2 y_1 z_3 + x_1 y_2 z_3$, $z_1 (x_3 z_2 + x_2 z_3)$ and so on. Since the action is trivial, $H^1(Q, H^3(N, A)) = H^1(Q, \langle \gamma \rangle)$ is just the number of homomorphisms, and using Mathematica we find there are 2 independent ones, meaning that $H^1(Q, H^3(N, A)) = \mathbb{Z}_2^2$.

$$\begin{array}{c|cccccc}
q = 4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
q = 3 & 1 & 2 & 4 & H^3(Q, H^3(N, A)) & H^4(Q, H^3(N, A)) & \cdots \\
q = 2 & \mathbf{1} & 3 & 5 & 8 & H^4(Q, H^3(N, A)) & \cdots \\
q = 1 & \mathbf{0} & 1 & 3 & H^3(Q, H^1(N, A)) & H^4(Q, H^3(N, A)) & \cdots \\
q = 0 & \mathbf{1} & \mathbf{2} & \mathbf{4} & 7 & 10 & \cdots \\
\hline
E_2^{p,q} & p = 0 & p = 1 & p = 2 & p = 3 & p = 4 & \cdots\cdots
\end{array} \tag{133}$$

where the calculation of $E_2^{2,1} = H^2(Q, H^1(N, A))$, $E_2^{2,2} = H^2(Q, H^2(N, A))$ and $E_2^{2,3} = H^2(Q, H^3(N, A))$ is calculated using brutal force Mathematica and will be mentioned around Eqs. (142), (144), and (146).

$$\begin{array}{c|cccccc}
q = 4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
q = 3 & ? & ? & ? & ? & ? & \cdots \\
q = 2 & \mathbf{1} & ? & ? & ? & ? & \cdots \\
q = 1 & \mathbf{0} & \mathbf{0} & \boldsymbol{k} = \mathbf{0} & ? & ? & \cdots \\
q = 0 & \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{6} & 7 + k = 7 & \cdots \\
\hline
E_3^{p,q} & p = 0 & p = 1 & p = 2 & p = 3 & p = 4 & \cdots\cdots
\end{array} \tag{134}$$

Conjecture for the ∞ page:

$$\begin{array}{c|cccccc}
q = 4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
q = 3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & ? & ? & \cdots \\
q = 2 & \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{6} & ? & \cdots \\
q = 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & ? & ? & \cdots \\
q = 0 & \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{7} & \cdots \\
\hline
E_\infty^{p,q} & p = 0 & p = 1 & p = 2 & p = 3 & p = 4 & \cdots\cdots
\end{array} \tag{135}$$

Which is that

$$\begin{aligned}
&E_\infty^{p,3} = 0,\\
&E_\infty^{p,2} = \dim(H^p, \mathcal{A}_2) - \dim(H^{p-2}, \mathcal{A}_3), E_\infty^{p,1} \ = 0,\\
&E_\infty^{p,0} = \dim(H^p, \mathcal{A}_0) - \dim(H^{p-2}, \mathcal{A}_1),
\end{aligned} \tag{136}$$

which means that

$$\begin{aligned} \dim E_\infty^{n,0} &= \sum_{i=0}^{n} f(i) - \left[1+(n-2)+\sum_{i=0}^{n-2} f(i-2)\right],\\ \dim E_\infty^{n-2,2} &= 1+(n-2)+\sum_{i=0}^{n-2} f(i-1) - \left[\sum_{i=0}^{n-4} f(i)\right], \end{aligned} \tag{137}$$

Table 4: Conjectured stabilized spectral sequence $E_3^{p,q}=E_\infty^{p,q}$ for $Fd\bar{3}m$ (I am very confident this is the corret one).

$E_3^{p,3}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$E_3^{p,2}$	1	3	4	6	8	9	11	13	14	16	18	19	21	23	24	26	28	29	31	33	34
$E_3^{p,1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$E_3^{p,0}$	1	2	4	6	7	9	11	12	14	16	17	19	21	22	24	26	27	29	31	32	34
p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

so that for sufficiently large n, we conjecture that

$$\begin{aligned} \dim(Fd\bar{3}m,\mathbb{Z}_2) &= \dim(E_\infty^{n,0}\oplus E_\infty^{n-1,1}\oplus E_\infty^{n-2,2}\oplus E_\infty^{n-3,3})\\ &= \dim E_\infty^{n,0}+\dim E_\infty^{n-2,2}\\ &= \sum_{i=0}^{n} f(i) - \left[1+(n-2)+\sum_{i=0}^{n-2} f(i-2)\right] + 1+(n-2)+\sum_{i=0}^{n-2} f(i-1) - \left[\sum_{i=0}^{n-4} f(i)\right]\\ &= 2f(n-3)+f(n-2)+f(n-1)+f(n)-f(-2), \end{aligned} \tag{138}$$

for $n\geq 2$, the above gives $5, 9, 11, 15, 19, 21, 25, 29, 31...$ which is the correct dimension given in GAP.

Let us try to calculate the above $E_2^{p,q}=H^p(S_4\times\mathbb{Z}_2,\mathcal{A}_q)$ where we have denoted $\mathcal{A}_q=H^q(\mathbb{Z}^3,\mathbb{Z}_2)$ the q-th cohomology of just the translation group. We know $\mathcal{A}_1=\langle\chi_1,\chi_2,\chi_3\rangle\cong\mathbb{Z}_2^3$, $\mathcal{A}_2=\langle b_1,b_2,b_3\rangle\cong\mathbb{Z}_2^3$ and $\mathcal{A}_3=\langle\gamma\rangle\cong\mathbb{Z}_2$. Let's attemp to calculate $H^*(S_4\times\mathbb{Z}_2,\mathcal{A}_q)$ using spectral sequence. Generally, using $0\to S_4\to O_\mathrm{h}=Q\to\langle P\rangle=\mathbb{Z}_2\to 0$ we have

$$\begin{array}{c|cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ q=3 & H^3(S_4,\mathcal{A})^{\langle P\rangle} & H^1(\langle P\rangle,H^3(S_4,\mathcal{A})) & H^2(\langle P\rangle,H^3(S_4,\mathcal{A})) & H^3(\langle P\rangle,H^3(S_4,\mathcal{A})) & H^4(\langle P\rangle,H^3(S_4,\mathcal{A})) & \cdots \\ q=2 & H^2(S_4,\mathcal{A})^{\langle P\rangle} & H^1(\langle P\rangle,H^2(S_4,\mathcal{A})) & H^2(\langle P\rangle,H^2(S_4,\mathcal{A})) & H^3(\langle P\rangle,H^2(S_4,\mathcal{A})) & H^4(\langle P\rangle,H^3(S_4,\mathcal{A})) & \cdots \\ q=1 & H^1(S_4,\mathcal{A})^{\langle P\rangle} & H^1(\langle P\rangle,H^1(S_4,\mathcal{A})) & H^2(\langle P\rangle,H^1(S_4,\mathcal{A})) & H^3(\langle P\rangle,H^1(S_4,\mathcal{A})) & H^4(\langle P\rangle,H^3(S_4,\mathcal{A})) & \cdots \\ q=0 & \mathcal{A}^{S_4} & H^1(\langle P\rangle,\mathcal{A}^{S_4}) & H^2(\langle P\rangle,\mathcal{A}^{S_4}) & H^3(\langle P\rangle,\mathcal{A}^{S_4}) & H^4(\langle P\rangle,\mathcal{A}^{S_4}) & \cdots \\ \hline E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots \end{array} \tag{139}$$

where $\mathcal{A}=\mathcal{A}_{1,2,3}$ for the rows $q=1,2,3$ in Eq. (133). The $\mathcal{A}_q^{S_4}=\mathcal{A}_q^Q$ is just the first column of Eq. (133), which is $\mathcal{A}_1^{S_4}=\{0\}$, $\mathcal{A}_2^{S_4}=\mathcal{A}_2^Q=\langle b_1+b_2+b_3\rangle$, and $\mathcal{A}_3^{S_4}=\mathcal{A}_3^Q=\langle\gamma\rangle$. Also, since $\langle P\rangle\cong\mathbb{Z}_2$ commutes with every element in S_4, it is easy to see that its action on $H^p(S_4,\mathcal{A})$ is always trivial. Using the fact that

$$H^n(\mathbb{Z}_2^h,\mathcal{A}=(\mathbb{Z}_2)^m)=\mathcal{A}^h/(h.\mathcal{A}-\mathcal{A}),\quad \text{for } n\geq 1, \tag{140}$$

where we the notation $\mathbb{Z}_2^h=\langle h\rangle$ means the group of order two generated by h, and we used the short hand $\mathcal{A}^{\langle h\rangle}=\mathcal{A}^h$. If we further have trivial h-action, then the denominator is trivial while the numerator is just $\mathcal{A}$, and we have

$$H^n(\mathbb{Z}_2^h,\mathcal{A})=\mathcal{A}.$$

Using this, we can simplify the spectral sequence Eq. (141) to a form that $E_2^{p,q}=E_2^{0,q}$ for any p:

$$\begin{array}{c|cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ q=3 & H^3(S_4,\mathcal{A}) & H^3(S_4,\mathcal{A}) & H^3(S_4,\mathcal{A}) & H^3(S_4,\mathcal{A}) & H^3(S_4,\mathcal{A}) & \cdots \\ q=2 & H^2(S_4,\mathcal{A}) & H^2(S_4,\mathcal{A}) & H^2(S_4,\mathcal{A}) & H^2(S_4,\mathcal{A}) & H^2(S_4,\mathcal{A}) & \cdots \\ q=1 & H^1(S_4,\mathcal{A}) & H^1(S_4,\mathcal{A}) & H^1(S_4,\mathcal{A}) & H^1(S_4,\mathcal{A}) & H^1(S_4,\mathcal{A}) & \cdots \\ q=0 & \mathcal{A}^{S_4} & \mathcal{A}^{S_4} & \mathcal{A}^{S_4} & \mathcal{A}^{S_4} & \mathcal{A}^{S_4} & \cdots \\ \hline E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots \end{array} \tag{141}$$

So the first task is to compute $H^*(S_4,\mathcal{A})$ for $\mathcal{A}=\mathcal{A}_{1,2,3}$. Let's try to use GAP. First, the command

```
G:=Image(IsomorphismPcpGroup(SymmetricGroup(4)));
```

will give

```
Pcp-group with orders [ 2, 3, 2, 2 ],
```

and it is important to know the correspondence with our definition of generator. Let's list as many group relations as possible: $C_3C_2C_3^{-1} = C_2C_2'$, $C_3C_2'C_3^{-1} = C_2$, $C_3^{-1}C_2C_3 = C_2'$, $C_3^{-1}C_2'C_3 = C_2C_2'$, $MC_3M = C_3^{-1}$, $MC_2M = C_2$, $MC_2'M = C_2C_2'$, and so on. Every of these relations indicate that we have

$$\texttt{G.1} = M, \quad \texttt{G.2} = C_3, \quad \texttt{G.3} = C_2', \quad \texttt{G.4} = C_2C_2',$$

where `G.1`,`G.2`,`G.3`,`G.4` corresponds to in order the four generators listed in `Pcp-group with orders [ 2, 3, 2, 2 ]`. I wanted to use the code I used in our projects with Yuan-Ming and Yanqi, but it seems the code does not work: the step `h2:=TwoCohomologyCR(cr);;` raises some error.

For $\mathcal{A} = \mathcal{A}_1$, we have $E_2^{0,q=0} = \mathcal{A}_1^{S_4} = \{0\}$ (from the action of S_4 on $\mathcal{A}_1$), $E_2^{p,q=1} = \mathbb{Z}_2$ (from the fact that $\mathbb{Z}_2 = H^1(Q, \mathcal{A}_1) = E_3^{0,1} \cup E_2^{1,0} = H^1(S_4, \mathcal{A}_1)$ since $E_2^{1,0} = \mathcal{A}_1^{S_4}$ is trivial so $E_3^{0,1} = E_2^{1,0}$), $E_2^{p,q=2} = H^2(S_4, \mathcal{A}_1) = \mathbb{Z}_2^2$ (through the Mathematica brutal force calculation). Now, the d_2 maps coming from the $q = 1$ row are zero maps, meaning the $(p, q) = (1, 1)$ element on E_2 is stablized. We still need to see if the $E_2^{0,2}$ element is stabilized.

Using Mathematica brutal force, we actually find

$$H^2(O_\mathrm{h}, H^1(T, \mathbb{Z}_2)) \equiv H^2(Q, H^1(N, \mathbb{Z}_2)) \equiv H^2(O_\mathrm{h}, \mathcal{A}_1) \equiv H^2(O_\mathrm{h}, \langle \chi_1, \chi_2, \chi_3 \rangle) = \mathbb{Z}_2^3, \tag{142}$$

this means that the spectral sequence (141) for the case of $\mathcal{A} = \mathcal{A}_1 = H^1(T, \mathbb{Z}_2) = \langle \chi_1, \chi_2, \chi_3 \rangle$ has the following form

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=3 & 2 & H^3(S_4,\mathcal{A}_1) & H^3(S_4,\mathcal{A}_1) & H^3(S_4,\mathcal{A}_1) & H^3(S_4,\mathcal{A}_1) & \cdots \\
q=2 & \mathbf{2} & H^2(S_4,\mathcal{A}_1) & H^2(S_4,\mathcal{A}_1) & H^2(S_4,\mathcal{A}_1) & H^2(S_4,\mathcal{A}_1) & \cdots \\
q=1 & \mathbf{1} & \mathbf{1} & H^1(S_4,\mathcal{A}_1) & H^1(S_4,\mathcal{A}_1) & H^1(S_4,\mathcal{A}_1) & \cdots \\
q=0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots
\end{array} \tag{143}$$

(this E_2 page actually collapses – see below)

Next let's check the case of $\mathcal{A}_2$. We know that $\mathcal{A}_2^{S_4} = \mathbb{Z}_2$ and $H^1(O_\mathrm{h}, \mathcal{A}_2) = \mathbb{Z}_2^3$. Using brutal force in Mathematica, we also find that $H^1(S_4, \mathcal{A}_2) = \mathbb{Z}_2^2$, and $H^2(S_4, \mathcal{A}_2) = \mathbb{Z}_2^2$. We also calculated in Mathematica that

$$H^2(O_\mathrm{h}, H^2(T, \mathbb{Z}_2)) \equiv H^2(Q, H^2(N, \mathbb{Z}_2)) \equiv H^2(O_\mathrm{h}, \mathcal{A}_2) \equiv H^2(O_\mathrm{h}, \langle b_1, b_2, b_3 \rangle) = \mathbb{Z}_2^5, \tag{144}$$

This shows that the the $(p, q) = (0, 1)$, $(1, 1)$, $(2, 0)$ and $(0, 2)$ elements are all stabilized at E_2, and we have the spectral sequence revealed as in Eq. (145).

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=3 & H^3(S_4,\mathcal{A}_2) & H^3(S_4,\mathcal{A}_2) & H^3(S_4,\mathcal{A}_2) & H^3(S_4,\mathcal{A}_2) & H^3(S_4,\mathcal{A}_2) & \cdots \\
q=2 & \mathbf{2} & H^2(S_4,\mathcal{A}_2) & H^2(S_4,\mathcal{A}_2) & H^2(S_4,\mathcal{A}_2) & H^2(S_4,\mathcal{A}_2) & \cdots \\
q=1 & \mathbf{2} & \mathbf{2} & H^1(S_4,\mathcal{A}_2) & H^1(S_4,\mathcal{A}_2) & H^1(S_4,\mathcal{A}_2) & \cdots \\
q=0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots
\end{array} \tag{145}$$

(this E_2 page actually collapses – see below)

Finally, similar to the calculation of $H^2(O_\mathrm{h}, \mathcal{A}_1)$ and $H^2(O_\mathrm{h}, \mathcal{A}_2)$, we used Mathematica to get

$$H^2(O_\mathrm{h}, H^3(T, \mathbb{Z}_2)) \equiv H^2(Q, H^3(N, \mathbb{Z}_2)) \equiv H^2(O_\mathrm{h}, \mathcal{A}_3) \equiv H^2(O_\mathrm{h}, \langle \gamma \rangle) = \mathbb{Z}_2^4, \tag{146}$$

Update: note that after we did the systematic computation for all the 230 groups, we know something more: in Table 6 we give our guess of $H^*(O_\mathrm{h}, \mathcal{A}_1)$ and $H^*(O_\mathrm{h}, \mathcal{A}_2)$. Taking $\mathcal{A}_1 = \langle \chi_1, \chi_2, \chi_3 \rangle$ for example (corresponding to the action t_1, d_7), the cohomology dimenson for $H^n(O_\mathrm{h}, \mathcal{A}_1)$ is

$$1 + n + \sum_{i=0}^{n} f(i-2)$$

(this is a conjecture that I think it's correct); note that

$$f(n) = [n/2] + [(n-3)/6] + 2 - \delta_{n \equiv 4 \bmod 6},$$

and we have

$$f(-2)=-1,\quad f(-1)=0,\quad f(0)=1,\quad f(1)=1,\quad f(2)=2,\quad f(3)=3,\quad f(4)=3,\quad f(5)=4,\quad f(6)=5,...$$

Compare with the spectral sequence (143), we see that the prediction is that

$$H^n(S_4,\mathcal{A}_1)=\mathbb{Z}_2^{1+f(n-2)},$$

which has also been listed in Table 6. Explictly,

$$H^{n=0,1,2,3,4,5,6,...}(S_4,\mathcal{A}_1)=\mathbb{Z}_2^{0,1,2,2,3,4,4...}$$

We see that it would be very beneficial to obtain the ring structure of $H^n(S_4,\mathcal{A}_1)$ and $H^n(S_4,\mathcal{A}_2)$.

First, note that $H^1(S_4,\mathcal{A}_1)=\mathbb{Z}_2=\langle x_2\rangle$, here x_2 reduces to nontrivial elements of V_1 and V_2 both. We see this is very different from the case of $H^1(S_4,\mathbb{Z}_2)$.

Is $H^*(S_4,\mathcal{A}_1)$ a ring? `https://mathoverflow.net/questions/243061/ring-structure-on-cohomology-of-groups` `https://math.stackexchange.com/questions/3025528/how-does-group-cohomology-behaves-where-coefficient-is-dire` `https://math.stackexchange.com/questions/1835972/troubles-to-understanding-notation-and-some-terminology-on-` `1835982#1835982` See below: we are indeed able to give it a ring structure. However, it seems we cannot give a ring structure to $H^*(S_4,\mathcal{A}_2)$.

The place to learn cup product is Ken Brown's book P110. Recall that before when we take cup product we directly take the product on $\mathbb{Z}_2$, we were implicitly using the fact that $\mathbb{Z}_2\otimes\mathbb{Z}_2\cong\mathbb{Z}_2$ where $1\otimes 0\to 0\otimes 1\to 0$ and $1\otimes 1\to 1$, so the tensor structure actually can be simply treated as integer multiplication in $\mathbb{F}_2$. MOre generally, cup product is a map $H^p(G,M)\times H^q(G,N)\to H^{p+q}(G,M\otimes N)$, so what we need to work out is $\mathcal{A}_1\otimes\mathcal{A}_1$, where $\mathcal{A}_1=\mathbb{Z}_2^3$ as abelian group but $\mathcal{A}_1$ contains more structure since it is a $\mathbb{Z}Q$ module.

Write $\mathcal{A}_1=\mathbb{F}_2[\chi_1,\chi_2,\chi_3]=\mathbb{Z}_2(\chi_1)\oplus\mathbb{Z}_2(\chi_2)\oplus\mathbb{Z}_2(\chi_3)$. First, using distribution we see that $\mathcal{A}_1\otimes\mathcal{A}_1=\oplus_{i,j=1,2,3}\mathbb{Z}_2(\chi_i\otimes\chi_j)$. Using the generator S_4 which is $C_3=(123)$ and $C_2M=(12)(34).(12)=(34)$, we have

$$C_3\colon \chi_i\otimes\chi_j\mapsto C_3\chi_i\otimes C_3\chi_j=\chi_{i+1}\otimes\chi_{j+1},$$

$$C_2M\colon \chi_1\otimes\chi_1\mapsto C_2M\chi_1\otimes C_2M\chi_1=\chi_2\otimes\chi_2,\quad \chi_2\otimes\chi_2\mapsto\chi_1\otimes\chi_1,$$

and so on. We find that

$$\mathcal{A}_1\otimes\mathcal{A}_1\cong\mathcal{A}_1 \tag{147}$$

as a $\mathbb{Z}Q$ module under the map

$$\chi_i\otimes\chi_j\to\chi_i,\quad i,j=1,2,3, \tag{148}$$

In other words, the tensor product induces a "product" for the coefficients

$$(\alpha_1,\beta_1,\gamma_1).(\alpha_2,\beta_2,\gamma_2):=(\alpha_1(\alpha_2+\beta_2+\gamma_2),\beta_1(\alpha_2+\beta_2+\gamma_2),\gamma_1(\alpha_2+\beta_2+\gamma_2)). \tag{149}$$

Under the above defined product we can endow $H^*(S_4,\mathcal{A}_1)$ a ring structure.

However, note that we have not found a ring structure for $H^*(S_4,\mathcal{A}_2)$. Below is our attempt:

$$C_3\colon b_i\otimes b_j\mapsto(C_3b_i)\otimes(C_3b_j)=b_{i+1}\otimes b_{j+1},$$

$$C_2M=(34)\colon(b_1,b_2,b_3)\mapsto(b_1,b_1+b_2,b_1+b_3),$$

so

$$C_2M\colon b_1\otimes b_1\mapsto(C_2Mb_1)\otimes(C_2Mb_1)=b_1\otimes b_1,$$

$$C_2M\colon b_1\otimes b_2\mapsto b_1\otimes b_1+b_1\otimes b_2,\quad b_1\otimes b_3\mapsto b_1\otimes b_1+b_1\otimes b_3,\quad b_2\otimes b_1\otimes b_1\otimes b_1+b_2\otimes b_1,$$

$$C_2M\colon b_2\otimes b_2\mapsto b_1\otimes b_1+b_1\otimes b_2+b_2\otimes b_1+b_2\otimes b_2,\quad b_2\otimes b_3\mapsto b_1\otimes b_1+b_2\otimes b_1+b_1\otimes b_3+b_2\otimes b_3,$$

$$C_2M\colon b_3\otimes b_2\mapsto b_1\otimes b_1+b_3\otimes b_1+b_1\otimes b_2+b_3\otimes b_2,\quad b_3\otimes b_3\mapsto b_1\otimes b_1+b_3\otimes b_1+b_1\otimes b_3+b_3\otimes b_3,$$

and whether there is an isomorphism between $\mathcal{A}_2\otimes\mathcal{A}_2$ and $\mathcal{A}_2$ as a $\mathbb{Z}S_4$ module is left for future work.

Below we will try to work out the ring struture of $H^*(S_4,\mathcal{A}_1)$. Using mathematica, we obtain the representative cochain for the nontrivial element υ of $H^1(S_4,\mathcal{A}_1)\cong\mathbb{Z}_2=\langle\upsilon\rangle$: for $g=C_2^cC_2'^{c'}C_3^tM^m\in S_4$,

$$\upsilon(g)=\boldsymbol{a}_{12c+6c'+2t+m}, \tag{150}$$

where

$$\begin{aligned}\boldsymbol{a}_{0,\ldots,23} =&(0,0,0),(1,1,1),(1,0,1),(0,1,0),(0,1,1),(1,0,0),(1,1,0),(1,0,0),\\&(0,1,1),(0,0,1),(1,0,1),(1,1,1),(0,1,1),(0,1,0),(1,1,0),(1,1,1),\\&(0,0,0),(0,0,1),(1,0,1),(0,0,1),(0,0,0),(1,0,0),(1,1,0),(0,1,0).\end{aligned} \tag{151}$$

and under the above cup product (and specifically, the product on the coefficients (149)) we define

$$\omega_1(g_1, g_2) := \upsilon(g_1).(g_1.\upsilon(g_2)), \tag{152}$$

where $g_1.\upsilon(g_2)$ denote the matrix action of g_1 on the vector $\upsilon(g_2)$. we can verify that

$$g_1.\omega_1(g_2, g_3) - \omega_1(g_1g_2, g_3) + \omega_1(g_1, g_2g_3) - \omega_1(g_1, g_2) = (0, 0, 0), \tag{153}$$

so $\omega_1(g_1, g_2) := \upsilon(g_1).(g_1.\upsilon(g_2))$ is satisfies the cocycle condition. We also checked in mathematica that it is not a coboundary, so $\omega_1(g_1, g_2)$ is a geniuine cocycle and lives in $H^2(S_4, \mathcal{A}_1) \cong \mathbb{Z}_2^2$. We write

$$H^2(S_4, \mathcal{A}_1) \cong \mathbb{Z}_2^2 = \langle \upsilon^2, \psi \rangle, \tag{154}$$

where

$$\psi(g_1, g_2) = \left\{ \begin{array}{ll} 0 & \text{if } c_1 = c_1' = t_1 = m_1 = 0 \text{ or } c_2 = c_2' = t_2 = m_2 = 0, \\ \boldsymbol{a}_{(23(12c_1+6c_1'+2t_1+m_1-1)+12c_2+6c_2'+2t_2+m_2-1)+1}, & \text{otherwise}, \end{array} \right. \tag{155}$$

where

$\boldsymbol{a}_{1,2,\ldots,529}$ =(0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 0), (0, 0, 1), (0, 0, 0), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 0), (1, 1, 1), (1, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1), (0, 1, 0), (1, 0, 1), (1, 1, 0), (1, 0, 0), (0, 0, 1), (1, 1, 1), (1, 1, 0), (0, 0, 0), (0, 1, 1), (0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 1, 0), (0, 0, 1), (1, 1, 0), (0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 0), (0, 1, 0), (1, 0, 1), (1, 1, 1), (1, 0, 1), (1, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 0, 0), (0, 1, 0), (1, 1, 1), (0, 0, 0), (1, 1, 1), (1, 0, 1), (0, 0, 0), (0, 1, 0), (1, 1, 1), (1, 0, 1), (0, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1), (1, 1, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 0), (1, 1, 1), (0, 0, 0), (1, 1, 0), (0, 1, 1), (0, 0, 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0), (1, 1, 1), (0, 1, 1), (0, 0, 0), (0, 1, 1), (0, 0, 1), (0, 1, 1), (0, 1, 1), (0, 0, 0), (1, 1, 0), (1, 0, 0), (0, 1, 1), (0, 0, 0), (1, 0, 0), (0, 0, 0), (1, 0, 1), (1, 1, 1), (1, 0, 0), (0, 0, 1), (1, 0, 0), (0, 0, 1), (0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1), (1, 0, 1), (0, 0, 0), (1, 0, 0), (0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 0, 0), (0, 1, 0), (1, 0, 1), (1, 1, 1), (0, 0, 1), (1, 0, 1), (0, 0, 1), (1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1), (1, 0, 1), (0, 0, 1), (0, 0, 0), (1, 1, 1), (1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 0, 0), (1, 0, 1), (0, 0, 1), (1, 0, 1), (1, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0), (0, 0, 1), (1, 1, 0), (0, 0, 1), (1, 1, 1), (0, 1, 1), (1, 1, 0), (0, 0, 1), (1, 1, 1), (0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 1, 1), (0, 1, 1), (1, 1, 1), (0, 0, 1), (1, 1, 0), (0, 0, 0), (1, 1, 1), (0, 1, 1), (0, 0, 1), (1, 0, 0), (1, 1, 0), (0, 1, 1), (0, 0, 0), (1, 1, 1), (0, 0, 1), (1, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 1), (1, 1, 0), (0, 0, 1), (0, 0, 0), (0, 1, 0), (1, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 0), (1, 1, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1), (0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0), (0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 1), (0, 1, 0), (0, 0, 0), (0, 0, 1), (1, 1, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (0, 0, 0), (0, 1, 0), (0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1), (0, 1, 1), (0, 0, 1), (1, 0, 0), (1, 0, 1), (0, 0, 0), (1, 1, 0), (0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 0, 0), (0, 0, 0), (1, 0, 0), (0, 0, 0), (1, 0, 1), (1, 1, 0), (0, 1, 0), (1, 1, 1), (0, 1, 0), (1, 0, 1), (0, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 0, 0), (1, 0, 0), (0, 1, 1), (0, 0, 0), (1, 1, 1), (1, 0, 0), (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 0), (1, 1, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1), (0, 1, 1), (1, 1, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0), (1, 1, 1), (1, 0, 0), (1, 1, 0), (1, 0, 0), (0, 0, 0), (1, 0, 0), (1, 1, 1), (0, 0, 1), (0, 1, 1), (1, 0, 1), (0, 0, 0), (1, 0, 1), (1, 1, 1), (0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1), (0, 1, 1), (1, 0, 0), (1, 1, 1), (1, 0, 1), (1, 1, 1), (0, 1, 1), (1, 0, 0), (0, 1, 1), (1, 1, 1), (1, 0, 0), (1, 1, 1), (0, 0, 1), (1, 0, 1), (1, 0, 0), (0, 0, 1), (0, 1, 1), (0, 0, 1), (0, 0, 0), (1, 1, 1), (1, 0, 0), (0, 0, 0), (1, 0, 0), (0, 0, 0), (1, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 1).

We checked that υ^3 is a true cocycle but $\upsilon^3 - \upsilon\psi$ is a coboundary. Therefore

$$H^3(S_4, \mathcal{A}_1) \cong \mathbb{Z}_2^2 = \langle \upsilon^3, \tau \rangle \tag{156}$$

for some 3-cocycle τ.

`https://www.gap-system.org/ForumArchive2/2015/005108.html` lists some code that can be used for our purpose:

```
LoadPackage("HAP");
G:=SymmetricGroup(4);
V:=AbelianGroup(IsPcGroup,[2,2,2]);;
gensV:=GeneratorsOfGroup(V);
w:=GroupHomomorphismByImages(V,V,gensV,gensV{[2,3,1]});
x:=GroupHomomorphismByImages(V,V,gensV,[gensV[1],gensV[2],gensV[1]*gensV[2]*gensV[3]]);
G1:=Group([w,x]);
iso:=GroupHomomorphismByImages(G,G1,[(1,2,3),(3,4)],[w,x]);
action:=function(g,v); return v^Image(iso,g^-1); end;;
A:=GOuterGroup();;
SetActedGroup(A,V);;
SetActingGroup(A,G);;
SetOuterAction(A,action);;
R:=ResolutionFiniteGroup(G,12);;
C:=HomToGModule(R,A);;
Cohomology(C,1);Cohomology(C,2);Cohomology(C,3);Cohomology(C,4);Cohomology(C,5);
Cohomology(C,6);Cohomology(C,7);Cohomology(C,8);Cohomology(C,9);Cohomology(C,10);
```

(157)

The output is of the form `[ 2 ]` and so on. They give the $\mathbb{Z}_2$ dimension $1, 2, 2, 3, 4, 4, 5, 6, 6, 7...$ for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10...$, we see our conjectured form $1 + f(n-2)$ is correct.

Note that in the above w and x correspond to the generators $(1, 2, 3)$ and $(3, 4)$ in S_4, and operation-wise they correspond to C_3 and C_2M, see analysis in the previous section. In the above code we need to use the transformation on the basis functions, not coordinates. We have $C_2 = (12)(34)$ and $M = (12)$, and that $C_3\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_2, \chi_3, \chi_1)$, $C_2\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_2, \chi_1, \chi_1+\chi_2+\chi_3)$, $M\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_2, \chi_1, \chi_3)$, so that $(34) = C_2M\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_1, \chi_2, \chi_1 + \chi_2 + \chi_3)$.

Similarly, we can look at $H^*(S_4, \mathcal{A}_2)$. We have $C_3\colon (b_1, b_2, b_3) \mapsto (b_2, b_3, b_1)$ and $C_2\colon (b_1, b_2, b_3) \mapsto (b_1, b_1 + b_3, b_1 + b_2)$ and $M\colon (b_1, b_3, b_2)$ so that $C_2M\colon (b_1, b_1 + b_2, b_1 + b_3)$, so we just have to change the above operator x to

```
x:=GroupHomomorphismByImages(V,V,gensV,[gensV[1],gensV[1]*gensV[2],gensV[1]*gensV[3]]);
```

(158)

we get dimension $2, 2, 3, 4, 4, 5, 6, 6, 7, 8$ for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10...$, so we see our formula $1+f(n-1)$ again is correct.

$(1243) = (34)(123) = (C_2M)C_3\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_2, \chi_3, \chi_1) \mapsto (\chi_2, \chi_3, \chi_1+\chi_2+\chi_3)$, so that $(14)(23) = (1243)^2\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_3, \chi_1 + \chi_2 + \chi_3, \chi_1)$, and $(13)(24) = C_2'\colon (\chi_1, \chi_2, \chi_3) \mapsto (\chi_1 + \chi_2 + \chi_3, \chi_3, \chi_2)$.

We have also checked the restriction of S_4 to A_4 and $V_2 = \langle C_2, C_2' \rangle = \langle (12)(34), (13)(24) \rangle$. The relevant lines are

```
G:=Group((1,2)(3,4),(1,3)(2,4));
G:=AlternatingGroup(4);
c3:=GroupHomomorphismByImages(V,V,gensV,[gensV[2],gensV[3],gensV[1]]);
c2:=GroupHomomorphismByImages(V,V,gensV,[gensV[2],gensV[1],gensV[1]*gensV[2]*gensV[3]]);
cp2:=GroupHomomorphismByImages(V,V,gensV,[gensV[1]*gensV[2]*gensV[3],gensV[3],gensV[2]]);
```

we have

$$H^n(V_2, \mathcal{A}_1) = \mathbb{Z}_2^{n+2},$$

$$H^n(A_4, \mathcal{A}_1) = \mathbb{Z}_2^{m[n+1]},$$

where $m[n]$ is the function defined before. These results are in agreement with Table 6.

Changing the above c2 to

```
c2A2:=GroupHomomorphismByImages(V,V,gensV,[gensV[1],gensV[1]*gensV[3],gensV[1]*gensV[2]]);
G1:=Group([c3,c2A2]);
iso:=GroupHomomorphismByImages(G,G1,[(1,2,3),(1,2)(3,4)],[c3,c2A2]);
```

we check the following is true for $n = 1, ..., 10$

$$H^n(A_4, \mathcal{A}_1) = \mathbb{Z}_2^{m[n+1]},$$

$$H^n(A_4, \mathcal{A}_2) = \mathbb{Z}_2^{\delta_{n=0}+\delta_{n>0}m[n-1]},$$

We can further check O_h: using (

```
Oh:=Group((1,2,3),(3,4),(5,6));
```

as a side comment, we can construct its resolution using

```
ROh:=ResolutionFiniteGroup(Oh,6);
```

Then we define

```
w:=GroupHomomorphismByImages(V,V,gensV,gensV{[2,3,1]});
x:=GroupHomomorphismByImages(V,V,gensV,[gensV[1],gensV[2],gensV[1]*gensV[2]*gensV[3]]);
y:=GroupHomomorphismByImages(V,V,gensV,gensV);
oh:=Group([w,x,y]);
iso:=GroupHomomorphismByImages(Oh,oh,[(1,2,3),(3,4),(5,6)],[w,x,y]);
R:=ResolutionFiniteGroup(Oh,10);;
for n in [0,1,2,3,4,5,6,7,8,9] do
Print(Cohomology(C,n),";;");
od;
```

we see that dimension is $0, 1, 3, 5, 8, 12, 16, 21, 27, ...$ which agrees with $1 + n + \sum_{i=0}^{n} f(i-2)$, so we see that the spectral sequence (143) does collapse at E_2. Similarly, by using the `x` for $\mathcal{A}_2$ (see (158)) we get dimension $1, 3, 5, 8, 12, 16, 21, 27, ...$ which agrees with $1 + n + \sum_{i=0}^{n} f(i-1)$, so (145) also collapses at E_2.

As another example, let us calculate the cohomology for $C_{2v} \cong \mathbb{Z}_2^2$ under the action d_5, d_6 and d_4, d_3, relevant for the group No. 42, $Fmm2$ (we are interested in as the E_2 page of the LHS spectral sequence does not collapse). The action d_5, d_6 applies to the $q = 1$ line and d_4, d_3 applies to the $q = 2$ line of the spectral sequence.

The code for $q = 1$ is

```
LoadPackage("HAP");
G:=Group((1,2)(3,4),(1,3)(2,4));
V:=AbelianGroup(IsPcGroup,[2,2,2]);;
gensV:=GeneratorsOfGroup(V);
c2:=GroupHomomorphismByImages(V,V,gensV,[gensV[2],gensV[1],gensV[1]*gensV[2]*gensV[3]]);
m:=GroupHomomorphismByImages(V,V,gensV,[gensV[3],gensV[1]*gensV[2]*gensV[3],gensV[1]]);
G1:=Group([c2,m]);
iso:=GroupHomomorphismByImages(G,G1,[(1,2)(3,4),(1,3)(2,4)],[c2,m]);
action:=function(g,v); return v^Image(iso,g^-1); end;;
A:=GOuterGroup();;
SetActedGroup(A,V);;
SetActingGroup(A,G);;
SetOuterAction(A,action);;
R:=ResolutionFiniteGroup(G,12);;
C:=HomToGModule(R,A);;
Cohomology(C,1);Cohomology(C,2);Cohomology(C,3);Cohomology(C,4);Cohomology(C,5);
Cohomology(C,6);Cohomology(C,7);Cohomology(C,8);Cohomology(C,9);Cohomology(C,10);
```

and we get $H^n(C_{2v}, H^1(\mathbb{Z}^3, \mathbb{Z}_2)) = \mathbb{Z}_2^{n+2}$.

The code for $q = 2$:

$C_2T_1C_2 = T_2T_3^{-1}$, $C_2T_2C_2 = T_1T_3^{-1}$, $C_2T_3C_2 = T_3^{-1}$, $MT_1M = T_2T_3^{-1}$, $MT_2M = T_2$, $MT_3M = T_2T_1^{-1}$,

For $g_i = T_1^{x_i}T_2^{y_i}T_3^{z_i}$ with $i = 1, 2$, define the three translation 2-cocycles as $b_1(g_1, g_2) = y_1z_2$, $b_2(g_1, g_2) = z_1x_2$, and $b_3(g_1, g_2) = x_1y_2$. We have

$$(C_2.b_1)(g_1,g_2) = b_1(C_2g_1C_2, C_2g_2C_2) = b_1((T_2T_3^{-1})^{x_1}(T_1T_3^{-1})^{y_1}T_3^{-z_1}, (T_2T_3^{-1})^{x_2}(T_1T_3^{-1})^{y_2}T_3^{-z_2}) = x_1(x_2{+}y_2{+}z_2) = (b_2{+}b_3)(g_1,g_2)$$

$$(C_2.b_2)(g_1,g_2) = b_2(C_2g_1C_2, C_2g_2C_2) = b_2((T_2T_3^{-1})^{x_1}(T_1T_3^{-1})^{y_1}T_3^{-z_1}, (T_2T_3^{-1})^{x_2}(T_1T_3^{-1})^{y_2}T_3^{-z_2}) = (x_1{+}y_1{+}z_1)y_2 = (b_1{+}b_3)(g_1,g_2)$$

$$(C_2.b_3)(g_1,g_2) = b_3(C_2g_1C_2, C_2g_2C_2) = b_3((T_2T_3^{-1})^{x_1}(T_1T_3^{-1})^{y_1}T_3^{-z_1}, (T_2T_3^{-1})^{x_2}(T_1T_3^{-1})^{y_2}T_3^{-z_2}) = y_1x_2 = b_3(g_1,g_2),$$

$$(M.b_1)(g_1,g_2) = b_1(Mg_1M, Mg_2M) = b_1((T_2T_3^{-1})^{x_1}T_2^{y_1}(T_2T_1^{-1})^{-z_1}, (T_2T_3^{-1})^{x_2}T_2^{y_2}(T_2T_1^{-1})^{-z_2}) = (x_1{+}y_1{+}z_1)x_2 = (b_2{+}b_3)(g_1,g_2),$$

$$(M.b_2)(g_1,g_2) = b_2(Mg_1M, Mg_2M) = b_2((T_2T_3^{-1})^{x_1}T_2^{y_1}(T_2T_1^{-1})^{-z_1}, (T_2T_3^{-1})^{x_2}T_2^{y_2}(T_2T_1^{-1})^{-z_2}) = x_1z_2 = b_2(g_1,g_2),$$

$$(M.b_3)(g_1,g_2) = b_3(Mg_1M, Mg_2M) = b_3((T_2T_3^{-1})^{x_1}T_2^{y_1}(T_2T_1^{-1})^{-z_1}, (T_2T_3^{-1})^{x_2}T_2^{y_2}(T_2T_1^{-1})^{-z_2}) = z_1(x_2{+}y_2{+}z_2) = (b_1{+}b_2)(g_1,g_2),$$

In summary, we have $C_2.(b_1, b_2, b_3) = (b_2 + b_3, b_1 + b_3, b_3)$, and $M.(b_1, b_2, b_3) = (b_2 + b_3, b_2, b_1 + b_2)$, writing the general form as $j_1b_1 + j_2b_2 + j_3b_3$, we have $C_2.(j_1b_1 + j_2b_2 + j_3b_3) = j_2b_1 + j_1b_2 + (j_1 + j_2 + j_3)b_3$ and $M.(j_1b_1 + j_2b_2 + j_3b_3) = j_3b_1 + (j_1 + j_2 + j_3)b_2 + j_1b_3$, therefore

$$C_2.\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}, \qquad M.\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix},$$

```
G:=Group((1,2)(3,4),(1,3)(2,4));
V:=AbelianGroup(IsPcGroup,[2,2,2]);;
gensV:=GeneratorsOfGroup(V);
c2:=GroupHomomorphismByImages(V,V,gensV,[gensV[2]*gensV[3],gensV[1]*gensV[3],gensV[3]]);
m:=GroupHomomorphismByImages(V,V,gensV,[gensV[2]*gensV[3],gensV[2],gensV[1]*gensV[2]]);
G1:=Group([c2,m]);
iso:=GroupHomomorphismByImages(G,G1,[(1,2)(3,4),(1,3)(2,4)],[c2,m]);
action:=function(g,v); return v^Image(iso,g^-1); end;;
A:=GOuterGroup();;
SetActedGroup(A,V);;
SetActingGroup(A,G);;
SetOuterAction(A,action);;
R:=ResolutionFiniteGroup(G,12);;
C:=HomToGModule(R,A);;
Cohomology(C,1);Cohomology(C,2);Cohomology(C,3);Cohomology(C,4);Cohomology(C,5);
Cohomology(C,6);Cohomology(C,7);Cohomology(C,8);Cohomology(C,9);Cohomology(C,10);
```

and we get $H^n(C_{2v}, H^2(\mathbb{Z}^3, \mathbb{Z}_2)) = \mathbb{Z}_2^{\delta_{n=0}+n}$.

The HAP package documentation chater 7 contains many useful comments: `https://gap-packages.github.io/hap/tutorial/chap7.html`, `https://gap-packages.github.io/hap/tutorial/chap11.html`, and this paper by Ellis `https://www.sciencedirect.com/science/article/pii/S0747717104000343`; a Chinese Physics Letters paper is a very good complement to it:`https://arxiv.org/pdf/2005.06572.pdf`

Now we study along the line of the Chinese Physics Letters paper, for the S_4 group. In GAP, using the HAP package one can obtian a resolution for S_4. There are basically two commands we can use:

```
R:=ResolutionFiniteGroup(G,15);
```

or

```
Rsmall:=ResolutionSmallGroup(G,15);
```

I think the `ResolutionFiniteGroup` command uses Algorithm 3.2.2 (see Graham's book P230), which is the classifying space/CW-space method; while the `ResolutionSmallGroup` command uses Algorithm 3.2.1 (P228), which is more related to the bar resolution (but only keeping the linearly independent basis). First all, checking `R!.elts=Rsmall!.elts;` gives true, so the elements of S_4 are numbered the same way in both resolutions.

The `R` has first dimensions (for $n = 0, 1, ..., 12$) $1, 3, 6, 10, 15, 21, 26, 29, 30, 32, 37, 42, 48...$ while `Rsmall` has first dimensions $1, 2, 3, 4, 6, 7, 7, 7, 7, 7, 7, 9, 10....$ Since for S_4 the `ResolutionSmallGroup` command gives significantly smaller chain, we will work with it.

We have $f \colon F \to \bar{F}$, where F is the small resolution from GAP. F is stored in the form of $\cdots \xrightarrow{\partial_{k+1}} R_k \xrightarrow{\partial_k} R_{k-1} \xrightarrow{\partial_{k-1}} \cdots$, where each R_k is several copies of $\mathbb{Z}G$ with basis denoted as $e_k^1, ..., e_k^{\ell_k}$, where $\ell_k =$ `Rsmall!.dimension(k)`. The boundary map ∂_k is accessed via `Rsmall!.boundary(k,j)` for $j = 1, 2, ..., \ell_k$, and the output is `Rsmall!.boundary(k,j)` $= \{\mathbf{a}_1^{k,j}, ..., \mathbf{a}_{\ell_k}^{k,j}\}$ telling the basis and the group element the basis e_k^j maps to.

First, note that S_4 is now labeled as: $g_1 = ()$, $g_2 = (34)$, $g_3 = (23)$, $g_4 = (234)$, $g_5 = (243)$, $g_6 = (24)$, $g_7 = (12)$, $g_8 = (12)(34)$ $g_9 = (123)$, $g_{10} = (1234)$ $g_{11} = (1243)$ $g_{12} = (124)$, $g_{13} = (132)$, $g_{14} = (1342)$, $g_{15} = (13)$, $g_{16} = (134)$ $g_{17} = (13)(24)$, $g_{18} = (1324)$, $g_{19} = (1432)$, $g_{20} = (142)$, $g_{21} = (143)$, $g_{22} = (14)$, $g_{23} = (1423)$, $g_{24} = (14)(23)$.

To work out f for the first few degrees: note that the basis for `Rsmall` are e_0^1, $e_1^{1,2}$, $e_2^{1,2,3}$, $e_3^{1,2,3,4}$, $e_4^{1,2,3,4,5,6}$ and so on. The chain map can be extracted from `Rsmall!.boundary(k,j)`, and we list the first few of them:

$$\begin{aligned}
\partial e_1^1 &= (-g_1 + g_5)e_0^1, \\
\partial e_1^2 &= (-g_1 + g_{23})e_0^1, \\
\partial e_2^1 &= (g_{12} + g_{16} + g_{24})e_1^1, \\
\partial e_2^2 &= (g_9 + g_{10})e_1^1 - (g_5 + g_6 + g_9 + g_{10} + g_{13} + g_{14})e_1^2, \\
\partial e_2^3 &= (g_4 + g_{12} + g_{15} + g_{19})e_1^2, \\
\partial e_3^1 &= (g_{10} - g_{14})e_2^1, \\
\partial e_3^2 &= (-g_3 + g_5)e_2^2, \\
\partial e_3^3 &= (-g_4 + g_7)e_2^3, \\
\partial e_3^4 &= -(g_9 + g_{10} + g_{12} + g_{15} + g_{16} + g_{18} + g_{23} + g_{24})e_2^1 \\
&\quad + (g_1 + g_3 + g_4 + g_7 + g_9 + g_{10} + g_{13} + g_{14} + g_{15} + g_{16} + g_{17} + g_{18})e_2^2 \\
&\quad + (g_7 + g_8 + g_9 + g_{10} + g_{11} + g_{12} + g_{13} + g_{14} + g_{15} + g_{16} + g_{17} + g_{18} + g_{19} + g_{20} + g_{21} + g_{22} + g_{23} + g_{24})e_2^3, \\
\partial e_4^1 &= -(g_{10} + g_{24})e_3^2, \\
\partial e_4^2 &= -(g_1 + g_{12} + g_{20})e_3^1, \\
\partial e_4^3 &= (g_1 + g_{10} + g_{17} + g_{19})e_3^3, \\
\partial e_4^4 &= (-g_3 + g_4)e_3^1 - (g_3 + g_{13})e_3^2 + (-g_1 + g_2)e_3^4, \\
\partial e_4^5 &= (-g_2 + g_5 + g_{13} - g_{15})e_3^1 - (g_2 + g_{16})e_3^2 + (-g_1 + g_3)e_3^4, \\
\partial e_4^6 &= (-g_3 + g_{13})e_3^1 - (g_{11} + g_{12})e_3^2 + (g_1 + g_2 - g_7 - g_8 - g_{11} - g_{12} - g_{13} - g_{14})e_3^3 + (-g_1 + g_7)e_3^4,
\end{aligned} \tag{159}$$

and so on. We checked that $\partial^2 = 0$. For example:

$$\partial^2 e_2^2 = \partial e_1^1(g_9 + g_{10}) - \partial e_1^2(g_5 + g_6 + g_9 + g_{10} + g_{13} + g_{14}) = (-g_1 + g_5)(g_9 + g_{10}) - (-g_1 + g_{23})(g_5 + g_6 + g_9 + g_{10} + g_{13} + g_{14}) = 0.$$

Note: here it is very important to formally put the basis e_k^i on the left of the group elements: only this way after taking differential the newly obtained group elements appear on the left of the existing group elements, and only this way the product of group elements agree with the usual convention. In other words, it seems to be more appropriate to write

$$\begin{aligned}
\partial e_1^1 &= e_0^1(-g_1 + g_5), \\
\partial e_1^2 &= e_0^1(-g_1 + g_{23}), \\
\partial e_2^1 &= e_1^1(g_{12} + g_{16} + g_{24}), \\
\partial e_2^2 &= e_1^1(g_9 + g_{10}) - e_1^2(g_5 + g_6 + g_9 + g_{10} + g_{13} + g_{14}), \\
\partial e_2^3 &= e_1^2(g_4 + g_{12} + g_{15} + g_{19})
\end{aligned} \tag{160}$$

and so on. Accordingly,

Using $f_k(e_k^i) = s_{k-1}f_{k-1}(\partial e_k^i)$, we have (note below has used mod-2 reduction)

$$\begin{aligned}
f_1(e_1^1) &= s_0 f_0(\partial e_1^1) = s_0 f_0((-g_1+g_5)e_0^1) = s_0((-g_1+g_5)[]) = -[g_1]+[g_5],\\
f_1(e_1^2) &= s_0 f_0(\partial e_1^2) = s_0 f_0((-g_1+g_{23})e_0^1) = s_0((-g_1+g_{23})[]) = -[g_1]+[g_{23}],\\
f_2(e_2^1) &= s_1 f_1(\partial e_2^1) = s_1 f_1((g_{12}+g_{16}+g_{24})e_1^1) = s_1((g_{12}+g_{16}+g_{24})(-[g_1]+[g_5]))\\
&= -[g_{12}|g_1]-[g_{16}|g_1]-[g_{24}|g_1]+[g_{12}|g_5]+[g_{16}|g_5]+[g_{24}|g_5],\\
f_2(e_2^2) &= s_1 f_1(\partial e_2^2) = s_1 f_1((g_9+g_{10})e_1^1-(g_5+g_6+g_9+g_{10}+g_{13}+g_{14})e_1^2)\\
&= s_1((g_9+g_{10})(-[g_1]+[g_5])-(g_5+g_6+g_9+g_{10}+g_{13}+g_{14})(-[g_1]+[g_{23}]))\\
&= [g_5|g_1]+[g_6|g_1]+[g_{13}|g_1]+[g_{14}|g_1]+[g_9|g_5]+[g_{10}|g_5]\\
&\quad -[g_5|g_{23}]-[g_6|g_{23}]-[g_9|g_{23}]-[g_{10}|g_{23}]-[g_{13}|g_{23}]-[g_{14}|g_{23}],\\
f_2(e_2^3) &= s_1 f_1(\partial e_2^3) = s_1 f_1((g_4+g_{12}+g_{15}+g_{19})e_1^2) = s_1((g_4+g_{12}+g_{15}+g_{19})(-[g_1]+[g_{23}]))\\
&= -[g_4|g_1]-[g_{12}|g_1]-[g_{15}|g_1]-[g_{19}|g_1]+[g_4|g_{23}]+[g_{12}|g_{23}]+[g_{15}|g_{23}]+[g_{19}|g_{23}],
\end{aligned} \tag{161}$$

We will also need the other contracting homotopy. However, the resolution above, constructed via `ResolutionSmallGroup`, does not give a contracting homotopy. Instead, the othe rresolution `ResolutionFiniteGroup` does give a contracting homotopy. We will use that to construct the map – see later.

For $[\bar{\alpha}] \in H^k(G, M)$ corresponding to the (normalized) bar resolution, we have $\bar{\alpha} \in \mathrm{Hom}_{\mathbb{ZG}}(\bar{F}_k, M)$. Using $f\colon F \to \bar{F}$ we have $f^*\colon \mathrm{Hom}_{\mathbb{ZG}}(\bar{F}_k, M) \to \mathrm{Hom}_{\mathbb{ZG}}(F_k, M)$, then for $x \in F_k$, $(f^*(\bar{\alpha}))(x) = \bar{\alpha}(f(x))$, this way we can convert a cocycle that is a inhomogeneous function (i.e. obtained from the normalized bar resolution) to a cocycle in F.

We can also use the map f to see if a cocycle is a genuine one one a coboundary. we can construct the most general coboundary β using the (normalized) bar resolution: write the map $\bar{\beta}$ to get $\bar{\alpha}_0(y) = (d\bar{\beta})(y)$ for $y \in \bar{y}$ Then the question is if there is a solution for $\bar{\alpha}_0(f(x)) = \bar{\alpha}(f(x))$.

We tried using the coboundary of the (normalized) bar resultion $\bar{F}$ (Basically any set-theoretical function $\bar{\omega}(g_1, ..., g_{n-1})$ and make $(d\bar{\omega})(g_1, g_2, ..., g_n))$ and map it to F via $f\colon F \to \bar{F}$ (using the induced map $f^*\bar{\omega}$). We hope that the condition we used to check whether a cocycle $\bar{\alpha}$ is a coboundary (whether there exists $\bar{\beta}$ s.t. $\bar{\alpha} = d\bar{\beta}$) can be directly mapped to F and solve it. However, it seems that the image of the function in F cannot serve as the criterion that a cocycle is a coboundary (see mathematica for detail).

To check whether a cocycle is a coboundary (in F), let's use the criterion outlined in the Chinese Physics Letters paper: suppose $\alpha \in \mathrm{Hom}_{\mathbb{Z}G}(F_k, M)$, the task is to determine whether there exists $\beta \in \mathrm{Hom}_{\mathbb{Z}G}(F_{k-1}, M)$ s.t. $\alpha = d\beta$. We take $\langle \alpha, e_k^i \rangle = \langle d\beta, e_k^i \rangle$, which gives $\langle \alpha, e_k^i \rangle = \langle \beta, \partial e_k^i \rangle$. Recall that $\partial e_k^i = \sum_j e_{k-1}^j g_{s_j^i}$, so $\langle \alpha, e_k^i \rangle = \sum_j \langle \beta, e_{k-1}^j g_{s_j^i} \rangle$, which is $\langle \alpha, e_k^i \rangle = \sum_j R(g_{s_j^i}) \langle \beta, e_{k-1}^j \rangle$, where $R(g)$ is the matrix corresponding to g, obtained from the action of G on M. Note that we have used the fact that $\langle \alpha, gx \rangle = g.\langle \alpha, x \rangle$ for $x \in F_k$ and $\alpha \in \mathrm{Hom}_{\mathbb{Z}G}(F_k, M)$. From this we can construct the matrix A as mentioned in Eq. (47) of the paper. To be more precise: we construct the following matrix $A_{\deg(k-1),\deg(k)}$, where for any GAP entry `[j,l]` that lies in `Rsmall!.boundary(k,i);`, we write

$$[A_{\deg(k-1),\deg(k)}]_{3j:3j+3,3i:3i+3} \texttt{ += } R(g_l), \tag{162}$$

note the `+=` sign as in `Rsmall!.boundary(k,i);` there are usually several entries with the same j. j runs over the basis of F_{k-1} while i runs over the basis of F_k. After obtaining $A_{\deg(k-1),\deg(k)}$, we do Smith decomposition for the transpose of $A_{\deg(k-1),\deg(k)}$:

$$L\left(A_{\deg(k-1),\deg(k)}\right)^T R = \Lambda, \tag{163}$$

where Λ is diagonal and L, R are matrices with integer entries. Following the Chinese paper, we check the zero diagonals of Λ, say the last r entries of the diagonal of Λ; then we look at the last r entries of the vector

$$L\boldsymbol{a}, \tag{164}$$

where $\boldsymbol{a}$ is the value of the cocycle $\bar{\alpha}$ (that we want to check wether is a genuine one), i.e. $\boldsymbol{a} = \langle \bar{\alpha}, e_k^i \rangle$ for $i = 1, 2, ...$, (note that for our $H^*(S_4, \mathcal{A}_1)$ calculation each $\langle \bar{\alpha}, e_k^i \rangle$ is a three component vector so we just flatten all these vectors as $i = 1, 2, ...$ to get $\boldsymbol{a}$). If the last r entries of $L\boldsymbol{a}$ are all zero, then $\bar{\alpha}$ is a coboundary, not a cocycle; if at least one of the last r entries of $L\boldsymbol{a}$ is nonzero then $\bar{\alpha}$ is a genuine cocycle.

Using the above method, we have checked that v^3 and $v\psi$ are the same cocycle, and that v^4 and ψ^2 are also the same cocycle.

So we have $v^3 = v\psi$ and $v^4 = \psi^2$.

General lesson learned: using contracting homotopy, one can establish the mapping between the bar resolution and the resolution given by GAP. The latter has the advantage of being very small while the former has the advantage of having an explicit cup product. Furthermore, there is an effiicient algorithm (given above) in the latter to check whether a cocycle is a coboundary. By combining all these tools (and furthermore, the spectral sequences), we have a complete way to determine the cohomology ring structure.

Now let us construct the map from the bar resolutino to the resolution given by `ResolutionFiniteGroup`, using the contracting homotopy provided by the latter. Denote this map by $g\colon \bar{F} \to F$. Then we have

$$g([g_i]) = s_0 g_0(\partial[g_i]) = s_0 g_0(g_i[] - []) = s_0(g_i e_0^1 - e_0^1) = s_0(g_i e_0^1) - s_0(e_0^1),$$

note that the contracting homotopy in GAP is given in the form of $s_k(g_i e_k^j) =$ `R!.homotopy(k,[j,i])`, where $i = 1, 2, ..., 24$ runs over all the elements of S_4.

We have written code in Mathematica so that a cocycle in F (obtained now from `ResolutionFiniteGroup`) can be converted using the contracting homotopy of F to an inhomogeneous function associated with the bar resolution $\bar{F}$. We first find the most general form for degree-3-cocycle in F, then map it to $\bar{F}$, then map it to F' which is obtained from `ResolutionSmallGroup`. Then we were able to check that τ is a genuine cocycle and that $\upsilon\tau$ is also a genuine cocycle. So so far we have

$$n = 1\colon \langle \upsilon \rangle; \quad n = 2\colon \langle \upsilon^2, \psi \rangle; \quad n = 3\colon \langle \upsilon^3 = \upsilon\psi, \tau \rangle; \quad n = 4\colon \langle \upsilon^4 = \psi^2, \upsilon\tau, \kappa \rangle,$$

where we have introduced the 4-cocycle κ. For $n = 5$, upon checking we guess

$$n = 5\colon \langle \upsilon^5, \upsilon\kappa = \upsilon^2\tau = \psi\tau, \varpi, \omega \rangle$$

For $n = 6$, we should have $\upsilon^6, \upsilon^2\kappa, \psi\kappa, \upsilon\varpi, \upsilon\omega, \tau^2$, we know only four of them survives. Let's see. We get $\tau^2 = \upsilon\omega = \upsilon\varpi$ abd $\upsilon^2\kappa = \psi\kappa$, so we have

$$n = 6\colon \langle \upsilon^6, \upsilon^2\kappa = \psi\kappa, \tau^2 = \upsilon\omega = \upsilon\varpi, \xi \rangle$$

where we have introduced a new generator ξ. Then for $n = 7$: we need to check υ^7, $\upsilon^3\kappa$, $\upsilon\tau^2$, $\upsilon\xi$, $\psi\varpi$, $\psi\omega$, and $\tau\kappa$. It turns out we have $\upsilon\tau^2 = \upsilon\xi = \psi\varpi = \psi\omega = \tau\kappa$, so we have

$$n = 7\colon \langle \upsilon^7, \upsilon^3\kappa, \upsilon\tau^2 = \upsilon\xi = \psi\varpi = \psi\omega = \tau\kappa, \sigma, \varsigma \rangle,$$

where there are two new generators, σ and ς.

We find at $n = 8$ the following relation: $\tau\omega = \tau\varpi$, $\upsilon^2\tau^2 = \kappa^4$; the following four are independent: $\upsilon^8, \upsilon^4\kappa, \upsilon^2\tau^2, \tau\omega$. For the other two, $\upsilon\sigma$ and $\upsilon\varsigma$, we have not checked whether any relation exists among them and $\upsilon^8, \upsilon^4\kappa, \upsilon^2\tau^2, \tau\omega$ exist as the computation takes too much time. We will come back to it later.

Conjecture (the $n = 8$ seems to be against this):

$$H^*(S_4, \mathcal{A}_1) = \mathbb{F}_2[\upsilon, \psi, \tau] \cup [\mathbb{F}_2[a, b]^{\mathbb{Z}_3}]^{\deg \to \deg + 2}/(\sim),$$

where the action $\mathbb{Z}_3$ on a, b is $a \mapsto b, b \mapsto a + b$; the relation $\sim$ is $\psi\upsilon = \upsilon^3$ and $\psi^2 = \upsilon^2$ and that that any elements of the $[\mathbb{F}_2[a, b]^{\mathbb{Z}_3}]^{\deg \to \deg + 2}$ must equal to the element in $\mathbb{F}_2[\upsilon, \tau]$ with the highest power of τ.

For our convenience it's idea to construct the space group from scratch:

```
T1:=[[1, 0, 0, 0], [0, 1, 0, 1/2], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T2:=[[1, 0, 0, 1/2], [0, 1, 0, 0], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T3:=[[1, 0, 0, 1/2], [0, 1, 0, 1/2], [0, 0, 1, 0], [0, 0, 0, 1]];
C3:=[[0, 0, 1, 0], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 0, 1]];
P:=[[-1, 0, 0, 0], [0, -1, 0, 0], [0, 0, -1, 0], [0, 0, 0, 1]];
C2:=[[-1, 0, 0, 1/4], [0, -1, 0, 1/4], [0, 0, 1, 0], [0, 0, 0, 1]];
C2p:=[[-1, 0, 0, 1/4], [0, 1, 0, 0], [0, 0, -1, 1/4], [0, 0, 0, 1]];
M:=[[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]];
G:=Group(T1,T2,T3,C2,C2p,C3,M,P);
Gp:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G))));
R:=ResolutionAlmostCrystalGroup(Image(Gp),5);
```

The last command gives a resolution for the space group of length 5, with contracting homotopy. The second last command gives

$[(PC_2')^T, (MC_3)^T, C_3^T, (C_2C_2)'^T, C_2^T, T_2^T, T_1^T, (T_1T_2T_3^{-1})^T]]$ `-> [ g1, g2, g3, g4, g5, g6, g7, g8 ]`.

So we have (using $MC_3 = C_3^2M$, $(PC_2')^{2m} = T_2^{-m}$ and $(PC_2')^{2m+1} = PT_2^mC_2'$ so that $(PC_2')^n = P^{\mathrm{Mod}_2(n)}T_2^{(2\mathrm{Mod}_2(n)-1)\frac{n-\mathrm{Mod}_2(n)}{2}}C$

$$\begin{aligned} & g_1^{n_1}g_2^{n_2}g_3^{n_3}g_4^{n_4}g_5^{n_5}g_6^{n_6}g_7^{n_7}g_8^{n_8} \\ = & \left(T_1^{n_8+n_7}T_2^{n_8+n_6}T_3^{-n_8}C_2^{n_5+n_4}C_2'^{n_4}C_3^{n_3+2n_2}M^{n_2}P^{\mathrm{Mod}_2(n_1)}T_2^{(2\mathrm{Mod}_2(n_1)-1)\frac{n_1-\mathrm{Mod}_2(n_1)}{2}}C_2'^{\mathrm{Mod}_2(n_1)}\right)^T \end{aligned} \tag{165}$$

which can be find with the help of Eq. (94). See Mathematica for result.

Note that when computing No. 141 ($I4_1/amd$), it turns out we have $[(PC_2'C_2)^T, (M)^T, C_2'^T, C_2^T, T_2^T, T_1^T, (T_1T_2T_3^{-1})^T]]$ `-> [ g1,` hence the only nontrivial task is $(PC_2'C_2)^{2m+1} = P^{\mathrm{Mod}_2(n)}T_1^{(2\mathrm{Mod}_2(n)-1)\frac{n-\mathrm{Mod}_2(n)}{2}}C_2C_2'$,

To get contracting homotopy, we use (below we have executed `R:=ResolutionAlmostCrystalGroup(Image(Gp),4);` which gives `R!.elts` of length 1621):

```
homotopy0:=List([1],j->List([1..1621],i->R!.homotopy(0,[j,i])));

homotopy1:=List([1..6],j->List([1..1621],i->R!.homotopy(1,[j,i])));

homotopy2:=List([1..18],j->List([1..1621],i->R!.homotopy(2,[j,i])));
```

It is worth looking at the function `ResolutionAlmostCrystalGroup` a bit in more detail, give it is the only one that is capable of producing (a partial list of) contracting homotopy. Such a function can be found in `https://github.com/gap-packages/hap/blob/e5e5a27ef66790490847b695924231d6a798744f/lib/Resolutions/resACgroup.gi#L4`. The only input is the crystallographic group (in pcp format) `G` and the order of resolution `K`. Then this function automatically gives the point group `P`, the translation group `T`, their respective resolutions `RP` and `RT`, the sujective mapping from space group to point group, `GhomP`. Note that the point group resolution `RP` is obtained using `RP:=ResolutionFiniteGroup(PC,K);`. Note that using `GhomP:=NaturalHomomorphismOnHolonomyGroup(Image(Gp))` and `P:=Image(GhomP);` and `PhomPC:=IsomorphismP` and `PC:=Image(PhomPC);` and `RP:=ResolutionFiniteGroup(PC,4);` we can get the resolution of point group used in `ResolutionAlmostCrystalGroup`. We see it's dimension `List([1..4],x->RP!.dimension(x));` is `[ 3, 6, 11, 18 ]`, which is a different resolution from the one obtained using `Oh:=Group((1,2,3),(3,4),(5,6))` and `Oh:=ResolutionFiniteGroup(Oh` which gives resolution dimensions `[4,10,20,35]` and so on. On the other hand, the finite group resolution associated with `oh:=SmallGroup(48,48);` has dimension `[4,10,20,36]`, so all these are different resolutions produced by Gap.

Back to the point group resolution given in `ResolutionAlmostCrystalGroup`, which has dimension for $p = 0, 1, 2, 3, 4$: `[1, 3, 6, 11, 18 ]`. The translation has dimension `[1,3,3,1]` so we have, for the tensor dimension at (p, q): $(0, 0)$: $1, (0, 1) = (1, 0) : 3, (0, 2) : 3, (1, 1) : 9, (2, 0) : 6, (0, 3) : 1, (1, 2) : 9, (2, 1) : 18, (3, 0) : 11$, and so on, and therefore the dimension for $n = p + q$ is `[1,6,18,39,72]`, and so on. This is indeed confirmed in gap by looking at `List([1..4],x->R!.dimension(x));`.

Then it uses the function `ResolutionExtension(GhomP,RT,RP,"Don't Test Finiteness");`) to output the desired resolution for G. Therefore our next focus is on this function `ResolutionExtension(GhomP,RT,RP,"Don't Test Finiteness");`). This function is given at `https://github.com/gap-packages/hap/blob/e5e5a27ef66790490847b695924231d6a798744f/lib/Perturbations/resExtension.gi#L4`. Note that the following website constains basic information about group cohomology of space groups: `https://docs.gap-system.org/pkg/hap/www/SideLinks/About/aboutSpaceGroup.html`.

Note that now we have `EEhomGG:=arg[1];`, `RN:=arg[2];`, and `RG:=arg[3];`. Now the total group has been called `E:=Source(EEhomGG);`, where the quotient group is called `G:=Image(EEhomGG);` and the translation is now called `N:=RN!.group;`.

We see that in `ResolutionExtension(GhomP,RT,RP,"Don't Test Finiteness");)` there is a line: `for gn in GeneratorsOfGr` which tells how elements of the total group are appended to the group element list. Then there is this `EltsE:=SSortedList(EltsE);` which sorts the elements, and the additional three lines (we omit them here) makes sure that the identity element is always the very first element.

The out put of `ResolutionExtension(GhomP,RT,RP,"Don't Test Finiteness");)` is `T`, which is calculated via `T:=TwistedTensorProduct(RG,RN,EhomG,GmapE,NhomE,NEhomN,EltsE,MultE,InvE);`.

The only thing between this line and the `return T;` line is appending element line `T!.appendToElts:=AppendToElts;` Therefore we will care about the function `TwistedTensorProduct`, which can be accessed in `https://github.com/gap-packages/hap/blob/e5e5a27ef66790490847b695924231d6a798744f/lib/Perturbations/twistedTensorProduct.gi`.

Note that in the function `TwistedTensorProduct`, there inputs are `R,S,EhomG,GmapE,NhomE,NEhomN,EltsE,Mult,InvE`. We see that `R` is the resolution `RG`, i.e. resolution of the point group, where `S` is the resolution `RN`, i.e. the resolution of the translation. The only type that the point group contracting homotopy is substantially used is this line `hty1:=HomotopyR(p,[r,g1]);` where there are three types the contracting homotopy of the space group is used: twice `r:=ShallowCopy(HomotopyS(q,[tensor[2],g]));`, and once `hty:=HomotopyS(q,[s,g2]);`.

Here the important function is `HomotopyGradedGen:=function(g,p,q,r,s,bool)`: here `p` and `q` specify the entry, and `r` and `s` specify the index of basis for the `p` and `q` entries, respectively; and `g` is an element of the total group `E`. The function starts with `g1:=EhomG(g);g2:=EmapN(g);Eg1:=GmapE(g1);Eg2:=NhomE(g2);`, where `g1` lives in the point group and `g2` lives in translation, and `Eg1`, `Eg2` are like their liftings.

Note that the final output for contracting homotopy is given in the line `homotopy:=FinalHomotopy`, which is a function, that outputs the function `Homtpy(n,x)` whenever we have the contracting homotopy available for both the translation and point groups. Therefore we just need to focus on the function `Homtpy`. Note that `FinalHomotopy` uses `Homtpy` once, which uses the function `HomotopyOfWord`, which makes use again the function `Homtpy`, i.e. `Homtpy` is applied recursively.

To start with, let's first take down the homotopy and boundary map for translations. The translations are obtaine by taking

`GhomP:=NaturalHomomorphismOnHolonomyGroup(Image(Gp));`

which out puts `[ g1, g2, g3, g4, g5, g6, g7, g8 ] -> [ g1, g2, g3, g4, g5, id, id, id ]`; then `P:=Image(GhomP);` and `T:=Kernel(GhomP);` gives the point group and translation group. Next we take nilpotent group resolution to get the resolution for translation:

`RT:=ResolutionNilpotentGroup(T,K);`

where we set e.g. `K=5`. the resolution `RT` has dimension 1,3,3,1; and the group elements `RT!.elts` gives `[ id, g6, g7, g8 ]`; We then have `RT!.boundary(3,1)` gives `[ [ -3, 1 ], [ 3, 2 ], [ 2, 1 ], [ -2, 3 ], [ -1, 1 ], [ 1, 4 ] ]`, meaning that

$$e_3 \xrightarrow{\partial} (g_8-1)e_2^1 + (g_7-1)e_2^2 + (g_6-1)e_2^3, \tag{166}$$

and `List([1..3],x->RT!.boundary(2,x));` gives `[ [ [ -2, 1 ], [ 2, 2 ], [ 1, 1 ], [ -1, 3 ] ], [ [ -3, 1 ], [ 3,` which are

$$\begin{aligned} e_2^1 &\xrightarrow{\partial} (-g_7+1)e_1^1 + (g_6-1)e_1^2, \\ e_2^2 &\xrightarrow{\partial} (-g_8+1)e_1^1 + (g_6-1)e_1^3, \\ e_2^3 &\xrightarrow{\partial} (-g_8+1)e_1^2 + (g_7-1)e_1^3, \end{aligned} \tag{167}$$

Then `List([1..3],x->RT!.boundary(1,x));` gives `[ [ [ -1, 1 ], [ 1, 2 ] ], [ [ -1, 1 ], [ 1, 3 ] ], [ [ -1, 1 ],` i.e.

$$\begin{aligned} e_1^1 &\xrightarrow{\partial} (g_6-1)e_0, \\ e_1^2 &\xrightarrow{\partial} (g_7-1)e_0, \\ e_1^3 &\xrightarrow{\partial} (g_8-1)e_0, \end{aligned} \tag{168}$$

and finally one can check `RT!.boundary(0,1);` that $e_0 \xrightarrow{\partial} 0$.

Then, contracting homotopy. Using `List([1..4],x->RT!.homotopy(0,[1,x]));` and so on we get

$$\begin{aligned} e_0 \to 0, \quad g_6e_0 \to e_1^1, \quad g_7e_0 \to e_1^2, \quad g_8e_0 \to e_1^3, \\ e_1^1 \to 0, \quad g_6e_1^1 \to 0, \quad g_7e_1^1 \to -e_2^1, \quad g_8e_1^1 \to -e_2^2, \\ e_1^2 \to 0, \quad g_6e_1^2 \to 0, \quad g_7e_1^2 \to 0, \quad g_8e_1^2 \to -e_2^3, \\ e_1^3 \to 0, \quad g_6e_1^3 \to 0, \quad g_7e_1^3 \to 0, \quad g_8e_1^3 \to 0, \\ e_2^1 \to 0, \quad g_6e_2^1 \to 0, \quad g_7e_2^1 \to 0, \quad g_8e_2^1 \to e_3, \\ e_2^2 \to 0, \quad g_6e_2^2 \to 0, \quad g_7e_2^2 \to 0, \quad g_8e_2^2 \to 0, \\ e_2^3 \to 0, \quad g_6e_2^3 \to 0, \quad g_7e_2^3 \to 0, \quad g_8e_2^3 \to 0, \\ e_3 \to 0, \quad g_6e_3 \to 0, \quad g_7e_3 \to 0, \quad g_8e_3 \to 0. \end{aligned} \tag{169}$$

We can extend ot to obtain a full contracting homotopy:

$$ge_0 \to \begin{cases} e_1^3, & T_3 \in g \\ e_1^2, & T_3 \notin g, T_2 \in g \\ e_1^1, & T_3 \notin g, T_2 \notin g, T_1 \in g \\ 0, & g = id \end{cases} \tag{170}$$

$$ge_1^1 \to \begin{cases} -e_2^2, & T_3 \in g \\ -e_2^1, & T_3 \notin g, T_2 \in g \\ 0, & g = \text{rest} \end{cases} \qquad ge_1^2 \to \begin{cases} -e_2^3, & T_3 \in g \\ 0, & g = \text{rest} \end{cases} \qquad ge_1^3 \to 0 \tag{171}$$

$$ge_2^1 \to \begin{cases} -e_3, & T_3 \in g \\ 0, & g = \text{rest} \end{cases} \qquad ge_2^2 \to 0, \qquad ge_2^3 \to 0. \tag{172}$$

I think if we use the polycyclic notation (i.e. if we define symmetry matrices and then convert to pcp group and use `ResolutionAlmostCrystalGroup` then the translation part should always be as above (the only change is the subscript of translation generators $g_{i+1}, g_{i+2}, g_{i+3}$, where i is the number of point group generators).

With this it is obvious for arbitrary $g \in T$, how the homotopy looks like.

A few more functions: first set the notation as in `twistedTensorProduct`, `https://github.com/gap-packages/hap/blob/e5e5a27ef66790490847b695924231d6a798744f/lib/Perturbations/twistedTensorProduct.gi`. Recall that total group is called `E`, translation `N` and point group `G`. `p` labels row which runs the resolution `R` for the point group `G` and translation resolution is called `S`. The resolution for the total group `E` at place `n` is formed by all the tensor product of basis of the two smaller resolutions at all $p+q = n$. The indexing can be see in the functions `DimPQ`, `Int2Pair` and `Pair2Int`: the sum if over `for j in [0..q] do`, meaning that the increasing indexing gomes for $q = 0$ first, then $q = 1$, then $q = 2$, and so on; for the m'th basis, $0 \leq m \leq n-1$, we have the unique decomposition $m =$ `DimPQ(p+1,q-1)` $+ r \cdot$ `DimensionS(q)` $+ s$, i.e. $m \to (p, q, r, s)$, where (r, s) describes the indexing within the $E_0^{p,q}$ entry, with the convention that it takes $e_p^r \otimes e_q^s$, where $e_p^r \in$ `R` is a basis vector for the point group resolution and $e_q^s \in$ `S` is a basis vector for translation group resolution. Note that gap indexes things from `1` onwards.

Note that the following commend can access the cocycles. As an example, consider F222:

```
T1:=[[1, 0, 0, 0], [0, 1, 0, 1/2], [0, 0, 1, 1/2], [0, 0, 0, 1]];;

T2:=[[1, 0, 0, 1/2], [0, 1, 0, 0], [0, 0, 1, 1/2], [0, 0, 0, 1]];;
T3:=[[1, 0, 0, 1/2], [0, 1, 0, 1/2], [0, 0, 1, 0], [0, 0, 0, 1]];;
C2:=[[-1, 0, 0, 1/4], [0, -1, 0, 1/4], [0, 0, 1, 0], [0, 0, 0, 1]];;
C2p:=[[-1, 0, 0, 1/4], [0, 1, 0, 0], [0, 0, -1, 1/4], [0, 0, 0, 1]];;
G:=Group(T1,T2,T3,C2,C2p);
Gp:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G))));
R:=ResolutionAlmostCrystalGroup(Image(Gp),10);
List([1..8],x->Cohomology(HomToIntegers(R),x));
```

which gives the cohomology with $\mathbb{Z}$ coefficient. The first ones are $n = 0$: $\mathbb{Z}$, $n = 1$: 0, $n = 2$: $\mathbb{Z}_2^4$, $n = 3$: $\mathbb{Z}_2^2\mathbb{Z}_4\mathbb{Z}$, $n = 4$: $\mathbb{Z}_2^6$ and so on. Let's continue towards getting the cocycles:

```
TR:=TensorWithIntegers(R);;

M1:=CocycleCondition(R,1);;
M2:=CocycleCondition(R,2);;
M3:=CocycleCondition(R,3);;
```

and so on. Note that these matrices `M1,M2,M3` are <u>not</u> themselves the coboundary matrices. In fact, as we checked, these matrices are the row reduced versions of the boundary matrices that we get from the boundary maps, which give the same nullspaces, i.e. they do give the correct cocycles, including all the coboundaries. These cocycles are be obtained by

```
SolutionsMod2:=NullspaceModQ(TransposedMat(M3),2);
```

keep in mind again that these include all the coboundaries. But the true cocycles are easy to obtain once we transcribe these to Mathematica.

Then, after finding a cocycle which we call `uCocycle`, one can use the following to obtain the cup product of it with another cocycle say `vCocycle`. We use the basics of gap system to define such a function (See `https://docs.gap-system.org/doc/ref/chap4.html#X815F71EA7BC0EB6F` for how to define a function).

```
ucupv:=function(R,uCocycle,vCocycle,p,q)

local w,sw,i,x,uvCocycle;
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,p,q);
uvCocycle:=[];
for i in [1..R!.dimension(p+q)] do
```

```
w:=uChainMap([[i,1]]);
sw:=0;
for x in w do
sw:=sw+SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
od;
uvCocycle[i]:=sw;
od;
return uvCocycle;
end;;
```

where on the first line one substitute `uCocycle` with any cocycle to be cup with, and `p` is its degree; then the other cocycle to cup is denoted `vCocycle`, having degree `q`.

See the function `cupProduct` at the page `https://github.com/gap-packages/hap/blob/47189249ba64d39a4d4e4f1afeeb5a72` `lib/Rings/cupProduct.gi#L5` for details.

23.1 Final results for $Fd\bar{3}m$

Here we look at the maximal symmetry group $Fd\bar{3}m$. This group is generated by three translations $T_{1,2,3}$, two twofold rotations C_2 and C_2' along two perpendicular axes with axis $(1/8, 1/8, 0) + \hat{z}$ and $(1/8, 0, 1/8) + \hat{y}$, respectively, a C_3 rotation, a mirror reflection M and inversion I. They act on the coordinates (x, y, z) in the following manner:

$$\begin{aligned}
C_2\colon (x,y,z) &\to (-x+1/4, -y+1/4, z),\\
C_2'\colon (x,y,z) &\to (-x+1/4, y, -z+1/4),\\
C_3\colon (x,y,z) &\to (z,x,y),\\
M\colon (x,y,z) &\to (y,x,z),\\
I\colon (x,y,z) &\to (-x,-y,-z).
\end{aligned} \tag{173}$$

We have the following relations among the generators,

$$\begin{aligned}
&C_2T_1C_2 = T_3^{-1}T_2, \quad C_2T_2C_2 = T_3^{-1}T_1, \quad C_2T_3C_2 = T_3^{-1},\\
&C_2'T_1C_2' = T_2^{-1}T_3, \quad C_2'T_2C_2' = T_2^{-1}, \quad C_2'T_3C_2' = T_2^{-1}T_1, \quad C_2C_2' = C_2'C_2,\\
&C_3T_1C_3^{-1} = T_2, \quad C_3T_2C_3^{-1} = T_3, \quad C_3T_3C_3^{-1} = T_1, \quad C_3C_2C_3^{-1} = C_2C_2', \quad C_3C_2'C_3^{-1} = C_2,\\
&MT_1M = T_2, \quad MT_2M = T_1, \quad MT_3M = T_3, \quad MC_3M = C_3^{-1}, \quad MC_2M = C_2, \quad MC_2'M = C_2C_2',\\
&IT_1I = T_1^{-1}, \quad IT_2I = T_2^{-1}, \quad IT_3I = T_3^{-1}, \quad IC_2I = T_3^{-1}C_2, \quad IC_2'I = T_2^{-1}C_2', \quad IC_3I = C_3, \quad IMI = M.
\end{aligned} \tag{174}$$

An arbitrary element in $Fd\bar{3}m$ can be written as $g = T_1^xT_2^yT_3^zC_2^c(C_2')^{c'}(C_3)^dM^mI^i$, with $x, y, z \in \mathbb{Z}$, $c, c', m, i \in \{0, 1\}$ and $d \in \{0, 1, 2\}$.

The $\mathbb{Z}_2$ cohomology ring of $Fd\bar{3}m$ is given by

$$H^*(Fd\bar{3}m, \mathbb{Z}_2) = \mathbb{Z}_2[A_m, A_i, B_\delta, B_{\beta+\phi}, C_{s\psi}, C_{n\gamma}]/\langle \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6\rangle, \tag{175}$$

where the relations are

$$\mathcal{R}_3\colon \quad A_iB_\delta = 0, \tag{176a}$$

$$\mathcal{R}_4\colon \quad A_iC_{s\psi} = A_mC_{s\psi} = 0, \quad B_{\beta+\phi}^2 = B_{\beta+\phi}A_i(A_m + A_i), \quad B_{\beta+\phi}B_\delta + A_mC_{n\gamma} = A_iC_{n\gamma} = 0, \tag{176b}$$

$$\mathcal{R}_5\colon \quad B_{\beta+\phi}C_{s\psi} = B_{\beta+\phi}C_{\gamma'} = 0, \tag{176c}$$

$$\mathcal{R}_6\colon \quad C_{\gamma'}^2 = C_{s\psi}C_{\gamma'}. \tag{176d}$$

6+4+4+2+2+2+5+5 = 30-5-3-2-2-1 = 17 - 2 = 15, so 2 relations at R5.

When restricting to the subgroup $F\bar{4}3m$, we have

$$B_{\beta+\phi}|_{F\bar{4}3m} = B_\beta + B_\phi, \quad C_{n\gamma}|_{F\bar{4}3m} = C_\gamma + C_{\gamma'} + C_{s\omega}, \quad . \tag{177}$$

We list the elements in each degree:

$$H^1(Fd\bar{3}m, \mathbb{Z}_2) = \langle A_m, A_i\rangle \cong (\mathbb{Z}_2)^2, \tag{178a}$$

$$H^2(Fd\bar{3}m,\mathbb{Z}_2) = \langle A_m^2, A_mA_i, A_i^2;\quad B_\beta, B_\delta\rangle \cong (\mathbb{Z}_2)^5, \tag{178b}$$

$$H^3(Fd\bar{3}m,\mathbb{Z}_2) = \langle A_m^3, A_m^2A_i, A_mA_i^2, A_i^3;\quad A_mB_\delta;\quad C_{s\psi};\quad A_mB_\beta, A_iB_\beta;\quad C_{\gamma'}\rangle \cong (\mathbb{Z}_2)^9, \tag{178c}$$

$$H^4(Fd\bar{3}m,\mathbb{Z}_2) = \langle A_m^4, A_m^3A_i, A_m^2A_i^2, A_mA_i^3, A_i^4;\quad A_m^2B_\delta, B_\delta^2;\quad A_m^2B_\beta, A_mA_iB_\beta, A_i^2B_\beta;\quad B_\delta B_\beta\rangle \cong (\mathbb{Z}_2)^{11}; \tag{178d}$$

$$\begin{aligned} H^5(Fd\bar{3}m,\mathbb{Z}_2) = {}& \langle A_m^5, A_m^4A_i, A_m^3A_i^2, A_m^2A_i^3, A_mA_i^4, A_i^5;\quad A_m^3B_\delta, A_mB_\delta^2;\quad B_\delta C_{s\psi}\rangle \\ & \cup \langle A_m^3B_\beta, A_m^2A_iB_\beta, A_mA_i^2B_\beta, A_i^3B_\beta;\quad A_mB_\delta B_\beta;\quad B_\delta C_{\gamma'}\rangle \cong (\mathbb{Z}_2)^{15}, \end{aligned} \tag{178e}$$

$$\begin{aligned} H^{k,k\geq 3}(Fd\bar{3}m,\mathbb{Z}_2) = {}& \langle A_m^k, A_m^{k-1}A_i, ..., A_i^k\rangle \cup \left(\bigcup_{\substack{p+2q=k\\ q\geq 1}} \langle A_m^pB_\delta^q\rangle\right)\left(\bigcup_{\substack{2p+3q=k\\ q\geq 1}} \langle B_\delta^pC_{s\psi}^q\rangle\right) \\ & \cup \langle A_m^{k-2}B_\beta, A_m^{k-3}A_iB_\beta, ..., A_i^{k-2}B_\beta\rangle \cup \left(\bigcup_{\substack{p+2q+2=k\\ q\geq 1}} \langle A_m^pB_\delta^qB_\beta\rangle\right) \cup \left(\bigcup_{\substack{2p+3q+2=k\\ q\geq 1}} \langle B_\beta B_\delta^pC_{\gamma'}^q\rangle\right) \\ & \cup \left(\bigcup_{k\equiv 0 \bmod 3} \langle C_{s\psi}^{k/3}C_{\gamma'}\rangle\right) \cong (\mathbb{Z}_2)^{1+f(k)+f(k-1)+f(k-2)+2f(k-3)}, \end{aligned} \tag{178f}$$

The first line gives the $q=0$ row of the spectral sequence $E_\infty^{k,0} = E_3^{k,0} = (\mathbb{Z}_2)^{P(n)}$: with $P(k)$ given by the number of solutions to the first line

$$P(k) = k + 2 + \left[\frac{k}{2}\right] + \left[\frac{k}{6}\right] - \delta_{k\equiv 1 \bmod 6} - \delta_{k\equiv 0 \bmod 2} = 2 - k + f(k) + f(k-1) + f(k-2) + f(k-3),$$

while the second line gives the $q=2$ row of the spectral sequence $E_\infty^{k-2,2} = E_3^{k-2,0} = (\mathbb{Z}_2)^{Q(n)}$ with

$$Q(k) = k - 1 + \left(1 + \left[\frac{k-2}{2}\right] - 1\right) + \left(1 + \left[\frac{k-2}{6}\right] - \delta_{k-2\equiv 1 \bmod 6} - \delta_{k\equiv 0 \bmod 2}\right) + \delta_{k\equiv 0 \bmod 3} = k - 1 + f(k-3)$$

see Eq. (**??**) for definition of $f(k)$.

24 About differentials

24.1 Pyrochlore d_2

Recall that $H^1(O_\mathrm{h}, H^1(T,\mathbb{Z}_2)) = \mathbb{Z}_2 = \langle \upsilon\rangle$, and that $H^*(O_\mathrm{h},\mathbb{Z}_2) = \mathbb{F}_2[x, x_1, y, c]/(yc)$, with $H^3(O_\mathrm{h},\mathbb{Z}_2) = \mathbb{Z}_2^7 = \langle x^3, x^2x_1, xx_1^2, x_1^3, xy, x_1y, c\rangle$. It turns out that

$$d_2^{1,1}\colon H^1(O_\mathrm{h}, H^1(T,\mathbb{Z}_2)) \to H^3(O_\mathrm{h},\mathbb{Z}_2) \tag{179}$$

sends

$$d_2^{1,1}\colon \upsilon \mapsto x_1y, \tag{180}$$

which was worked out in Mathematica.

We also checked that

$$d_2^{2,1}\colon H^2(O_\mathrm{h}, H^1(T,\mathbb{Z}_2)) \to H^4(O_\mathrm{h},\mathbb{Z}_2), \tag{181}$$

with $H^2(O_\mathrm{h}, H^1(T,\mathbb{Z}_2)) = \langle \upsilon^2, \psi, \upsilon\iota\rangle$, where we defined $\upsilon\iota$ to be the third cocycle that lives in $H^1(\langle P\rangle, H^1(O_\mathrm{h},\mathcal{A}_1))$. We also have $H^4(O_\mathrm{h},\mathbb{Z}_2) = \mathbb{Z}_2^{10} = \langle x^4, x^3x_1, x^2x_1^2, xx_1^3, x_1^4, y^2, x^2y, x_1^2y, xx_1y, x_1c\rangle$. The map $d_2^{2,1}$ has the form

$$\mathrm{Ker} d_2^{2,1} = 0, \tag{182}$$

$$d_2^{2,1}\colon \upsilon^2 \mapsto xx_1y, \quad \psi \mapsto xx_1y + x_1c, \quad \upsilon\iota \mapsto x_1^2y. \tag{183}$$

Using the method of Qing-Rui Wang, Cheng Meng, arXiv2104.13233 (see the next subsection), i.e. Eq. (192), we were able to verify our conjecture for $Fd\bar{3}m$ for the following entries:

$$\ker(d_2^{0,3}\colon E_2^{0,3} \to E_2^{2,2}) = 0, \quad \ker(d_2^{1,3}\colon E_2^{1,3} \to E_2^{3,2}) = 0, \quad \ker(d_2^{2,3}\colon E_2^{2,3} \to E_2^{4,2}) = 0, \tag{184}$$

This uses the following code:
Application to $d_2^{3,1}\colon H^p(O_\mathrm{h}, H^3(T,\mathbb{Z}_2)) \to H^{p+2}(O_\mathrm{h}, H^2(T,\mathbb{Z}_2))$:

```
LoadPackage("HAP");
Oh:=Group((1,2,3),(3,4),(5,6));
ROh:=ResolutionFiniteGroup(Oh,8);
List([0..8],ROh!.dimension);
```

which gives

```
[ 1, 4, 10, 20, 35, 55, 77, 101, 131 ]
```

To print the boundary, take degree-7 for example:

```
for i in [1..101] do
Print(ROh!.boundary(7,i),",");
od;
```

As a side note, we can print the contracting homotopy. Take degree-4 for example:

```
for i in [1..35] do
for j in [1..48] do
Print(ROh!.homotopy(4,[i,j]),",");
od;od;
```

24.2 General d_2

The paper by Qing-Rui Wang, Cheng Meng, arXiv2104.13233, gives a very explicit account for the differentials. Now let's look at $d_2^{p,q}$ for simplicity. First of all, for $1 \to N \to G \xrightarrow{p} Q \to 1$, define the notation $n_i \in N$, $q_j \in Q$, and the element of G is written as (n, q). This is basically the way that Rotman writes, and in 2104.13233 this is written as n_q. First note that $(m, p)(n, q) = (m\rho_p(n)\nu(p, q), pq)$, and $(n, q)^{-1} = (q^{-1}n^{-1}\nu^{-1}(q, q^{-1})q, q^{-1})$, where we have assumed that the group action $\rho\colon Q \to N$ is $\rho_q(n) = qnq^{-1}$. Then, one can easily check that $(n, q) \cdot (n, q)^{-1} = (1, 1)$, while $(n, q)^{-1} \cdot (n, q) = (1, 1)$ requires that $q^{-1}n^{-1}\nu^{-1}(q, q^{-1})nq\nu(q^{-1}, q) = 1 \in N$. If further N is abelian as is our case, then we have $q^{-1}\nu(q, q^{-1})q = \nu(q^{-1}, q)$.

Recall a lifting $\ell\colon Q \to G$ is any function (not necessarily homomorphism) that satisfies $p\ell = 1 \in Q$. It's easy to see that $\ell(q) = (1, q)$ is a (actually the standard) lifting. This lifting is also denoted $\ell(q) = q^*$ in Bill's notes and Serre's original paper. Using this lifting, one can then check that

$$(pq)^{*-1} = (l(pq))^{-1} = (1, pq)^{-1} = (q^{-1}p^{-1}\nu^{-1}(pq, q^{-1}p^{-1})pq, q^{-1}p^{-1}), \tag{185}$$

therefore

$$\begin{aligned} p^*q^*(pq)^{*-1} = l(p)l(q)(l(pq))^{-1} &= (1, p)(1, q)(l(pq))^{-1} = (\nu(p, q), pq)(l(pq))^{-1} \\ &= (\nu(p, q), pq)(q^{-1}p^{-1}\nu^{-1}(pq, q^{-1}p^{-1})pq, q^{-1}p^{-1}) \\ &= (\nu(p, q)\nu^{-1}(pq, q^{-1}p^{-1})\nu(pq, q^{-1}p^{-1}), 1) \\ &= (\nu(p, q), 1), \end{aligned} \tag{186}$$

and

$$\begin{aligned} (pq)^{*-1}p^*q^* = (l(pq))^{-1}l(p)l(q) = (l(pq))^{-1}(\nu(p, q), pq) \\ (q^{-1}p^{-1}\nu^{-1}(pq, q^{-1}p^{-1})pq, q^{-1}p^{-1})(\nu(p, q), pq) \\ = (q^{-1}p^{-1}\nu^{-1}(pq, q^{-1}p^{-1})pq \cdot q^{-1}p^{-1}\nu(p, q)pq \cdot \nu(q^{-1}p^{-1}, pq), 1) \\ = (q^{-1}p^{-1}\nu^{-1}(pq, q^{-1}p^{-1})pq \cdot q^{-1}p^{-1}\nu(p, q)pq \cdot q^{-1}p^{-1}\nu(pq, q^{-1}p^{-1})pq, 1) \\ = (q^{-1}p^{-1}\nu(p, q)pq, 1), \end{aligned} \tag{187}$$

where $\nu(p, q)$ is the factor set as defined by Rotman, and it is an element of $H^2(Q, Z(N))$. In the above we calculated $l(p)l(q)(l(pq))^{-1}$ and $(l(pq))^{-1}l(p)l(q)$. It is crucial to realize that they are very different, which is crucial to the form of differential below.

ArXiv2104.13233 gives the differential d_2 in Eqs. (22-23) and Appendix B. Note that his δ_2 is basically our d_2. For $f \in H^i(Q, H^j(N, M))$, we have $(f(q_1, q_2 ..., q_i))(n_1, ..., n_j) \in M$. $(d_2^{p,q} f)$ should have argument $(q_1, ..., q_{i+2})$ and $(n_1, ..., n_{j-1})$. We have

$$
\begin{aligned}
&[(d_2^{i,j} f)(q_1, ..., q_{i+2})](n_1, ..., n_{j-1}) \\
&= [(q_1q_2).\,(f(q_3, ..., q_{i+2}))]\,(\nu(q_1, q_2), n_1, ..., n_{j-1}) - [(q_1q_2).\,(f(q_3, ..., q_{i+2}))]\,(n_1, \nu(q_1, q_2), n_2, ..., n_{j-1}) + \cdots + (-1)^{j-1}\,[(q_1q_2).\,(f(q_3, \\
&= (q_1q_2).\,\{[(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{-1}\nu(q_1, q_2)(q_1q_2), (q_1q_2)^{-1}n_1(q_1q_2), ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2))\} \\
&\quad - (q_1q_2).\,\{[(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{-1}n_1(q_1q_2), (q_1q_2)^{-1}\nu(q_1, q_2)(q_1q_2), ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2))\} \\
&+ \cdots + (-1)^{j-1}(q_1q_2).\,\{[(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{-1}n_1(q_1q_2), ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2), (q_1q_2)^{-1}\nu(q_1, q_2)(q_1q_2))\}\,.
\end{aligned}
\tag{188}
$$

For $j = 1$: this reduces to

$$
\begin{aligned}
(d_2^{i,j=1} f)(q_1, ..., q_{i+2}) &= [(q_1q_2).\,(f(q_3, ..., q_{i+2}))]\,(\nu(q_1, q_2)) \\
&= (q_1q_2).\,\{[(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{-1}\nu(q_1, q_2)(q_1q_2))\}\,,
\end{aligned}
\tag{189}
$$

For us, since the action of G on M is trivial, we have

$$
(d_2^{i,j=1} f)(q_1, ..., q_{i+2}) = [(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{-1}\nu(q_1, q_2)(q_1q_2)),
\tag{190}
$$

This seems to differ from the one we have been using (first introduced by Bill in his notes), which is

$$
(d_2^{i,j=1} f)(q_1, ..., q_{i+2}) = [(f(q_3, ..., q_{i+2}))]\,((q_1q_2)^{*-1}q_1^*q_2^*),
\tag{191}
$$

But they the above two equations are actually equal, by virtue of Eq. (187). This checks that all our differential calculations are so far correct.

To summarize: the d_2 differential that we will be using is: for $f \in H^i(Q, H^j(N, M))$,

$$
\begin{aligned}
[(d_2^{i,j} f)(q_1, ..., q_{i+2})](n_1, ..., n_{j-1}) = {} & f(q_3, ..., q_{i+2})((q_1q_2)^{*-1}q_1^*q_2^*, (q_1q_2)^{-1}n_1(q_1q_2), ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2)) \\
& - f(q_3, ..., q_{i+2})((q_1q_2)^{-1}n_1(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*, ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2)) \\
& + \cdots \\
& + (-1)^{j-1} f(q_3, ..., q_{i+2})((q_1q_2)^{-1}n_1(q_1q_2), ..., (q_1q_2)^{-1}n_{j-1}(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*).
\end{aligned}
\tag{192}
$$

We further list a few that will be useful to use: for $(i, j) = (0, 3)$: $f \in H^0(Q, H^3(N, M)) = H^3(N, M)^Q$ is nothing but a three-cocycle of N. Denote it as $\omega_3(n_1, n_2, n_3) \in M$ for $n_1, n_2, n_3 \in N$. We have

$$
\begin{aligned}
[(d_2^{0,3} f)(q_1, q_2)](n_1, n_2) = {} & \omega_3((q_1q_2)^{*-1}q_1^*q_2^*, (q_1q_2)^{-1}n_1(q_1q_2), (q_1q_2)^{-1}n_2(q_1q_2)) \\
& - \omega_3((q_1q_2)^{-1}n_1(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*, (q_1q_2)^{-1}n_2(q_1q_2)) \\
& + \omega_3((q_1q_2)^{-1}n_1(q_1q_2), (q_1q_2)^{-1}n_2(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*),
\end{aligned}
\tag{193}
$$

Similarly, for $(i, j) = (0, 2)$: $f \in H^0(Q, H^2(N, M)) = H^2(N, M)^Q$ is a two-cocycle of N. denote it as $\omega_2(n_1, n_2) \in M$ for $n_1, n_2 \in N$, we have

$$
[(d_2^{0,2} f)(q_1, q_2)](n) = \omega_2((q_1q_2)^{*-1}q_1^*q_2^*, (q_1q_2)^{-1}n(q_1q_2)) - \omega_2((q_1q_2)^{-1}n(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*).
\tag{194}
$$

Note that $(q_1q_2)^{-1}n(q_1q_2)$ should be calculated as $(q_1^*q_2^*)^{-1}n(q_1^*q_2^*)$.

Finally, we list $(i, j) = (1, 2)$:

$$
[(d_2^{1,2} f)(q_1, q_2, q_3)](n) = f(q_3)((q_1q_2)^{*-1}q_1^*q_2^*, (q_1q_2)^{-1}n(q_1q_2)) - f(q_3)((q_1q_2)^{-1}n(q_1q_2), (q_1q_2)^{*-1}q_1^*q_2^*).
\tag{195}
$$

24.3 Some d_3

We will be using the expression of $d_3^{p,q}$ for $(p,q)=(0,3)$. Using (B39) of Qing-Rui's paper, we have

$$\begin{aligned}
&[(d_3^{0,3}f)(q_1,q_2,q_3)](n)\\
&\quad=[(q_1q_2q_3).f](n,\nu(q_1q_2,q_3),\nu(q_1,q_2))-[(q_1q_2q_3).f](\nu(q_1q_2,q_3),n,\nu(q_1,q_2))+[(q_1q_2q_3).f](\nu(q_1q_2,q_3),\nu(q_1,q_2),n)\\
&\quad-[(q_1q_2q_3).f](n,\nu(q_1,q_2q_3),\nu(q_2,q_3))+[(q_1q_2q_3).f](\nu(q_1,q_2q_3),n,\nu(q_2,q_3))-[(q_1q_2q_3).f](\nu(q_1,q_2q_3),\nu(q_2,q_3),n)\\
&=f((q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_1q_2,q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_1,q_2)q_1q_2q_3)\\
&\quad-f(\nu(q_1q_2,q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{-1},(q_1q_2q_3)^{-1}\nu(q_1,q_2)q_1q_2q_3)\\
&\quad+f((q_1q_2q_3)^{-1}\nu(q_1q_2,q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_1,q_2)q_1q_2q_3,(q_1q_2q_3)^{-1}nq_1q_2q_3)\\
&\quad-f((q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_1,q_2q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_2,q_3)q_1q_2q_3)\\
&\quad+f((q_1q_2q_3)^{-1}\nu(q_1,q_2q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_2,q_3)q_1q_2q_3)\\
&\quad-f((q_1q_2q_3)^{-1}\nu(q_1,q_2q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}\nu(q_2,q_3)q_1q_2q_3,(q_1q_2q_3)^{-1}nq_1q_2q_3)\\
&=f((q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{*-1}(q_1q_2)^*q_3^*,q_3^{-1}(q_1q_2)^{*-1}q_1^*q_2^*q_3)\\
&\quad-f((q_1q_2q_3)^{*-1}(q_1q_2)^*q_3^*,(q_1q_2q_3)^{-1}nq_1q_2q_3,q_3^{-1}(q_1q_2)^{*-1}q_1^*q_2^*q_3)\\
&\quad+f((q_1q_2q_3)^{*-1}(q_1q_2)^*q_3^*,q_3^{-1}(q_1q_2)^{*-1}q_1^*q_2^*q_3,(q_1q_2q_3)^{-1}nq_1q_2q_3)\\
&\quad-f((q_1q_2q_3)^{-1}nq_1q_2q_3,(q_1q_2q_3)^{*-1}q_1^*(q_2q_3)^*,q_3^{*-1}q_2^{*-1}q_1^{*-1}q_2^*q_3^*(q_2q_3)^{*-1}q_1^*q_2^*q_3^*)\\
&\quad+f((q_1q_2q_3)^{*-1}q_1^*(q_2q_3)^*,(q_1q_2q_3)^{-1}nq_1q_2q_3,q_3^{*-1}q_2^{*-1}q_1^{*-1}q_2^*q_3^*(q_2q_3)^{*-1}q_1^*q_2^*q_3^*)\\
&\quad-f((q_1q_2q_3)^{*-1}q_1^*(q_2q_3)^*,q_3^{*-1}q_2^{*-1}q_1^{*-1}q_2^*q_3^*(q_2q_3)^{*-1}q_1^*q_2^*q_3^*,(q_1q_2q_3)^{-1}nq_1q_2q_3),
\end{aligned}\tag{196}$$

where we have used $(q_1q_2q_3)^{-1}\nu(q_2,q_3)q_1q_2q_3=q_3^{-1}q_2^{-1}q_1^{-1}q_2q_3(q_2q_3)^{*-1}q_2^*q_3^*q_3^{-1}q_2^{-1}q_1q_2q_3)$ $=q_3^{*-1}q_2^{*-1}q_1^{*-1}q_2^*q_3^*(q_2q_3)^{*-1}q_2^*q_3^*q_3^{*-1}q_2^{*-1}q_1^*q_2^*q_3^*=q_3^{*-1}q_2^{*-1}q_1^{*-1}q_2^*q_3^*(q_2q_3)^{*-1}q_1^*q_2^*q_3^*$.

Parallel to Eq. (B38-B40) in Qing-rui's paper,we should have

$$(\delta_3F_{1,2})(\mathbf{h},\mathbf{k},\mathbf{l},\mathbf{m})=\iota_{(\nu\otimes\nu)(-\Delta_1(\mathbf{h},\mathbf{k},\mathbf{l}))}\left({}^{1_{\mathbf{hkl}}}F\right)^{\cdot,\cdot,\cdot,1_{\mathbf{m}}}=\frac{\iota_{\nu(\mathbf{h},\mathbf{kl}),{}^{\mathbf{h}}\nu(\mathbf{k},\mathbf{l})}\left({}^{1_{\mathbf{hkl}}}F\right)^{\cdot,\cdot,1_{\mathbf{m}}}}{\iota_{\nu(\mathbf{hk},\mathbf{l}),\nu(\mathbf{h},\mathbf{k})}\left({}^{1_{\mathbf{hkl}}}F\right)^{\cdot,\cdot,1_{\mathbf{m}}}}=\frac{\left({}^{1_{\mathbf{hkl}}}F\right)^{\nu(\mathbf{h},\mathbf{kl}),{}^{\mathbf{h}}\nu(\mathbf{k},\mathbf{l}),1_{\mathbf{m}}}}{\left({}^{1_{\mathbf{hkl}}}F\right)^{\nu(\mathbf{hk},\mathbf{l}),\nu(\mathbf{h},\mathbf{k}),1_{\mathbf{m}}}}\tag{197}$$

We also have

$$(\delta_2F_{2,1})(\mathbf{h},\mathbf{k},\mathbf{l},\mathbf{m})=\iota_{[-\nu(\mathbf{h},\mathbf{k})]}\left({}^{1_{\mathbf{hk}}}F\right)^{\cdot,1_{\mathbf{l}},1_{\mathbf{m}}}=\frac{1}{\left({}^{1_{\mathbf{hk}}}F\right)^{\nu(\mathbf{h},\mathbf{k}),1_{\mathbf{l}},1_{\mathbf{m}}}}\tag{198}$$

and notice that for our case we have $\left({}^{1_{\mathbf{hk}}}F\right)^{\nu(\mathbf{h},\mathbf{k}),1_{\mathbf{l}},1_{\mathbf{m}}}=F^{(\mathbf{hk}^{-1}\nu(\mathbf{h},\mathbf{k})\mathbf{hk},1_{\mathbf{l}},1_{\mathbf{m}})}\to F(q_1,q_2)((q_1q_2)^{*-1}q_1^*q_2^*)$.

24.4 SPT

New discovery: in a split extension as in the symmorphic group case: $0\to T\to G\to Q\to 0$ with $G=T\rtimes Q$, where T is translation and Q is point group; as mentioned before, the equivalent statement of symmorphism is that there is an isomorphic copy of $Q:=G/T$ in G. In this case, actually all the $d_2\colon E_2^{p,q=1}\to E_2^{p+2,q=0}$ maps are trivial. This is very nicely explained in the book "Cohomology of Number Fields". But note that in general, even for this split case, d_2 maps that starts from $q>1$ rows can be nontrivial, as explained in the Burt Totaro paper "Cohomology of semidirect product groups".

Note that Bill's definition of $d_2\colon E_2^{1,1}\to E_2^{3,0}$ has been applied in the $Fd\bar{3}m$ case (with $H_2^{p,q}=H^p(O_\mathrm{h},H^q(T,\mathbb{Z}_2))$ to show that $E_2^{1,1}=\mathbb{Z}_2$ while $E_3^{1,1}=0$. See above.

According to the LHS spectral sequence, associated with the group extension $1\to N\to G\to Q=G/N\to 1$ we have the E_2 page $E_2^{p,q}=H^p(Q,H^q(N,A))$. Now we consider the special case of $A=U(1)$ and a central extension $1\to\mathbb{Z}_n\to G_f\to G_b\to 1$, i.e. $N:=\mathbb{Z}_n$, $G:=G_f$ and $QP:G_b$, and G_b acts trivially on $\mathbb{Z}_n$. Any such group extension is labeled by an element $[\lambda]\in H^2(G_b,\mathbb{Z}_n)$. Our goal is to have an expression for $H^2(G_f,U(1))$ and study its connection to the Fermionic SPT result given in Eqs. (8)-(11) in `https://journals.aps.org/prx/pdf/10.1103/PhysRevX.10.031055`.

The E_2 page can be written explicitly

$$\begin{tikzcd}
q=2 & 0 & 0 & 0 & 0 \\
q=1 & \mathbb{Z}_n \arrow[rrd, "d_2^{0,1} = \lambda\cup"] & H^1(G_b,\mathbb{Z}_n) \arrow[rrd, "d_2^{1,1}\lambda\cup"] & & \\
q=0 & U(1) & H^1(G_b,U(1)) & H^2(G_b,U(1)) & H^3(G_b,U(1)) \\
 & p=0 & p=1 & p=2 & p=3
\end{tikzcd} \tag{199}$$

where we have used the crucial result that the two differential maps $d_2^{0,1}$ and $d_2^{1,1}$ have the form of cup product $\lambda\cup$, where $\lambda \in Z^2(Q, N/[N,N])$ is the element that determines the group extension $1 \to \mathbb{Z}_n \to G_f \to G_b \to 1$. Such a result is standard and can be found in e.g. the book Cohomology of Number Fields by Neukirch, Schmidt, and Wingberg, Theorem 2.4.4, P114.

Below we define $\mathbb{Z}_n = \{z = e^{\frac{2\pi i}{n}}, z^2, ..., z^n = 1\}$ and the characters $\chi_i(z) = z^i$ for $H^1(\mathbb{Z}_n, U(1)) = \{\chi_1, ..., \chi_n\} \cong \mathbb{Z}_n$. Here we are using the multiplicative notation for $\mathbb{Z}_n$ as Kyle did. We have

$$\begin{aligned}
H^2(G_f,U(1)) &= E_3^{1,1} \oplus E_3^{2,0} = \ker(d_2^{1,1}) \oplus (E_2^{2,0}/\mathrm{Im}(d_2^{0,1})) \\
&= \{n_1 \in Z^1(G_b,\mathbb{Z}_n) | \lambda \cup n_1 = d\nu_2 \text{ for some } \nu_2 \in C^2(G_b,U(1))\} \\
&\quad \oplus H^2(G_b,U(1))/\{\lambda \cup \chi_i | \chi_i \in H^1(\mathbb{Z}_n,U(1)) \cong \mathbb{Z}_n\} \\
&= \{(n_1,\nu_2) | \lambda \cup n_1 = d\nu_2, n_1 \in Z^1(G_b,\mathbb{Z}_n), \nu_2 \in C^2(G_b,U(1))\}/\{(0,\nu_2) | \nu_2 \in Z^2(G_b,U(1))\} \\
&\quad \oplus H^2(G_b,U(1))/\Gamma,
\end{aligned} \tag{200}$$

Note that for any $g_1, g_2 \in G_b$ and $z^j \in \mathbb{Z}_n$, $\lambda \cup \chi_i(g_1,g_2,z^j) = \lambda(g_1,g_2) \otimes \chi_i(z^j) = \chi_i(\lambda(g_1,g_2)) = (\lambda(g_1,g_2))^i$. Therefore $\Gamma := \{\lambda \cup \chi_i | \chi_i \in H^1(\mathbb{Z}_n,U(1)) \cong \mathbb{Z}_n\} = \langle \lambda \rangle$ is the abelian group generated by λ, as defined in Eq. (11) of the PRX paper (note that their ω_2 is what we call λ).

In the last line of Eq. (200) we have noted that for a given n_1 there are multiple solutions of ν for $\lambda \cup n_1 = d\nu$ which differ by the elements in the cocycle $Z^2(G_b,U(1))$ and has to be quotient out (this is the equivalence relation that Kyle defined). These elements nevertheless appear in the second summand. Therefore the last line has an expression identical to Eqs. (8)-(11) of the PRX paper, and we have shown that the FSPT data is in one-to-one correspondence with the bosonic SPT data $H^2(G_f,U(1))$.

25 Combining GAP with spectral sequence calculation: a complete guide

Main reference:

- Computing classification of interacting fermionic symmetry-protected topological phases using topological invariants, China Physics Letters, by Ouyang, Wang, Gu, Qi, `https://arxiv.org/abs/2005.06572`. This paper gives way to map between the bar resolution and the resolution given by GAP. The crucial thing about this mapping is that (1) this can greatly reduce the computation cost, and (2) the cup product as a simple form only in the (computationally-costly) bar resolution/inhomogeneous resolution; the mapping allows us to examine whether the cup product is a true cocycle or a coboundary in the (computationally-cheap) resolution given by GAP, hence allowing us to determine the ring structure.

- Domain Wall Decorations, Anomalies and Spectral Sequences in Bosonic Topological Phases by Wang, Ning, Cheng, `https://arxiv.org/abs/2104.13233`. The key result is an explicit form for the differentials of the LHS spectral sequence. Also, the differential is given concrete physical meaning.

- To extract the resolution computed in GAP, a good guide is given in Ellis Graham's book, Chapter 3, section 1-3.

- In the future we will aim at completing all the computations within GAP and not resorting to Mathematica anymore. The tutorial for HAP contains further information: `https://gap-packages.github.io/hap/tutorial/chap7.html`.

For a finite group G with nontrivial action on M, the $H^*(G, M)$ can always be directly computed in GAP, see the examples starting from Eq. (157).

The very method has been implemented in our Mathematica file. We need the following information:

- Denote the resolution by `R`. We distinguish two cases, finite groups and crystallographic groups:
  - For finite groups, usually the resolution is obtained by

    ```
    ResolutionFiniteGroup(G,8);
    ```

    or

    ```
    ResolutionSmallGroup(G,8);
    ```

    The former computes a resolution and also outputing the contracting homotopy. The latter computes via the standard bar resolution (i.e. the degree n-cochain is a set-theoretic mapping $G^n \to M$) and hence is very slow; also, it does not give contracting homotopy. However, it usually gives the smallest size Resolution one can get.
  - If `SG` is a crystallographic group in pcp notation (see previous examples), then we can construct its resolution using

    ```
    ResolutionAlmostCrystalGroup(Image(SG),5)
    ```

    where SG is stored as a polycycle group, or

    ```
    R:=ResolutionSpaceGroup(SpaceGroupIT(3,Num),15);
    ```

    Note that the latter uses polymake and is such faster, however it does not give contracting homotopy. Nevertheless, we have not found the contracting homotopy of a crystallographic group of much use, as the bar resultion cannot be recovered from it for infinite groups.
- Resolution size; this can be accessed by e.g.

  ```
  List([0..8],R!.dimension);
  ```

- How the group elements are indexed in the resolution. Usually, we will prefer writing group elements as permutation cycles (for finite group such as S_4 or O_h or matrices (such as for crystallographic groups). Then, the index of the group elements can be found by

  ```
  R!.elts;
  ```

  Note tha the generator of the group can also be accessed by `G.1`, `G.2`, and so on. The product of elements $g_1 g_2$ is written as `G.1*G.2`, i.e. in GAP they also observe the "action from the left" rule. However, as we have noticed several times in Mathematica, this group multiplication often has to be swapped for some reason. One simply has to be cautious about it during the implementation, and only the correct order gives the correct coboundary and cocycles.
- Boundary. This is the most important information that determines a resolution. Here boundary is the differential map $\partial_k \colon C_k \to C_{k-1}$, where C_k is a $\mathbb{Z}G$ chain complex, meaning that C_k is several copies of $\mathbb{Z}G$. We will always work with a free $\mathbb{Z}G$ resolution, meaning that each C_k is a free $\mathbb{Z}G$-module that is several copies of the ring $\mathbb{Z}G$, and hence we only need to specify the basis (i.e. the number of copies of $\mathbb{Z}G$ so that C_k is completely determined. In the resolution constructed by GAP, this basis is formally denoted as $e_1, e_2, ..., e_{b_k}$ where b_k is the number of copies of $\mathbb{Z}G$ that C_k is ismomorphic to. On the other hand, in the reduced bar resolution, C_k has basis whose number (of copies of $\mathbb{Z}G$ is as large as $(|G|-1)^k$: the basis are denoted as $e_i = [g_1|g_2|\cdots|g_k]$ for $g_1, g_2, ..., g_k \in G$ and we usually choose to use the reduced bar resolution, meaning that $[g_1|g_2|\cdots|g_k] = 0$ when any of g_i is the identity in G. Back to gap. The number of basis $e_1, ..., e_{b_k}$ at C_k is specified by the size of resolution given above; in other words, the command `List([0..8],R!.dimension);` tells the number of basis b_k for $k = 0, 1, ..., 8$. Then, each basis e_i^k in C_k is sent to a linear combinations of group element-product with-basis $g_{s_j^{k-1}} e_j^{k-1}$ in C_k, and this is specified by

  ```
  R!.boundary(k,i);
  ```

  Quite often we will out put this differential map using the command

  ```
  for i in [1..6] do
  ```

```
Print(ROh!.boundary(7,i),",");od;
```

where in the above we take as example $k = 7$ and $b_k = 6$. The output of the above command is a collection of b_k 2D lists, where each of these 2D lists ($i = 1, 2, ..., b_k$) is a list of ordered pairs `[j,l]` where j is the basis e_j^{k-1} in C_{k-1} that is a summand in ∂e_i^k and l is the index of group element that appears before e_j^{k-1}. In other words, ∂e_i^k contains a term $g_l e_j^{k-1}$; and by summing over all these terms in this 2D list one gets the image ∂e_i^k; and then letting i to run from $1, 2, ...$ to b_k one gets the complete chain differential map $\partial\colon C_k \to C_{k-1}$. We always repeat the above command for $k = 1, 2, ...$ and copy the printed output on the screen to Mathematica as an association (e.g. in Mathematica, `ROhboundary1to8`).

- Contracting homotopy. This is needed for the following goal: if we want to obtain the inhomogeneous cocycle (as output) when given (as input) G a finite group, a resolution from GAP, and a cocycle expressed as a vector within in this resolution. In this case, we need to choose before hand the command that produces resolution along with a homotopy, i.e. the `ResolutionFiniteGroup`, and not `ResolutionSmallGroup` as it does not give a contracting homotopy. The contracting homotopy has been implemented in Mathematica, see the relevant codes there.

The above is the part relevant to GAP. After downloading (i.e. copying) the above information to Mathematica, the rest of the job is done in Mathematica. Important steps are:

- Writing the action $G \to M$ in matrix form.
- Writing the boundary map in matrix form $M_{k\to k-1}$. This uses the Mathematica association for the boundary `ROhboundary1to8` and the action in matrix form mentioned above. The matrix $M_{k\to k-1}$ will be having dimension number of rows times number of columns $mb_k \times mb_{k-1}$, where m is the vector space dimension of M. each $m \times m$ block of the matrix is filled with the representation matrix for the group action. Note that I think here we need to put the transpose of this matrix (while although in all the existing Mathematica codes so far we have been putting the matrices themselves which I think is the reason why we need to reverse the order of group multiplication). As an example, we have

  ```
  Table[Table[E21Ohmatinv\[LetterSpace]deg2\[LetterSpace]x\[LetterSpace]deg1[[3ii-2;;3ii,3Abs[jj[[1]]]-2;;

  3 Abs[jj[[1]]]]]+=GactionmatrixA2[Ohgindtoccptminv[jj[[2]]][[;;4]]],{jj,ROhboundary1to8[2][[ii]]}],{ii,1
  ```

  After that we should check that $M_{k\to k-1}.M_{k-1\to k-2}$ gives zero (in the mod 2 sense if it is the mod-2 cohomology that we calculate), and that the nullspace dimension of $M_{k\to k-1}$ minus the matrix rank (again, both in the mod 2 sense if necessary) gives the correct cohomology dimension, with the code being something like

  ```
  Length[NullSpace[E21Ohmatinv\[LetterSpace]deg2\[LetterSpace]x\[LetterSpace]deg1,Modulus->2]]

  -MatrixRank[E21Ohmatinv\[LetterSpace]deg1\[LetterSpace]x\[LetterSpace]deg0,Modulus->2]
  ```

- The information so far has enabled us to compute all the cocycle and coboundary within this resolution, hence from the GAP point of view the cohomology computation is completed. However, there are still several tasks one can do, especially if we are looking for the cup product/ring structure of the cohomology, or if this cohomology computation is just one row of the spectral sequence calculation. We list some of the further tasks below:
- To map an inhomogeneous cochain to a cochain in the resolution given by GAP: this is achievable thanks to the fact that the contracting homotopy for the bar resolution is always known. Recall that bar resoution is a $\mathbb{Z}G$ resolution for the chain complex, while the inhomogeneous resolution is a $\mathbb{Z}G$ resolution for the cochain complex. We know that taking the functor $\mathrm{Hom}_{\mathbb{Z}G}(\cdot, M)$ to the chain $\to C_k \xrightarrow{\partial_k} C_{k-1} \to$ gives the cochain resolution from which one computes the cohomology, and this functor has been implicitly applied in our Mathematica code, by constructing the matrix $M_{k\to k-1}$ (this matrix consists of blocks of the representation matrix for $G \to M$, and this exactly amounts to taking $\mathrm{Hom}_{\mathbb{Z}G}(\cdot, M)$). Back to the main question. To map an inhomogeneous cochain to a cochain in the resolution given by GAP, $f \mapsto f^*$, one needs to specify how the basis of the latter, e_i^k for $i = 1, 2, ..., b_k$, is mapped to the basis of the (reduced) bar resolution: $e_i^k \mapsto \sum [g_1|\cdots|g_k]$ so that $f^*(e_i^k) = f(\sum [g_1|\cdots|g_k]) = \sum f(g_1, ..., g_k)$. The basis map is implemented in Mathematica via the reduced bar resolution contracting homotopy, see the China Physics Letters paper for detail. Here we only present the code: it is something like

  ```
  Fbarhomotopyind[gind_,a__]:=Table[Join[{gind},b],{b,a}];

  OhFfntbasisdeg1ind = {{{1}, {2}}, {{1}, {3}}, {{1}, {5}}, {{1}, {13}}}

  OhFfntbasisdeg2ind = Table[Flatten[Table[Fbarhomotopyind[i[[2]],
  ```

```
OhFfntbasisdeg1ind[[Abs[i[[1]]]]]], {i, ROhboundary1to8[2][[j]]}], 1], {j, 1, 10}]
```

Then for any inhomogeneous function $f(g_1, g_2, ..., g_k)$, we can use the following code to get a vector that is the image of the map in the GAP resolution basis:

```
Table[Expand[Sum[f[Ohgindtoccptminv[gggg[[1]]],

Ohgindtoccptminv[gggg[[2]]]],{gggg,OhFfntbasisdeg2ind[[i]]}],Modulus->2],{i,1,20}]
```

- Other possible steps are: mapping a GAP cocycle to an inhomogeneous cocycle function using the contracting homotopy given in GAP; implementing the differential map in Mathematica. See corresponding codes.

26 Powerful code: checking ring structure for any space group

```
LoadPackage("HAP");
G:=SpaceGroupIT(3,227);
Gp:=IsomorphismPcpGroup(G);
R:=ResolutionAlmostCrystalGroup(Image(Gp),3);
```

This construction is very slow in GAP. So let's use back our old code:

```
T1:=[[1, 0, 0, 0], [0, 1, 0, 1/2], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T2:=[[1, 0, 0, 1/2], [0, 1, 0, 0], [0, 0, 1, 1/2], [0, 0, 0, 1]];
T3:=[[1, 0, 0, 1/2], [0, 1, 0, 1/2], [0, 0, 1, 0], [0, 0, 0, 1]];
C3:=[[0, 0, 1, 0], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 0, 1]];
P:=[[-1, 0, 0, 0], [0, -1, 0, 0], [0, 0, -1, 0], [0, 0, 0, 1]];
C2:=[[-1, 0, 0, 1/4], [0, -1, 0, 1/4], [0, 0, 1, 0], [0, 0, 0, 1]];
C2p:=[[-1, 0, 0, 1/4], [0, 1, 0, 0], [0, 0, -1, 1/4], [0, 0, 0, 1]];
M:=[[0, 1, 0, 0], [1, 0, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]];
G227:=Group(T1,T2,T3,C2,C2p,C3,M,P);
G216:=Group(T1,T2,T3,C2,C2p,C3,M);
G196:=Group(T1,T2,T3,C2,C2p,C3);
Gp227:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G227))));
Gp216:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G216))));
Gp196:=IsomorphismPcpGroup(AffineCrystGroupOnRight(GeneratorsOfGroup(TransposedMatrixGroup(G196))));
R227:=ResolutionAlmostCrystalGroup(Image(Gp227),4);
R216:=ResolutionAlmostCrystalGroup(Image(Gp216),4);
R196:=ResolutionAlmostCrystalGroup(Image(Gp196),4);
```

Then, execute

```
Read("~/Downloads/CR_Mod2CocyclesAndCoboundaries.gi");
```

which imports the file `CR_Mod2CocyclesAndCoboundaries.gi`. The code of this file is attached below. Then we execute

```
List([1..3],x->Mod2RingGenerators(R227,x));

List([1..3],x->Mod2RingGenerators(R216,x));
List([1..3],x->Mod2RingGenerators(R196,x));
```

To get the generator of the mod 2 cohomology ring.

Then, we can check if there are degree 4 generators for the mod-2 cohomology ring for small space groups, using the following code:

```
for i in [1..100] do
> G:=SpaceGroupBBNWZ(3,i);
> Gp:=IsomorphismPcpGroup(G);
> R:=ResolutionAlmostCrystalGroup(Image(Gp),5);
> Print([i,Length(Mod2RingGenerators(R,4))]);
> od;
```

Note that we have chosen to use `> G:=SpaceGroupBBNWZ(3,i);` instead of `> G:=SpaceGroupIT(3,i);` as the former appears significantly faster than the latter. For example, the former runs 4'46" while the latter runs 7'57". We find that there is no degree 4 generators. We can check that out of 17 wallpaper groups there is only one group having degree 3 generator;

```
for i in [1..17] do
> G:=SpaceGroupIT(2,i);
> Gp:=IsomorphismPcpGroup(G);
> R:=ResolutionAlmostCrystalGroup(Image(Gp),4);
> Print([i,Length(Mod2RingGenerators(R,3))]);
> od;
```

Output is

```
[ 1, 0 ][ 2, 0 ][ 3, 0 ][ 4, 0 ][ 5, 0 ][ 6, 0 ][ 7, 0 ][ 8, 0 ][ 9, 0 ][ 10, 0 ][ 11, 0 ][ 12, 1 ]
[ 13, 0 ][ 14, 0 ][ 15, 0 ][ 16, 0 ][ 17, 0 ]
```

Officially, let's run

```
for i in [1..230] do
if i=133 or i=138 or i=210 then
R:=ResolutionAlmostCrystalGroup(Image(IsomorphismPcpGroup(SpaceGroupIT(3,i))),6);
else
R:=ResolutionAlmostCrystalGroup(Image(IsomorphismPcpGroup(SpaceGroupBBNWZ(3,i))),6);
fi;
> Print([i,List([1..5],x->Mod2RingGenerators(R,x))]);Print("\n");
> od;
```

```
######################################################################
######################################################################

CR_Mod2CocyclesAndCoboundaries:=function(arg)
local
R, n, toggle, Dimension, Boundary,
M1, M2, row, sol,
kerdim, imgdim, cohdim, Mod2Cohomologydim,
    BasisKerd1, BasisImaged2, Rels, CobandCoc,
#Smith, SmithRecord, TorsionCoefficients,
ColMat, InvColMat,
RemoveRowsMat, InsertRowsList,
    GF2ToZ,
```

```
CycleToClass, ClassToCycle,
i, j, k, x, sum;


R:=arg[1];
n:=arg[2];
Dimension:=R!.dimension;
Boundary:=R!.boundary;
toggle := true;

if n <0 then return false; fi;
if n=0 then return [0]; fi;

####################################################################
GF2ToZ:=function(v)
local v0,k;

v0:=[];
for k in [1..Length(v)] do
    if v[k] = 0*Z(2) then
    v0[k]:=0;
    else
        v0[k]:=1;
    fi;
od;

return v0;
end;
####################################################################


################CONSTRUCT BOUNDARY MATRICES M1 AND M2########
M1:=[];
M2:=[];

#M1 is Dim(n) x Dim(n-1);
#M2 is Dim(n+1) x Dim(n);

for i in [1..Dimension(n)] do
row:=[];
        for j in [1..Dimension(n-1)] do
        sum:=0;
                for x in Boundary(n,i) do
                if AbsoluteValue(x[1])=j then
                sum := sum + SignInt(x[1]);
                fi;
                od;
        row[j]:= RemInt(sum,2);
        od;
M1[i]:=row;
od;

if Dimension(n+1)>0 then
for i in [1..Dimension(n+1)] do
row:=[];
        for j in [1..Dimension(n)] do
        sum:=0;
                for x in Boundary(n+1,i) do
                if AbsoluteValue(x[1])=j then
```

```
                sum := sum + SignInt(x[1]);
                fi;
                od;
        row[j]:= RemInt(sum,2);
        od;
M2[i]:=row;
od;

else

row:=[];
for j in [1..Dimension(n)] do
row[j]:=0;
od;
M2[1]:=row;
fi;
################MATRICES M1 AND M2 CONSTRUCTED###############

#M1 is Dim(n) x Dim(n-1);
#M2 is Dim(n+1) x Dim(n);
#BasisKerd1:=LLLReducedBasis(TransposedMat(M2),"linearcomb").relations;
#BasisImaged2:=LLLReducedBasis(TransposedMat(M1)).basis;


if M2 = [ [ ] ] then
BasisKerd1:=[];
else
BasisKerd1:=BasisNullspaceModN(TransposedMat(M2),2);
fi;

if M1 = [ ] then
BasisImaged2:=[];
else
BasisImaged2:=BaseMat(TransposedMat(M1)*Z(2));
fi;

#Print(BasisKerd1);
#Print(BasisImaged2);

imgdim:=Length(BasisImaged2);
kerdim:=Length(BasisKerd1);
cohdim:=kerdim-imgdim;


CobandCoc:=[];
for i in [1..imgdim] do
    Append(CobandCoc,[BasisImaged2[i]]);
od;


if cohdim >0 then
for i in [1..kerdim] do
        if imgdim = 0 then
        Append(CobandCoc,[BasisKerd1[i]*Z(2)]);
        else
        sol:=SolutionMat(CobandCoc,BasisKerd1[i]*Z(2));
        if sol=fail then
            Append(CobandCoc,[BasisKerd1[i]*Z(2)]);
        fi;
```

```
        fi;
od;
fi;

for i in [1..kerdim] do
    CobandCoc[i]:=GF2ToZ(CobandCoc[i]);
od;


if toggle=false then
return rec(
#cocyclesBasis:=BasisKerd1,
#boundariesCoefficients:=Rels,
#torsionCoefficients:=fail,
        cocyclesBasis:=CobandCoc,
        Mod2Cohomologydim:=cohdim,
cocycleToClass:=fail,
classToCocycle:=fail );
fi;


#####################################################################
CycleToClass:=function(v)
local u;

if cohdim = 0 then
return [];
fi;

u:=GF2ToZ(SolutionMat(CobandCoc*Z(2),v*Z(2)));
return List([1..cohdim],x->u[Length(u)-cohdim+x]);

end;
#####################################################################

#####################################################################
ClassToCycle:=function(u)
local v,w, i, temp;

w:=List([1..Dimension(n)],x->0);

if cohdim>0 then
for i in [1..Dimension(n)] do
temp := 0;
for j in [1..cohdim] do
temp := temp + CobandCoc[Length(CobandCoc)-cohdim+j][i]*u[j];
w[i] := temp mod 2;
od;
od;
fi;


return w;
end;
#####################################################################

return  rec(
#cocyclesBasis:=BasisKerd1,
  #boundariesCoefficients:=Rels,
```

```
        #torsionCoefficients:=TorsionCoefficients,
        cocyclesBasis:=CobandCoc,
        Mod2Cohomologydim:=cohdim,
  cocycleToClass:=CycleToClass,
  classToCocycle:=ClassToCycle );

end;
######################################################################
######################################################################



######################################################################
######################################################################

Mod2CupProduct:=function(arg)
local
    R, u, v, p, q, P, Q, N,
    uCocycle,
    vCocycle,
    uvCocycle,
    uChainMap,
    DimensionR,
    i, w, x, sw;

    ####################BEGIN TO READ THE INPUT##################
R:=arg[1];
DimensionR:=R!.dimension;
u:=arg[2];
v:=arg[3];
p:=arg[4];
q:=arg[5];

if Length(arg)>5 then P:=arg[6];
else
P:=CR_Mod2CocyclesAndCoboundaries(R,p,true);
fi;

if Length(arg)>6 then Q:=arg[7];
else
Q:=CR_Mod2CocyclesAndCoboundaries(R,q,true);
fi;

if Length(arg)>7 then N:=arg[8];
else
N:=CR_Mod2CocyclesAndCoboundaries(R,p+q,true);
fi;
    ####################FINISHED REAQDING THE INPUT#############

uCocycle:=P.classToCocycle(u);
vCocycle:=Q.classToCocycle(v);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,p,q);


uvCocycle:=[];
for i in [1..DimensionR(p+q)] do
w:=uChainMap([[i,1]]);
sw:=0;
    for x in w do
```

```
    sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
    od;
uvCocycle[i]:=sw mod 2;
od;

return N.cocycleToClass(uvCocycle);
end;
#####################################################################
#####################################################################



#####################################################################
#####################################################################

Mod2RingGenerators:=function(arg)
local
        R, n, GG, IT,
        Gens, GensBasis, Cups, Cupped, cupped, CuppedBasis, spacedim,
        uCocycle, vCocycle, uvCocycle, ww, uChainMap,
        sol, CB, CohomologyBasis, TR,
        BasisP, BasisQ, GF2ToZ,
        i, p, q, u, v, ln, iu, iv, w, x, sw;

n:=arg[2];
spacedim:=3;

if Length(arg)=3 then
spacedim:=arg[3];
fi;

if IsInt(arg[1]) then
IT := arg[1];
if IT = 133 or IT = 138 or IT = 210 or IT = 222 or IT = 224 then
    GG := Image(IsomorphismPcpGroup(SpaceGroupIT(spacedim,IT)));
else
    GG := Image(IsomorphismPcpGroup(SpaceGroupBBNWZ(spacedim,IT)));
fi;
R := ResolutionAlmostCrystalGroup(GG,n+1);

else if IsGroup(arg[1]) then
    GG := Image(IsomorphismPcpGroup(arg[1]));
    R := ResolutionAlmostCrystalGroup(GG,n+1);
    else
        R:=arg[1];
    fi;
fi;


TR:=HomToIntegersModP(R,2);
if Cohomology(TR,n) = 0 then return []; fi;
#####################################################################
CohomologyBasis:=function(Torsion)
local i, v, Basis;
Basis:=[];
for i in [1..Length(Torsion)] do
v:=List([1..Length(Torsion)], j->0);
v[i]:=Torsion[i];
Append(Basis, [v]);
```

```
od;
return Basis;
end;
######################################################################
GF2ToZ:=function(v)
local v0,k;

v0:=[];
for k in [1..Length(v)] do
    if v[k] = 0*Z(2) then
    v0[k]:=0;
    else
        v0[k]:=1;
    fi;
od;

return v0;
end;
######################################################################

Cups:=CohomologyBasis(List([1..Cohomology(TR,n)],i->1));

if n = 1 then

return Cups;

fi;

CB:=[];
for p in [1..n] do
CB[p]:=CR_Mod2CocyclesAndCoboundaries(R,p,true);
od;

Cupped :=[];

for p in [1..QuoInt(n,2)] do
q:=n-p;
BasisP:=CohomologyBasis(List([1..Cohomology(TR,p)],i->1));
BasisQ:=CohomologyBasis(List([1..Cohomology(TR,q)],i->1));


iu :=1;
for u in BasisP do

uCocycle:=CB[p].classToCocycle(u);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,p,q);
ww:=[];
for i in [1..(R!.dimension(n))] do
Append(ww, [uChainMap([[i,1]])]);
od;

iv :=1;
for v in BasisQ do

vCocycle:=CB[q].classToCocycle(v);

if ((p < q) or (p=q and iv>=iu)) then

    uvCocycle:=[];
```

```
    for i in [1..(R!.dimension(n))] do
        #w:=uChainMap([[i,1]]);
        w:=ww[i];
        sw:=0;
        for x in w do
            sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
        od;
        uvCocycle[i]:=sw mod 2;
    od;

    cupped := CB[n].cocycleToClass(uvCocycle);

    Append(Cupped,[cupped*Z(2)]);

    #cupped :=Mod2CupProduct(R,u,v,p,q,CB[p],CB[q],CB[n]);

fi;

iv := iv+1;
od;

iu := iu+1;
od;

od;

if Cupped = [] then
CuppedBasis := [];
else
CuppedBasis := List(BaseMat(Cupped),ShallowCopy);
fi;

#Append(CuppedBasis,[List([1..Cohomology(TR,n)],x->0*Z(2))]);

#Gens := List(BaseOrthogonalSpaceMat(CuppedBasis),ShallowCopy);

#Print(Cups);

Gens := BaseSteinitzVectors(Cups*Z(2),CuppedBasis)!.factorspace;

GensBasis :=[];

for i in [1..Length(Gens)] do
    GensBasis[i]:=GF2ToZ(Gens[i]);
od;


return GensBasis;
end;
#####################################################################
#####################################################################


#####################################################################
#####################################################################

Mod2RingGenerators1to5:=function(arg)
local
        R,n,GG,IT,Gen1,Gen2,Gen3,Gen4,Gen5,Gens,
```

```
        cupped, Cups,
        Letters, TR,
        i, p, q, r, u, v, ln, iu, iv, w, x, sw;


if Length(arg)=1 then
n:=5;
else
n:=arg[2];
fi;


if IsInt(arg[1]) then
IT := arg[1];
if IT = 133 or IT = 138 or IT = 210 or IT = 222 or IT = 224 then
    GG := Image(IsomorphismPcpGroup(SpaceGroupIT(3,IT)));
else
    GG := Image(IsomorphismPcpGroup(SpaceGroupBBNWZ(3,IT)));
fi;
R := ResolutionAlmostCrystalGroup(GG,n+1);

else if IsGroup(arg[1]) then
    GG := Image(IsomorphismPcpGroup(arg[1]));
    R := ResolutionAlmostCrystalGroup(GG,n+1);
    else
        R:=arg[1];
    fi;
fi;

TR:=HomToIntegersModP(R,2);

####################################################################



Letters := [["A1","A2","A3","A4","A5","A6","A7"],["B1","B2","B3","B4","B5","B6","B7"],["C1","C2","C3","C4","C

Gen1:=Mod2RingGenerators(R,1);
Gen2:=Mod2RingGenerators(R,2);
Gen3:=Mod2RingGenerators(R,3);
Gen4:=Mod2RingGenerators(R,4);
Gen5:=Mod2RingGenerators(R,5);

Gens:=[Gen1,Gen2,Gen3,Gen4,Gen5];


return [IT, Gens];
end;

####################################################################
####################################################################






####################################################################
####################################################################
```

```
Mod2RingGensAndRels:=function(arg)
local
        R,n,GG,IT,Gen1,Gen2,Gen3,Gen4,spacedim,
        Gens, Cupped, CuppedLetter, CupRels, CupRelsLetter,
        CupBase2, CupBase3,CupBase4,CupBase5,CupBase6,
        CupBase2Letter,CupBase3Letter,CupBase4Letter,CupBase5Letter,CupBase6Letter,
        CupRel2, CupRel2Letter, CupRel3Letter, CupRel4Letter, CupRel5Letter, CupRel6Letter,
        cupped, Cups,
        Letters, AddLetters,
        uCocycle, vCocycle, uvCocycle, uChainMap, ww,
        sol, cc, CB, CohomologyBasis, TR,
        BasisP, BasisQ, SmithRecord, GF2ToZ, IToPosition,
        i, p, q, r, u, v, ln, iu, iv, w, x, sw;


if Length(arg)=1 then
n:=6;
spacedim:=3;
else
n:=arg[2];
spacedim:=3;
fi;

if Length(arg)=3 then
n:=6;
spacedim:=arg[3];
fi;

if IsInt(arg[1]) then
IT := arg[1];
if IT = 133 or IT = 138 or IT = 210 or IT = 222 or IT = 224 then
    GG := Image(IsomorphismPcpGroup(SpaceGroupIT(spacedim,IT)));
else
    GG := Image(IsomorphismPcpGroup(SpaceGroupBBNWZ(spacedim,IT)));
fi;
R := ResolutionAlmostCrystalGroup(GG,n+1);

else if IsGroup(arg[1]) then
    GG := Image(IsomorphismPcpGroup(arg[1]));
    R := ResolutionAlmostCrystalGroup(GG,n+1);
    else
        R:=arg[1];
    fi;
fi;

TR:=HomToIntegersModP(R,2);

#####################################################################
CohomologyBasis:=function(Torsion)
local i, v, Basis;
Basis:=[];
for i in [1..Length(Torsion)] do
v:=List([1..Length(Torsion)], j->0);
v[i]:=Torsion[i];
Append(Basis, [v]);
od;
return Basis;
end;
```

```
#####################################################################
GF2ToZ:=function(v)
local v0,k;

v0:=[];
for k in [1..Length(v)] do
    if v[k] = 0*Z(2) then
    v0[k]:=0;
    else
        v0[k]:=1;
    fi;
od;

return v0;
end;
######################################################################
IToPosition:=function(v)
local v0,k;

v0:=[];
for k in [1..Length(v)] do
    if v[k] = 1 then
    Append(v0, [ k ]);
    fi;
od;
return v0;
end;
######################################################################
AddLetters:=function(v)
local v0,k;

if v = [] then
 return [];
fi;
v0:=v[1];
if Length(v)>1 then
for k in [2..Length(v)] do
v0:=Concatenation(v0,"+",v[k]);
od;
fi;
return [v0];
end;
######################################################################

Letters := [["A1","A2","A3","A4","A5","A6","A7"],["B1","B2","B3","B4","B5","B6","B7"],["C1","C2","C3","C4","C

Gen1:=Mod2RingGenerators(R,1,spacedim);
Gen2:=Mod2RingGenerators(R,2,spacedim);
Gen3:=Mod2RingGenerators(R,3,spacedim);
Gen4:=Mod2RingGenerators(R,4,spacedim);
Gens:=[Gen1,Gen2,Gen3,Gen4];

CB:=[];
for p in [1..n] do
CB[p]:=CR_Mod2CocyclesAndCoboundaries(R,p,true);
od;


####################### r = 2 ##########################
```

```
CupBase2 := [];
CupBase2Letter := [];

iu :=1;
for u in Gen1 do
iv :=1;
for v in Gen1 do
if iv>=iu then
    cupped :=Mod2CupProduct(R,u,v,1,1,CB[1],CB[1],CB[2]);
    Append(CupBase2,[cupped]);
    Append(CupBase2Letter,[Concatenation(Letters[1,iu],Letters[1,iv])]);
fi;
iv := iv+1;
od;
iu := iu+1;
od;


CupRel2 := [];
CupRel2Letter := [];

if not (CupBase2 = []) then
CupRel2 := List(BasisNullspaceModN(CupBase2,2),ShallowCopy)*Z(2);
for cc in CupRel2 do
    Append(CupRel2Letter,AddLetters(List(IToPosition(GF2ToZ(cc)),x->CupBase2Letter[x])));
od;
fi;


#CupBase2 := List(BaseOrthogonalSpaceMat(CupRels),ShallowCopy);
#CupBase2 := List(BaseMat(TransposedMat(Cupped)*Z(2)),ShallowCopy);
####Both are problematic!!!

#iu :=1;
#for cc in Gen2 do
#    Append(CupBase2, [cc]);
#    Append(CupBase2Letter,[Letters[2,iu]]);
#    iu :=iu+1;
#od;



####################### r = 3 ##########################



CupBase3 :=[];
CupBase3Letter :=[];
iu :=1;
for u in Gen1 do
iv :=1;
for v in CupBase2 do
    cupped :=Mod2CupProduct(R,u,v,1,2,CB[1],CB[2],CB[3]);
    Append(CupBase3,[cupped]);
    Append(CupBase3Letter,[Concatenation(Letters[1,iu],CupBase2Letter[iv])]);
    iv := iv+1;
od;
iu := iu+1;
```

```
od;


CupRel3Letter := [];

iu :=1;
for u in Gen1 do
iv :=1;
for cc in Gen2 do
    cupped :=Mod2CupProduct(R,u,cc,1,2,CB[1],CB[2],CB[3]);
    if cupped = List([1..Cohomology(TR,3)],x->0) then
        Append(CupRel3Letter,[Concatenation(Letters[1,iu],Letters[2,iv])]);
    else
    if not (CupBase3 = []) then
    sol :=SolutionMat(CupBase3*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase3,[cupped]);
        Append(CupBase3Letter,[Concatenation(Letters[1,iu],Letters[2,iv])]);
    else
        #Print(sol);
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase3Letter[x]);
        Append(sol,[Concatenation(Letters[1,iu],Letters[2,iv])]);
        Append(CupRel3Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;



####################### r = 4 ##########################

CupBase4 :=[];
CupBase4Letter :=[];
iu :=1;
for u in Gen1 do
iv :=1;
for v in CupBase3 do
    cupped :=Mod2CupProduct(R,u,v,1,3,CB[1],CB[3],CB[4]);
    Append(CupBase4,[cupped]);
    Append(CupBase4Letter,[Concatenation(Letters[1,iu],CupBase3Letter[iv])]);
    iv := iv+1;
od;
iu := iu+1;
od;


CupRel4Letter := [];



iu :=1;
for u in Gen1 do
iv :=1;
for cc in Gen3 do
    cupped :=Mod2CupProduct(R,u,cc,1,3,CB[1],CB[3],CB[4]);
```

```
    if cupped = List([1..Cohomology(TR,4)],x->0) then
        Append(CupRel4Letter,[Concatenation(Letters[1,iu],Letters[3,iv])]);
    else
    if not (CupBase4 = []) then
    sol :=SolutionMat(CupBase4*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase4,[cupped]);
        Append(CupBase4Letter,[Concatenation(Letters[1,iu],Letters[3,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase4Letter[x]);
        Append(sol,[Concatenation(Letters[1,iu],Letters[3,iv])]);
        Append(CupRel4Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;



iu :=1;
for u in Gen2 do
iv :=1;
for cc in Gen2 do
    if iv>=iu then
    cupped :=Mod2CupProduct(R,u,cc,2,2,CB[2],CB[2],CB[4]);
    if cupped = List([1..Cohomology(TR,4)],x->0) then
        Append(CupRel4Letter,[Concatenation(Letters[2,iu],Letters[2,iv])]);
    else
    if not (CupBase4 = []) then
    sol :=SolutionMat(CupBase4*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase4,[cupped]);
        Append(CupBase4Letter,[Concatenation(Letters[2,iu],Letters[2,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase4Letter[x]);
        Append(sol,[Concatenation(Letters[2,iu],Letters[2,iv])]);
        Append(CupRel4Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;


####################### r = 5 ##########################

CupBase5 :=[];
CupBase5Letter :=[];
iu :=1;
for u in Gen1 do
iv :=1;
for v in CupBase4 do
    cupped :=Mod2CupProduct(R,u,v,1,4,CB[1],CB[4],CB[5]);
```

```
    Append(CupBase5,[cupped]);
    Append(CupBase5Letter,[Concatenation(Letters[1,iu],CupBase4Letter[iv])]);
    iv := iv+1;
od;
iu := iu+1;
od;


CupRel5Letter := [];



iu :=1;
for u in Gen1 do

uCocycle:=CB[1].classToCocycle(u);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,1,4);
ww:=[];
for i in [1..(R!.dimension(5))] do
Append(ww, [uChainMap([[i,1]])]);
od;


iv :=1;
for cc in Gen4 do

    vCocycle:=CB[4].classToCocycle(cc);

    uvCocycle:=[];
    for i in [1..(R!.dimension(5))] do
        w:=ww[i];
        sw:=0;
        for x in w do
            sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
        od;
        uvCocycle[i]:=sw mod 2;
    od;

    cupped := CB[5].cocycleToClass(uvCocycle);
    #cupped :=Mod2CupProduct(R,u,cc,1,4,CB[1],CB[4],CB[5]);
    if cupped = List([1..Cohomology(TR,5)],x->0) then
        Append(CupRel5Letter,[Concatenation(Letters[1,iu],Letters[4,iv])]);
    else
    if not (CupBase5 = []) then
    sol :=SolutionMat(CupBase5*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase5,[cupped]);
        Append(CupBase5Letter,[Concatenation(Letters[1,iu],Letters[4,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase5Letter[x]);
        Append(sol,[Concatenation(Letters[1,iu],Letters[4,iv])]);
        Append(CupRel5Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;
```

```
iu :=1;
for u in Gen2 do

uCocycle:=CB[2].classToCocycle(u);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,2,3);
ww:=[];
for i in [1..(R!.dimension(5))] do
Append(ww, [uChainMap([[i,1]])]);
od;


iv :=1;
for cc in Gen3 do

    vCocycle:=CB[3].classToCocycle(cc);

    uvCocycle:=[];
    for i in [1..(R!.dimension(5))] do
        w:=ww[i];
        sw:=0;
        for x in w do
            sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
        od;
        uvCocycle[i]:=sw mod 2;
    od;

    cupped := CB[5].cocycleToClass(uvCocycle);
    #cupped :=Mod2CupProduct(R,u,cc,2,3,CB[2],CB[3],CB[5]);

    if cupped = List([1..Cohomology(TR,5)],x->0) then
        Append(CupRel5Letter,[Concatenation(Letters[2,iu],Letters[3,iv])]);
    else
    if not (CupBase5 = []) then
    sol :=SolutionMat(CupBase5*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase5,[cupped]);
        Append(CupBase5Letter,[Concatenation(Letters[2,iu],Letters[3,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase5Letter[x]);
        Append(sol,[Concatenation(Letters[2,iu],Letters[3,iv])]);
        Append(CupRel5Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;


####################### r = 6 ##########################

CupBase6 :=[];
CupBase6Letter :=[];
iu :=1;
for u in Gen1 do
iv :=1;
for v in CupBase5 do
```

```
    cupped :=Mod2CupProduct(R,u,v,1,5,CB[1],CB[5],CB[6]);
    Append(CupBase6,[cupped]);
    Append(CupBase6Letter,[Concatenation(Letters[1,iu],CupBase5Letter[iv])]);
    iv := iv+1;
od;
iu := iu+1;
od;

iu :=1;
for u in Gen2 do
iv :=1;
for v in CupBase4 do
    cupped :=Mod2CupProduct(R,u,v,2,4,CB[2],CB[4],CB[6]);
    Append(CupBase6,[cupped]);
    Append(CupBase6Letter,[Concatenation(Letters[2,iu],CupBase4Letter[iv])]);
    iv := iv+1;
od;
iu := iu+1;
od;


CupRel6Letter := [];



iu :=1;
for u in Gen2 do

uCocycle:=CB[2].classToCocycle(u);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,2,4);
ww:=[];
for i in [1..(R!.dimension(6))] do
Append(ww, [uChainMap([[i,1]])]);
od;


iv :=1;
for cc in Gen4 do

    vCocycle:=CB[4].classToCocycle(cc);

    uvCocycle:=[];
    for i in [1..(R!.dimension(6))] do
        w:=ww[i];
        sw:=0;
        for x in w do
            sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
        od;
        uvCocycle[i]:=sw mod 2;
    od;

    cupped := CB[6].cocycleToClass(uvCocycle);
    #cupped :=Mod2CupProduct(R,u,cc,2,4,CB[2],CB[4],CB[6]);

    if cupped = List([1..Cohomology(TR,6)],x->0) then
        Append(CupRel6Letter,[Concatenation(Letters[2,iu],Letters[4,iv])]);
    else
    if not (CupBase6 = []) then
    sol :=SolutionMat(CupBase6*Z(2),cupped*Z(2));
```

```
    if sol = fail then
        Append(CupBase6,[cupped]);
        Append(CupBase6Letter,[Concatenation(Letters[2,iu],Letters[4,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase6Letter[x]);
        Append(sol,[Concatenation(Letters[2,iu],Letters[4,iv])]);
        Append(CupRel6Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    iv := iv+1;
od;
iu := iu+1;
od;

iu :=1;
for u in Gen3 do

uCocycle:=CB[3].classToCocycle(u);
uChainMap:=CR_ChainMapFromCocycle(R,uCocycle,3,3);
ww:=[];
for i in [1..(R!.dimension(6))] do
Append(ww, [uChainMap([[i,1]])]);
od;

iv :=1;
for cc in Gen3 do
    if iv>=iu then
    vCocycle:=CB[3].classToCocycle(cc);

    uvCocycle:=[];
    for i in [1..(R!.dimension(6))] do
        w:=ww[i];
        sw:=0;
        for x in w do
            sw:=sw+ SignInt(x[1])*vCocycle[AbsoluteValue(x[1])];
        od;
        uvCocycle[i]:=sw mod 2;
    od;

    cupped := CB[6].cocycleToClass(uvCocycle);
    #cupped :=Mod2CupProduct(R,u,cc,3,3,CB[3],CB[3],CB[6]);

    if cupped = List([1..Cohomology(TR,6)],x->0) then
        Append(CupRel6Letter,[Concatenation(Letters[3,iu],Letters[3,iv])]);
    else
    if not (CupBase6 = []) then
    sol :=SolutionMat(CupBase6*Z(2),cupped*Z(2));
    if sol = fail then
        Append(CupBase6,[cupped]);
        Append(CupBase6Letter,[Concatenation(Letters[3,iu],Letters[3,iv])]);
    else
        sol:=List(IToPosition(GF2ToZ(sol)),x->CupBase6Letter[x]);
        Append(sol,[Concatenation(Letters[3,iu],Letters[3,iv])]);
        Append(CupRel6Letter,AddLetters(sol));
    fi;
    fi;
    fi;
    fi;
```

```
    iv := iv+1;
od;
iu := iu+1;
od;




return rec(GensAtDegN:=List([1..4],x->List([1..Length(Gens[x])],y->Letters[x,y])),RelsAtDegN:=[CupRel2Letter,
end;
######################################################################
######################################################################
```

Next we can start our work. For any space group, we use

27 Other space groups

27.1 The book: Crystallographic Groups of Four-Dimensional Sapce

$\mathbb{Q}$ classes: there are total $2+3+3+7+5+7+5=32$ classes, corresponding to the 32 different point groups (different in the physical, operational sense). [Note that on the level of abstract groups there are only 18 distinct ones.]

Family (T/M/O/T/H); crystal system (1-7); $\mathbb{Q}$ classes ($2+3+3+7+5+7+5=32$); $\mathbb{Z}$ classes.

- Triclinic: 1 crystal system, 2 $\mathbb{Q}$ classes
- Monoclinic: 1 crystal system, 3 $\mathbb{Q}$ classes
- Orthorhombic: 1 crystal system, 3 $\mathbb{Q}$ classes
- Tetragonal: 1 crystal system, 7 $\mathbb{Q}$ classes
- Hexagonal: 2 crystal systems, 5 $\mathbb{Q}$ classes + 7 $\mathbb{Q}$ classes
- Cubic: 1 crystal system, 5 $\mathbb{Q}$ classes

Below are Crystal system/$\mathbb{Q}$ class/$\mathbb{Z}$ classes, the first pair of parentheses give the dimension of cohomology for each $\mathbb{Z}$ class; the second pair of parentheses give the actual number of space groups determined from the $\mathbb{Z}$ class. One can check that the number of space groups add up to 219.

- 1/1-2
  - 1/1/1(1):(1)
  - 1/2/1(1):(1)
- 2/1-3
  - 2/1/1-2(2,1):(2,1)
  - 2/2/1-2(4,2):(2,2)
  - 2/3/1-2(8,2):(4,2)
- 3/1-3
  - 3/1/1-4(8,2,1,2):(4,2,1,2)
  - 3/2/1-5(16,4,4,2,4):(10,3,4,2,3)
  - 3/3/1-4(64,8,2,8):(16,6,2,4)
- 4/1-7
  - 4/1/1-2(4,2):(3,2)
  - 4/2/1-2(1,1):(1,1)
  - 4/3/1-2(4,2):(4,2)

  - 4/4/1-2(8,2):(6,2)
  - 4/5/1-2(8,4):(8,4)
  - 4/6/1-4(4,4,2,2):(4,4,2,2)
  - 4/7/1-2(16,4):(16,4)

- 5/1-5
  - 5/1/1-2(1,3):(1,2) 146 143,144
  - 5/2/1-2(1,1):(1,1) 148 147
  - 5/3/1-3(1,3,3):(1,2,2) 155 149,151 150,152
  - 5/4/1-3(2,2,2):(2,2,2) 160,161 156,158 157,159
  - 5/5/1-3(2,2,2):(2,2,2) 166,167 162,163 164,165

- 6/1-7
  - 6/1/1(6):(4)
  - 6/2/1(1):(1)
  - 6/3/1(2):(2)
  - 6/4/1(6):(4)
  - 6/5/1(4):(4)
  - 6/6/1-2(2,2):(2,2)
  - 6/7/1(4):(4)

- 7/1-5
  - 7/1/1-3(2,1,2):(2,1,2)
  - 7/2/1-3(4,2,2):(3,2,2)
  - 7/3/1-3(4,2,2):(3,2,2)
  - 7/4/1-3(2,2,2):(2,2,2)
  - 7/5/1-3(4,4,2):(4,4,2)

27.2 Result for all 230 groups

Isomorphic space groups: (76,78),(91,95),(92,96),(144,145),(151,153),(152,154),(169,170),(171,172),(178,179),(180,181),(213,212)

The command for space group is (need to import HAP using `LoadPackage("HAP");` and have polymake installed (note that loading HAP automatically load polymake)

```
for Num in [1..230] do
> R:=ResolutionSpaceGroup(SpaceGroupIT(3,Num),15);
> for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14] do
> Print("H^",n,"=",Cohomology(HomToIntegersModP(R,2),n),";;");
> od;
> Print("\n");
> od;
```

The command for point group is (since it takes time, we can just calculate the 32 distinct point group, using instead "for Num in [1,2,3,6,10,16,25,47,,75,81,83,89,99,111,123,143,147,149,156,162,168,174,175,177,183,187,191,195,200,207,215,221] do" for the first line)

```
gap> for Num in [1..230] do
> R:=ResolutionFiniteGroup(PointGroup(SpaceGroupIT(3,Num)),15);
> for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14] do
```

```
> Print("H^",n,"=",Cohomology(HomToIntegersModP(R,2),n),";;");
> od;
> Print("\n");
> od;
```

Then, the command to obtain the cohomology with $\mathbb{Z}^{\text{or}}$ coefficient, where the superscript "or" means that if the space group G is orientation-reversing then it acts on the coefficient $\mathbb{Z}$ by multiplying by -1. In other words, the action is multiplying by the determant. According to the crystalline equivalence principle by Ryan and Dominic, this is the necessary action for crystalline SPT when treating them as internal symmetries. To implement this action, we do:

```
for Num in [1..230] do
> G:=SpaceGroupIT(3,Num);
> R:=ResolutionSpaceGroup(G,15);
> ZZ:=GL(1,Integers);;
> Zor:= GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)]]);;
> C:=HomToIntegralModule(R,Zor);;
> for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14] do
> Print("H^",n,"=",Cohomology(C,n),";;");
> od;
> Print("\n");
> od;
```

For example, if we want to work out $H^4(G, U(1)^{\text{or}}) \sim H^5(G, \mathbb{Z}^{\text{or}})$ (note only their torsion part equal each other!), then we do

```
for Num in [1..230] do
> G:=SpaceGroupIT(3,Num);
> R:=ResolutionSpaceGroup(G,6);
> ZZ:=GL(1,Integers);;
> Zor:= GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)]]);;
> C:=HomToIntegralModule(R,Zor);;
> Print("H^5=",Cohomology(C,5),";;");
> Print("\n");
od;
```

Then, the command to obtain the 3rd cohomology with $\mathbb{Z}$ coefficients:

```
for Num in [1..219] do
> e:=Image(IsomorphismPcpGroup(SpaceGroup(3,Num)));
> R:=ResolutionAlmostCrystalGroup(e,4);
> Print(Num,":H^3=",Cohomology(HomToIntegers(R),3),"\n");
> od;
```

Note that this finishes at number 219, so this is not the international numbering. So, we should use this (although a bit slow):

```
for Num in [1..230] do
> Print(Num,":H^3=",GroupCohomology(SpaceGroupBBNWZ(3,Num),3),"\n");
> od;
```

But it turns out the command for point group “ResolutionFiniteGroup” is too slow. For these finite groups, since we already have results let us address them now. Note that in terms of the abstract group structure, there are only the following classes: $\mathbb{Z}_n$, Dih_n, A_4, S_4, $\mathbb{Z}_n \times \mathbb{Z}_2$, $Dih_n \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, $S_4 \times \mathbb{Z}_2$, where Dih_n is the dihedral group with order $2n$. Note that for n odd we have $\mathbb{Z}_{2n} = \mathbb{Z}_n \times \mathbb{Z}_2$ and $Dih_{2n} = Dih_n \times \mathbb{Z}_2$. We have $T \cong A_4$, $T_d \cong S_4$, $T_h \cong A_4 \times \mathbb{Z}_2$, $O \cong S_4$, and $O_h \cong S_4 \times \mathbb{Z}_2$.

Note that if we want we can use different action for the $\mathbb{Z}$ coefficient. For the pyrochlore case (No. 227), this is given in subsec. 22.2.

From the wikipedia page of Crystallographic_point_group, we see that the building blocks of point groups are $\mathbb{Z}_{1,2,3,4,6}$, and $Dih_{2,3,4,6}$.

For finite abelian group, we have

$$H^n(\mathbb{Z}_q, \mathbb{Z}_2) = \mathbb{Z}_{\gcd(q,2)}, \tag{201}$$

i.e. when q is even the answer is $\mathbb{Z}_2$, and when q is odd the result is trivial.

From Adem Milgram, we have (see P127 for dihedral; see P143 and P179, theorem 2 for symmetric groups)

$$\begin{aligned} H^n(Dih_3, \mathbb{Z}_2) = H^n(S_3, \mathbb{Z}_2) = H^n(\mathbb{Z}_2, \mathbb{Z}_2) = 1, \\ H^n(Dih_2, \mathbb{Z}_2) = H^n((Dih_4, \mathbb{Z}_2) = H^n(Dih_6, \mathbb{Z}_2) = n + 1. \end{aligned} \tag{202}$$

We have also calculated the mod-2 cohomology dimension for A_4 and S_4, which we denoted by $m(n)$ and $f(n)$ in Eq. (89) and (90). We copy them here:

$$\begin{aligned} m(n) &= [n/2] + [(n-3)/6] - [(n+2)/3] + 2 - \delta_{n \equiv 4 \bmod 6}, \\ f(n) &= [n/2] + [(n-3)/6] + 2 - \delta_{n \equiv 4 \bmod 6}. \end{aligned} \tag{203}$$

This has be verified up to $n \leq 40$ in GAP using “`R:=ResolutionSmallGroup(A4,41);  `”, “`for n in [0..40] do`”, “`Print(Cohomology(HomToIntegersModP(R,2),n),",")`”, “`od;`”.

Note that $f(n)$ is the number of nonnegative integer solutions to $a + 2b + 3c = n$ subject to the condition that when $c \geq 1$ then $a = 0$ (or equivalently, when $a \geq 1$ then $c = 0$).

Now we construct the normal subgroup series. First, we start from No. 221–230: we want to use $O_h = S_4 \times \mathbb{Z}_2 \supset S_4 \supset A_4 \supset$, where the first is to break the inversion symmetry.

Point group information:

There are in total 18 distinct abstract point groups:

$$1, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2, Dih_4, Dih_4 \times \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_2^2, Dih_3, Dih_3 \times \mathbb{Z}_2, Dih_3 \times \mathbb{Z}_2^2, A_4, A_4 \times \mathbb{Z}_2, S_4, S_4 \times \mathbb{Z}_2. \tag{204}$$

Using the code in Mathematica and python for computing the mod-2 rank of a sparse matrix over GF(2), we find the $E_2^{p,1}$ and $E_2^{p,2}$ terms in the translation–point group spectral LHS spectral sequence by checking for small p. Those dimensions with a $*$ symbol are the ones that should be checked more, but I think (with 99% confidence) that what is written is the final results.

First, look at $\mathbb{Z}_4$. Using Theorem 9.27 on Rotman's book, we define $D = x - 1$ and $N = 1 + x + x^2 + x^3$ for $x = d_1, q_4, q_5$, and $A = \mathbb{Z}_2^3$. We need to calculate

$$_NA = \{a \in A | Na = 0\}, \quad NA, \quad A^G, \quad DA,$$

We have

x	$_NA$	NA	A^G	DA	$H^0(G,A) = A^G$	$H^{2n-1}(G,A) = {}_NA/DA$	A^G/NA
d_1	A	Trivial	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	2	2	2
q_4	A	Trivial	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	1	1
q_5	A	Trivial	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	1	1

Then let's look at $\mathbb{Z}_3$, with $D = x - 1$ and $N = 1 + x + x^2$ for $x = t_2, t_{2r}, t_3, t_4$, and $A = \mathbb{Z}_2^3$. We have

x	$_NA$	NA	A^G	DA	$H^0(G,A) = A^G$	$H^{2n-1}(G,A) = {}_NA/DA$	A^G/NA
$t_{2(r)}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	0	0
t_3	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	0	0
t_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	0	0

Then for $\mathbb{Z}_2$, with $D = x - 1$ and $N = 1 + x = 1 - x$ for $x = I, d_1$, and $A = \mathbb{Z}_2^3$:

Table 5: Point group information. Note that for the group cohomology, x, e, a, b, σ_1 have degree 1, y and σ_2 has degree 2, and c has degree 3. For the form of the function $m(n)$ and $f(n)$, see Eq. (203).

No.	International	Schönflies	Abstract	Order	Mod-2 Coh. Ring	Coh. Dim. at n	Generators
1	1	C_1	Trivial	1	Trivial	0	Trivial
2	$\bar{1}$	C_i	$\mathbb{Z}_2$	2	$\mathbb{F}_2[x]$	1	(2)
3–5	2	C_2	$\mathbb{Z}_2$	2	$\mathbb{F}_2[x]$	1	(2)
6–9	m	C_s	$\mathbb{Z}_2$	2	$\mathbb{F}_2[x]$	1	(2)
10–15	$2/m$	C_{2h}	$\mathbb{Z}_2^2$	4	$\mathbb{F}_2[x_1, x_2]$	$n+1$	$(2), (3)$
16–24	222	D_2	$\mathbb{Z}_2^2$	4	$\mathbb{F}_2[x_1, x_2]$	$n+1$	$(2), (3)$
25–46	$mm2$	C_{2v}	$\mathbb{Z}_2^2$	4	$\mathbb{F}_2[x_1, x_2]$	$n+1$	$(2), (3)$
47–74	mmm	D_{2h}	$\mathbb{Z}_2^3$	8	$\mathbb{F}_2[x_1, x_2, x_3]$	$\frac{1}{2}(n+1)(n+2)$	$(2), (3), (5)$
75–80	4	C_4	$\mathbb{Z}_4$	4	$\mathbb{F}_2[x]$	1	(3)
81–82	$\bar{4}$	S_4	$\mathbb{Z}_4$	4	$\mathbb{F}_2[x]$	1	(3)
83–88	$4/m$	C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8	$\mathbb{F}_2[x_1, x_2]$	$n+1$	$(3), (5)$
89–98	422	D_4	Dih_4	8	$\mathbb{F}_2[e, \sigma_1, \sigma_2]/(e\sigma_1)$	$n+1$	$(3), (5)$
99–110	$4mm$	C_{4v}	Dih_4	8	$\mathbb{F}_2[e, \sigma_1, \sigma_2]/(e\sigma_1)$	$n+1$	$(3), (5)$
111–122	$\bar{4}2m$	D_{2d}	Dih_4	8	$\mathbb{F}_2[e, \sigma_1, \sigma_2]/(e\sigma_1)$	$n+1$	$(3), (5)$
123–142	$4/mmm$	D_{4h}	$Dih_4 \times \mathbb{Z}_2$	16	$\mathbb{F}_2[e, \sigma_1, \sigma_2, x]/(e\sigma_1)$	$\frac{1}{2}(n+1)(n+2)$	$(3), (5), (9)$
143–146	3	C_3	$\mathbb{Z}_3$	3	Trivial	0	(2)
147–148	$\bar{3}$	S_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6	$\mathbb{F}_2[x]$	1	$(2), (4)$
149–155	32	D_3	Dih_3	6	$\mathbb{F}_2[e]$	1	$(2), (4)$
156–161	$3m$	C_{3v}	Dih_3	6	$\mathbb{F}_2[e]$	1	$(2), (4)$
162–167	$\bar{3}m$	D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12	$\mathbb{F}_2[e, x]$	$n+1$	$(2)\cdot(7), (4)$
168–173	6	C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6	$\mathbb{F}_2[x]$	1	$(2), (4)$
174	$\bar{6}$	C_{3h}	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6	$\mathbb{F}_2[x]$	1	$(2), (4)$
175–176	$6/m$	C_{6h}	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	12	$\mathbb{F}_2[x_1, x_2]$	$n+1$	$(2)\cdot(4), (7)$
177–182	622	D_6	$Dih_3 \times \mathbb{Z}_2$	12	$\mathbb{F}_2[e, x]$	$n+1$	$(2)\cdot(4), (7)$
183–186	$6mm$	C_{6v}	$Dih_3 \times \mathbb{Z}_2$	12	$\mathbb{F}_2[e, x]$	$n+1$	$(2)\cdot(4), (7)$
187–190	$\bar{6}m2$	D_{3h}	$Dih_3 \times \mathbb{Z}_2$	12	$\mathbb{F}_2[e, x]$	$n+1$	$(2)\cdot(4), (7)$
191–194	$6/mmm$	D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	24	$\mathbb{F}_2[e, x_1, x_2]$	$\frac{1}{2}(n+1)(n+2)$	$(2)\cdot(4), (7), (13)$
195–199	23	T	A_4	12	$\mathbb{F}_2[a, b]^{\mathbb{Z}_3}$	$m(n)$	$(2), (5)$
200–206	$m\bar{3}$	T_h	$A_4 \times \mathbb{Z}_2$	24	$\mathbb{F}_2[a, b]^{\mathbb{Z}_3} \otimes \mathbb{F}_2[x]$	$\sum_{j=0}^{n} m(j)$	$(2), (5)\cdot(13)$
207–214	432	O	S_4	24	$\mathbb{F}_2[x, y, c]/(xc)$	$f(n)$	$(5), (13)$
215–220	$\bar{4}3m$	T_d	S_4	24	$\mathbb{F}_2[x, y, c]/(xc)$	$f(n)$	$(2)\cdot(13), (5)$
221–230	$m\bar{3}m$	O_h	$S_4 \times \mathbb{Z}_2$	48	$\mathbb{F}_2[x, x_1, y, c]/(xc)$	$\sum_{j=0}^{n} f(n)$	$(13), (5)\cdot(25)$

Table 6: Table for $E_2^{p,1}$ and $E_2^{p,2}$: point group cohomology for coefficients χ or b.

PG	No.	PG Coh. Dim.	Gen	Reps	Coh. Dim.
1	1	0	trivial	I	$3\delta_{n=0}$
$\mathbb{Z}_2$	2–9	1	(2)	I	3
				d_1	$\delta_{n=0}+1$
$\mathbb{Z}_2^2$	10–46	$n+1$	(2),(3)	I, I	$3(1+n)$
				d_1, I	$2+n$
				d_5, d_6	$2+n$
				d_4, d_3	$\delta_{n=0}+n$
				d_2, d_2	$2+n$
$\mathbb{Z}_2^3$	47–74	$\frac{1}{2}(n+1)(n+2)$	(2),(3),(5)	I, I, I	$\frac{3}{2}(n+1)(n+2)$
				I, d_1, I	$\frac{1}{2}(n+1)(n+4)$
				d_5, d_6, I	$\frac{1}{2}(n+1)(n+4)$
				d_4, d_3, I	$\frac{1}{2}n(n+1)+1$
$\mathbb{Z}_4$	75–82	1	(3)	d_1	2
				q_4	1
				q_5	1
$\mathbb{Z}_4\times\mathbb{Z}_2$	83–88	$n+1$	(3),(5)	d_1, I	$2(1+n)$
				q_4, I	$1+n$
				q_5, I	$1+n$
Dih_4	89–122	$n+1$	(3),(5)	d_1, I	$2(1+n)$
				q_4, d_3	$1+n$
				q_5, d_6	$1+n$
$Dih_4\times\mathbb{Z}_2$	123–142	$\frac{1}{2}(n+1)(n+2)$	(3),(5),(9)	d_1, I, I	$(n+1)(n+2)^*$
				q_4, d_3, I	$\frac{1}{2}(n+1)(n+2)^*$
				q_5, d_6, I	$\frac{1}{2}(n+1)(n+2)^*$
$\mathbb{Z}_3$	143–146	0	(2)	$t_{2(r)}$	$\delta_{n=0}$
				t_3	$\delta_{n=0}$
				t_4	$\delta_{n=0}$
$\mathbb{Z}_3\times\mathbb{Z}_2$	147–148, 168–174	1	(2),(4)	$t_{2(r)}, I$	1
				t_3, I	1
				t_4, I	1
$\mathbb{Z}_3\times\mathbb{Z}_2^2$	175–176	$n+1$	(2)·(4),(7)	$t_{2(r)}, I$	$n+1$
Dih_3	149–161	1	(2),(4)	$t_{2(r)}, d_1$	1
				t_3, d_4	1
				t_4, d_5	1
$Dih_3\times\mathbb{Z}_2$	162–167, 177–190	$n+1$	(2)·(7),(4); (2)·(4), (7)	$t_{2(r)}, d_1$	$n+1$
				t_3, d_4	$n+1$
				t_4, d_5	$n+1$
$Dih_3\times\mathbb{Z}_2^2$	191–194	$\frac{1}{2}(n+1)(n+2)$	(2)·(4),(7),(13)	$t_{2(r)}, d_1, I$	$\frac{1}{2}(n+1)(n+2)^*$
A_4	195–199	$m(n)$	(2),(5)	I, t_1	$1+n$
				d_5, t_1	$m(n+1)$
				d_4, t_1	$\delta_{n=0}+\delta_{n>0}m(n-1)$
$A_4\times\mathbb{Z}_2$	200–206	$\sum_{j=0}^{n} m(j)$	(2), (5)·(13)	t_1, I	$\frac{1}{2}(n+1)(n+2)^*$
				t_1, d_5	$\sum_{j=0}^{n} m(j+1)^*$
				t_1, d_4	$1+\sum_{j=1}^{n} m(j-1)^*$
S_4	207–214, 215–220	$f(n)$	(5),(13); (2)·(13), (5)	t_1, d_1	$1+n^*$
				t_1, d_7	$1+f(n-2)$
				t_1, d_8	$1+f(n-1)$
				d_1, t_1	$1+n^*$
				d_7, t_1	$1+f(n-2)$
				d_8, t_1	$1+f(n-1)$
$S_4\times\mathbb{Z}_2$	221–230	$\sum_{j=0}^{n} f(j)$	(13), (5)·(25)	t_1, d_1	$\frac{1}{2}(n+1)(n+2)^*$
				t_1, d_7	$1+n+\sum_{i=0}^{n} f(i-2)$
				t_1, d_8	$1+n+\sum_{i=0}^{n} f(i-1)$

Gen. X	$(X-1)(a,b,c)^T$	$(X-1).\langle\chi_{1,2,3}\rangle$
I	$(0,0,0)$	trivial
d_1	$(a+b,a+b,0)$	$\langle\chi_1+\chi_2\rangle$
d_5,d_6	$(a+b+c,a+b+c,0)$, $(a+b+c,0,a+b+c)$	$\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle$
d_4,d_3	$(a+b,a+b,a+b)$, $(a+c,a+c,a+c)$	$\langle\chi_1+\chi_2+\chi_3\rangle$
d_2,d_2	$(0,b+c,b+c)$	$\langle\chi_2+\chi_3\rangle$
q_4	$(b+c,b+c,a+c)$	$\langle\chi_1+\chi_2,\chi_3\rangle$
q_5	$(a+b,c,a+b+c)$	$\langle\chi_1+\chi_3,\chi_2+\chi_3\rangle$
q_4,d_3	$(b+c,b+c,a+c)$, $(a+c,a+c,a+c)$	$\langle\chi_1+\chi_2,\chi_3\rangle$
q_5,d_6	$(a+b,c,a+b+c)$, $(a+b+c,0,a+b+c)$	$\langle\chi_1+\chi_3,\chi_2+\chi_3\rangle$
t_2	$(b,a+b,0)$	$\langle\chi_1,\chi_2\rangle$
t_3	$(b,a+b,a+b)$	$\langle\chi_1,\chi_2+\chi_3\rangle$
t_{2r}	$(a+b,a,0)$	$\langle\chi_1,\chi_2\rangle$
t_4	$(a+b+c,a,0)$	$\langle\chi_1,\chi_2\rangle$
t_2,d_1	$(b,a+b,0)$, $(a+b,a+b,0)$	$\langle\chi_1,\chi_2\rangle$
t_3,d_4	$(b,a+b,a+b)$, $(a+b,a+b,a+b)$	$\langle\chi_1,\chi_2+\chi_3\rangle$
t_{2r},d_1	$(a+b,a,0)$, $(a+b,a+b,0)$	$\langle\chi_1,\chi_2\rangle$
t_4,d_5	$(a+b+c,a,0)$, $(a+b+c,a+b+c,0)$	$\langle\chi_1,\chi_2\rangle$
t_1	$(a+c,a+b,b+c)$	$\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle$
t_1,d_5	$(a+c,a+b,b+c)$, $(a+b+c,a+b+c,0)$	$\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle$
t_1,d_4	$(a+c,a+b,b+c)$, $(a+b,a+b,a+b)$	$\langle\chi_1,\chi_2,\chi_3\rangle$
t_1,d_1	$(a+c,a+b,b+c)$, $(a+b,a+b,0)$	$\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle$
t_1,d_7	$(a+c,a+b,b+c)$, $(c,c,0)$	$\langle\chi_1+\chi_2,\chi_1+\chi_3\rangle$
t_1,d_8	$(a+c,a+b,b+c)$, $(0,0,a+b)$	$\langle\chi_1,\chi_2,\chi_3\rangle$

x	${}_NA$	NA	A^G	DA	$H^0(G,A)=A^G$	$H^{2n-1}(G,A)={}_NA/DA$	A^G/NA
I	A	Trivial	A	Trivial	3	3	3
d_1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	2	1	1

Then for $\mathbb{Z}_3\times\mathbb{Z}_2=\mathbb{Z}_6$, with $D=x-1$ and $N=1+x+x^2+x^3+x^4+x^5$ for $x=t_2I,t_{2r}I,t_3I,t_4I=t_2,t_{2r},t_3,t_4$, and $A=\mathbb{Z}_2^3$:

x	${}_NA$	NA	A^G	DA	$H^0(G,A)=A^G$	$H^{2n-1}(G,A)={}_NA/DA$	A^G/NA
$t_{2(r)}$	A	Trivial	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	1	1
t_3	A	Trivial	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	1	1
t_4	A	Trivial	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	1	1	1

Then, for $\mathbb{Z}_2^2$. We see that there are five cases depending on the generators are (I,I), (d_1,I), (d_5,d_6), (d_4,d_3), and (d_2,d_2), but there are only three types of spectral sequence. We have $G=\mathbb{Z}_2^2$ so we set $N=\mathbb{Z}_2$ and $Q=\mathbb{Z}_2$, and examine $E_2^{p,q}=H^p(Q,H^q(N,A))$ where $A=\mathbb{Z}_2^3$ is either $\langle\chi_1,\chi_2,\chi_3\rangle$ or $\langle b_1,b_2,b_3\rangle$. For (I,I), the cohomology dimension is $3(1+n)$ so $E_2^{p,q}=3$ for all p,q. For (d_1,I), (d_5,d_6), and (d_2,d_2) which we have dimension $2+n$, and for (d_4,d_3) which we have dimension $\delta_{n=0}+n$, the respective spectral sequences are

$$\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=3 & 2 & 1 & 1 & 1 & \cdots \\
q=2 & 2 & 1 & 1 & 1 & \cdots \\
q=1 & 2 & 1 & 1 & 1 & \cdots \\
q=0 & 2 & 1 & 1 & 1 & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & \cdots
\end{array}\ ,\qquad
\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=3 & 1 & 1 & 1 & 1 & \cdots \\
q=2 & 1 & 1 & 1 & 1 & \cdots \\
q=1 & 1 & 1 & 1 & 1 & \cdots \\
q=0 & 1 & 0 & 0 & 0 & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & \cdots
\end{array} \tag{205}$$

We know these are the collapsed result, i.e. $E_2^{p,q}=E_\infty^{p,q}$, since the group $G=\mathbb{Z}_2^2$ now is a direct product of Q and N. This also gives the ring structure (for the row considered):

$$\mathbb{F}_2[x_1,x_2]\otimes\{a_1,a_2\}/x_1a_2,\qquad\qquad \mathbb{F}_2[x_1,x_2]\otimes\{a_1,a_2\}/(a_1x_1=a_2x_2), \tag{206}$$

here $\{a_1,a_2\}$ depends on the form of the generators. For example, for (d_5,d_6) we have $\{a_1,a_2\}=\{\chi_1+\chi_2,\chi_1+\chi_3\}$.

With this, one can show that for $\mathbb{Z}_2^3$ we already have the ring structure, which is what we have in the $\mathbb{Z}_2^2$ case cup product with x_3.

For $\mathbb{Z}_4 \times \mathbb{Z}_2$ and for $\mathbb{Z}_3 \times \mathbb{Z}_2^2$ we have $E_2^{p,q} = 2$ or 1 for all p, q, depending on the action. So these are also understood. Similarly, if we can understand Dih_4 and Dih_3, we see that $Dih_4 \times \mathbb{Z}_2$, $Dih_3 \times \mathbb{Z}_2$ and $Dih_3 \times \mathbb{Z}_2^2$ are just simple diagonal sum. So to understand all No.≤ 194 all we need to understand is Dih_4 and Dih_3.

For $Dih_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ (so that $N = \mathbb{Z}_4$ and $Q = \mathbb{Z}_2$), the Adem book shows that the spectral sequence collapses at E_2, so all $d_2 = 0$. And the dimension counting is actually obvious: $E_2^{p,q} = 2$ or 1 for all p, q depending on the action being (d_1, I) or (q_4, d_3) or (q_5, d_6).

For Dih_3, it is easy to see that we must have spectral sequence $E_2^{p,q} = \delta_{p=0}$.

We can now check the symmorphic space groups. We find that only the following six have a noncollapsing E_2 page (meaning E_3 page not the same as the E_2 page):

$$42, 69, 87, 107, 121, 139, 202, 217, 225, 229$$

Note that No. 139 space group's maximal space groups are precisely 69,87,107,121,139; and 69 has a maximal subgroup 42, so all these groups are connected; similarly we have that 202 has a maximal subgroup 69, 217 has a maximal subgroup of 121, and 225 of 139 and 202, and 229 of 139, 217. Therefore, in some sense the "problem" starts to appear at No. 42, which doesn't have a collapsing E_2 page. And this results in the noncollapsing of the other five space groups that is built upon it.

So let's focus on No. 42 space group, Fmm2. This space group is for FCC lattice type, generator (2) is a C_2 axis along z, and generator (3) is mirror that flips the y coordinate. Note that just keeping the generator (2) gives the subgroup No.5 C2, whose cohomology ring we have worked out. There, we the point generator C_2 is exactly the generator (2) here. So we just need to add another generator, which let's call M_y. M_y commutes with C_2, so in our original coordinate we have $M_y \colon (x, y, z) \to (x, 1/4 - y, z)$. We conjecture that the spectral sequence for Fmm2 has the form at the second page which gives total dimension $1, 4, 7, 9, 11, 13, ...$ for $n = 0, 1, 2, 3, 4, 5, ...$

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
q=4 & 3 & 2 & 2 & 2 & 2 & \cdots \\
q=3 & 3 & 2 & 2 & 2 & 2 & \cdots \\
q=2 & 3 & 2 & 2 & 2 & 2 & \cdots \\
q=1 & 3 & 3 & 3 & 3 & 3 & \cdots \\
q=0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots
\end{array} \tag{207}$$

where the $p = 0$ row is $H^q(C2, \mathbb{Z}_2)^{M_y} = (3^{M_y}, 4^{M_y}, 4^{M_y}, ...)$ for $q = 1, 2, 3, ...$ And we conjecture that $3^{M_y} = 3$, $4^{M_y} = 3$, $4^{M_y} = 3$ for $q = 1, 2, 3$, and so on.

With this, the No. 69 Fmmm collapses at the second page: it is easy to check that we have

$$H^n(Fmmm(No.69), \mathbb{Z}_2) = \sum_{i=0}^{n} H^i(Fmm2(No.42), \mathbb{Z}_2). \tag{208}$$

So the corresponding spectral sequence is just $E_\infty^{p,q} = E_2^{p,q} = H^q(Fmm2(No.42), \mathbb{Z}_2)$.

Similarly, we have $H^n(I4/mmm(No.139), \mathbb{Z}_2) = \delta_{n=0} + 3n = \sum_{i=0}^{n} H^i(S, \mathbb{Z}_2)$ for $S =$ No.87,107,121.

No. 42 group is an example of E_2 nondegenerate split groups. In Adem's paper `https://reader.elsevier.com/reader/sd/pii/S0021869308001014?token=08323CDB3D96BC53936272B403FE1BD768B73BDFB882AFA4EFD9DEA782CB6477FB5B519` `originRegion=us-east-1&originCreation=20221024230315` (see Sec. 5), and Ref. [16] therein, which is `https://reader.elsevier.com/reader/sd/pii/S0021869396901817?token=AECC9E42D43AAFA984EE16301FF39656E5FE44CBF05E91789EC` `originRegion=us-east-1&originCreation=20221024231218`. According to Sec. 5, for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, there exist $\mathbb{Z}G$-modules M which are not realizable as the cohomology of a G-space, and Ref. [16] construsted concrete examples, but takes $n = 2$, so $p^n = p^2 = 9$. It is not clear if setting $n = 3$ gives also the result.

So the last task now is to understand the three groups No.87,107,121. The maximal subgruops (of index 2) that have the smallest cohomology are the ones that have $(1, 2, 3, 4, 4, 4, 4, ...)$ for $n = 0, 1, 2, 3, 4, 5, 6....$ To get the corret dimension $\delta_{n=0} + 3n$, it seems the E_2 page does not collapse.

Note that we have the theorem (see P247-248 of Graham's book) that if the point group has periodic cohomology then so does the space group. For us, these point groups are $P = \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_2, Dih_3$. We see that indeed they have periodic cohomology (of period one).

- $P2_1/m$ (No. 11), nonsymmorphic. Note that the cohomology ring of finite abelian group (such as $\mathbb{Z}_2^3$) is also complicated, see `https://central.bac-lac.gc.ca/.item?id=TC-OWTU-7276&op=pdf&app=Library&oclc_number=835911846`.

- $C2/m$ (No. 12), symmorphic. Point group is $2/m = C_{2h} = \langle C_2, M \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that in the TaTe$_2$ coordinate, we write C_2 as C_{2y}, whose rotation axis is along y, and we write M as M_y, as it flips (x, y, z) to $(x, -y, z)$.

- $I4_1/amd$ (No. 141), nonsymmorphic. $\mathrm{LiYbO_2}$, point group D_{4h}.
- $P\bar{3}m1$ (No. 164), symmorphic. 1T phase TMD and $\mathrm{EuCd_2P_2}$. Point group D_{3d}.
- $R\bar{3}m$ (No. 166), symmorphic. Delafossites. Point group D_{3d}.
- $P6_3cm$ (No. 185), nonsymmorphic. BYBO; BErBO. Point group C_{6v}.
- $P6_3/mmc$ (No. 194), nonsymmorphic. $\mathrm{Mn_3Sn}$; 2H phase TMD; $\mathrm{EuIn_2As_2}$. Point group D_{6h}.

Define the following matrices

$$\begin{gathered}
d_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, d_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, d_3=\begin{pmatrix}0&0&1\\1&1&1\\1&0&0\end{pmatrix} d_4=\begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix},\\
d_5=\begin{pmatrix}0&1&1\\1&0&1\\0&0&1\end{pmatrix}, d_6=\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}, d_7=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}, d_8=\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix},\\
t_1=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, t_2=\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}, t_{2r}=t_2^2=\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}, t_3=\begin{pmatrix}1&1&0\\1&0&0\\1&1&1\end{pmatrix},\\
t_4=\begin{pmatrix}0&1&1\\1&1&0\\0&0&1\end{pmatrix}, q_4=\begin{pmatrix}1&1&1\\0&0&1\\1&0&0\end{pmatrix}, q_5=\begin{pmatrix}0&1&0\\0&1&1\\1&1&0\end{pmatrix}, I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}
\end{gathered}\tag{209}$$

and define $\chi_{1,2,3}$ and $b_{1,2,3}$ as usual. (Note that here we define $b_1(T_1^{x_1}T_2^{y_1}T_3^{z_1}, T_1^{x_2}T_2^{y_2}T_3^{z_2}) = y_1 z_2$ and so on.)

Note that the transitive action of point group on $\chi_{1,2,3}$ and $b_{1,2,3}$ are only of the following eight kinds (trivial one not counted):

$$1, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_4, \mathbb{Z}_3, Dih_3, Dih_4, A_4, S_4. \tag{210}$$

No.	Name	PG Generators	$\rtimes$?	$E_{2=\infty}$?	PG Sch.	PG. abs.	Ind.	$PG_{\chi,b}$	$\langle PG\rangle_\chi$	$\langle PG\rangle_b$	dim$[H^{1-10}(\text{Space Group}, \mathbb{Z}_2)]$
1	$P1$	Trivial	✓	✓	C_1	–	1:1	–	I	I	(3,3,1,0,0,0,0,0,0,0)
2	$P\bar{1}$	(2)	✓	✓	C_i	$\mathbb{Z}_2$	2:1	–	I	I	(4,7,8,8,8,8,8,8,8,8)
3	$P2$	(2)	✓	✓	C_2	$\mathbb{Z}_2$	2:1	–	I	I	(4,7,8,8,8,8,8,8,8,8)
4	$P2_1$	(2)			C_2	$\mathbb{Z}_2$	2:1	–	I	I	(3,3,1,0,0,0,0,0,0,0)
5	$C2$	(2)	✓	✓	C_2	$\mathbb{Z}_2$	2:2	$\mathbb{Z}_2$	d_1	d_1	(3,4,4,4,4,4,4,4,4,4)
6	Pm	(2)	✓	✓	C_s	$\mathbb{Z}_2$	2:1	–	I	I	(4,7,8,8,8,8,8,8,8,8)
7	Pc	(2)			C_s	$\mathbb{Z}_2$	2:1	–	I	I	(3,3,1,0,0,0,0,0,0,0)
8	Cm	(2)	✓	✓	C_s	$\mathbb{Z}_2$	2:2	$\mathbb{Z}_2$	d_1	d_1	(3,4,4,4,4,4,4,4,4,4)
9	Cc	(2)			C_s	$\mathbb{Z}_2$	2:2	$\mathbb{Z}_2$	d_1	d_1	(2,2,1,0,0,0,0,0,0,0)
10	$P2/m$	(2), (3)	✓	✓	C_{2h}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(5,12,20,28,36,44,52,60,68,76)
11	$P2_1/m$	(2), (3)			C_{2h}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)
12	$C2/m$	(2), (3)	✓	✓	C_{2h}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,8,12,16,20,24,28,32,36,40)
13	$P2/c$	(2), (3)			C_{2h}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)
14	$P2_1/c$	(2), (3)			C_{2h}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
15	$C2/c$	(2), (3)			C_{2h}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
16	$P222$	(2), (3)	✓	✓	D_2	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(5,12,20,28,36,44,52,60,68,76)
17	$P222_1$	(2), (3)			D_2	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)
18	$P2_12_12$	(2), (3)			D_2	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
19	$P2_12_12_1$	(2), (3)			D_2	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(2,2,1,0,0,0,0,0,0,0)
20	$C222_1$	(2), (3)			D_2	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	I, d_1	I, d_1	(3,4,4,4,4,4,4,4,4,4)
21	$C222$	(2), (3)	✓	✓	D_2	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	I, d_1	I, d_1	(4,8,12,16,20,24,28,32,36,40)
22	$F222$	(2), (3)	✓	✓	D_2	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_5, d_6	d_4, d_3	(4,7,10,14,18,22,26,30,34,38)
23	$I222$	(2), (3)	✓	✓	D_2	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_4, d_3	d_5, d_6	(3,6,10,14,18,22,26,30,34,38)
24	$I2_12_12_1$	(2), (3)			D_2	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_4, d_3	d_5, d_6	(3,5,6,6,6,6,6,6,6,6)
25	$Pmm2$	(2), (3)	✓	✓	C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(5,12,20,28,36,44,52,60,68,76)
26	$Pmc2_1$	(2), (3)			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)
27	$Pcc2$	(2), (3)			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)
28	$Pma2$	(2), (3)			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(4,7,8,8,8,8,8,8,8,8)

29	$Pca2_1$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,3,1,0,0,0,0,0,0,0)
30	$Pnc2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
31	$Pmn2_1$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
32	$Pba2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
33	$Pna2_1$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(2,2,1,0,0,0,0,0,0,0)
34	$Pnn2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:1	–	I, I	I, I	(3,4,4,4,4,4,4,4,4,4)
35	$Cmm2$	$(2), (3)$	✓	✓	C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	I, d_1	I, d_1	(4,8,12,16,20,24,28,32,36,40)
36	$Cmc2_1$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	I, d_1	I, d_1	(3,4,4,4,4,4,4,4,4,4)
37	$Ccc2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	I, d_1	I, d_1	(3,5,6,6,6,6,6,6,6,6)
38	$Amm2$	$(2), (3)$	✓	✓	C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_2, d_2	d_2, d_2	(4,8,12,16,20,24,28,32,36,40)
39	$Aem2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_2, d_2	d_2, d_2	(4,7,8,8,8,8,8,8,8,8)
40	$Ama2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_2, d_2	d_2, d_2	(3,5,6,6,6,6,6,6,6,6)
41	$Aea2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:2	$\mathbb{Z}_2$	d_2, d_2	d_2, d_2	(3,3,2,2,2,2,2,2,2,2)
42	$Fmm2$	$(2), (3)$	✓		C_{2v}	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_5, d_6	d_4, d_3	(4,7,9,11,13,15,17,19,21,23)
43	$Fdd2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_5, d_6	d_4, d_3	(2,2,2,2,2,2,2,2,2,2)
44	$Imm2$	$(2), (3)$	✓	✓	C_{2v}	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_4, d_3	d_5, d_6	(3,6,10,14,18,22,26,30,34,38)
45	$Iba2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_4, d_3	d_5, d_6	(3,4,4,4,4,4,4,4,4,4)
46	$Ima2$	$(2), (3)$			C_{2v}	$\mathbb{Z}_2^2$	4:4	$\mathbb{Z}_2^2$	d_4, d_3	d_5, d_6	(3,5,6,6,6,6,6,6,6,6)
47	$Pmmm$	$(2), (3), (5)$	✓	✓	D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(6,18,38,66,102,146,198,258,326,402)
48	$Pnnn$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,8,12,16,20,24,28,32,36,40)
49	$Pccm$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(5,12,20,28,36,44,52,60,68,76)
50	$Pban$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,8,12,16,20,24,28,32,36,40)
51	$Pmma$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(5,12,20,28,36,44,52,60,68,76)
52	$Pnna$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,5,6,6,6,6,6,6,6,6)
53	$Pmna$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,8,12,16,20,24,28,32,36,40)
54	$Pcca$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,7,8,8,8,8,8,8,8,8)
55	$Pbam$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,8,12,16,20,24,28,32,36,40)
56	$Pccn$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,5,6,6,6,6,6,6,6,6)
57	$Pbcm$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,7,8,8,8,8,8,8,8,8)
58	$Pnnm$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,6,10,14,18,22,26,30,34,38)
59	$Pmmn$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(4,8,12,16,20,24,28,32,36,40)
60	$Pbcn$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,4,4,4,4,4,4,4,4,4)
61	$Pbca$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,3,2,2,2,2,2,2,2,2)
62	$Pnma$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:1	–	I, I, I	I, I, I	(3,5,6,6,6,6,6,6,6,6)
63	$Cmcm$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(4,8,12,16,20,24,28,32,36,40)
64	$Cmce$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(4,7,9,11,13,15,17,19,21,23)
65	$Cmmm$	$(2), (3), (5)$	✓	✓	D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(5,13,25,41,61,85,113,145,181,221)
66	$Cccm$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(4,9,15,21,27,33,39,45,51,57)
67	$Cmme$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(5,12,20,28,36,44,52,60,68,76)
68	$Ccce$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:2	$\mathbb{Z}_2$	I, d_1, I	I, d_1, I	(4,7,9,11,13,15,17,19,21,23)
69	$Fmmm$	$(2), (3), (5)$	✓		D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_5, d_6, I	d_4, d_3, I	(5,12,21,32,45,60,77,96,117,140)
70	$Fddd$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_5, d_6, I	d_4, d_3, I	(3,5,7,9,11,13,15,17,19,21)
71	$Immm$	$(2), (3), (5)$	✓	✓	D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_4, d_3, I	d_5, d_6, I	(4,10,20,34,52,74,100,130,164,202)
72	$Ibam$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_4, d_3, I	d_5, d_6, I	(4,8,12,16,20,24,28,32,36,40)
73	$Ibca$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_4, d_3, I	d_5, d_6, I	(4,7,8,8,8,8,8,8,8,8)
74	$Imma$	$(2), (3), (5)$			D_{2h}	$\mathbb{Z}_2^3$	8:4	$\mathbb{Z}_2^2$	d_4, d_3, I	d_5, d_6, I	(4,9,15,21,27,33,39,45,51,57)
75	$P4$	(3)	✓	✓	C_4	$\mathbb{Z}_4$	4:2	$\mathbb{Z}_2$	d_1	d_1	(3,5,6,6,6,6,6,6,6,6)
76	$P4_1$	(3)			C_4	$\mathbb{Z}_4$	4:2	$\mathbb{Z}_2$	d_1	d_1	(2,2,1,0,0,0,0,0,0,0)
77	$P4_2$	(3)		✓	C_4	$\mathbb{Z}_4$	4:2	$\mathbb{Z}_2$	d_1	d_1	(3,5,6,6,6,6,6,6,6,6)
78	$P4_3$	(3)			C_4	$\mathbb{Z}_4$	4:2	$\mathbb{Z}_2$	d_1	d_1	(2,2,1,0,0,0,0,0,0,0)
79	$I4$	(3)	✓	✓	C_4	$\mathbb{Z}_4$	4:4	$\mathbb{Z}_4$	q_4	q_5	(2,3,4,4,4,4,4,4,4,4)
80	$I4_1$	(3)			C_4	$\mathbb{Z}_4$	4:4	$\mathbb{Z}_4$	q_4	q_5	(2,2,2,2,2,2,2,2,2,2)
81	$P\bar{4}$	(3)	✓	✓	S_4	$\mathbb{Z}_4$	4:2	$\mathbb{Z}_2$	d_1	d_1	(3,5,6,6,6,6,6,6,6,6)
82	$I\bar{4}$	(3)	✓	✓	S_4	$\mathbb{Z}_4$	4:4	$\mathbb{Z}_4$	q_4	q_5	(2,3,4,4,4,4,4,4,4,4)
83	$P4/m$	$(3), (5)$	✓	✓	C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
84	$P4_2/m$	$(3), (5)$			C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,7,11,15,19,23,27,31,35,39)
85	$P4/n$	$(3), (5)$			C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
86	$P4_2/n$	$(3), (5)$			C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
87	$I4/m$	$(3), (5)$	✓		C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	8:4	$\mathbb{Z}_4$	q_4, I	q_5, I	(3,6,9,12,15,18,21,24,27,30)

88	$I4_1/a$	(3), (5)			C_{4h}	$\mathbb{Z}_4\times\mathbb{Z}_2$	8:4	$\mathbb{Z}_4$	q_4, I	q_5, I	(2,3,4,4,4,4,4,4,4,4)
89	$P422$	(3), (5)	✓	✓	D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
90	$P42_12$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
91	$P4_122$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
92	$P4_12_12$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(2,2,2,2,2,2,2,2,2,2)
93	$P4_222$	(3), (5)		✓	D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
94	$P4_22_12$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
95	$P4_322$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
96	$P4_32_12$	(3), (5)			D_4	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(2,2,2,2,2,2,2,2,2,2)
97	$I422$	(3), (5)	✓	✓	D_4	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,6,10,14,18,22,26,30,34,38)
98	$I4_122$	(3), (5)			D_4	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,5,7,9,11,13,15,17,19,21)
99	$P4mm$	(3), (5)	✓	✓	C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
100	$P4bm$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
101	$P4_2cm$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,6,8,10,12,14,16,18,20,22)
102	$P4_2nm$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
103	$P4cc$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,6,6,6,6,6,6,6,6)
104	$P4nc$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(2,3,4,4,4,4,4,4,4,4)
105	$P4_2mc$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,7,11,15,19,23,27,31,35,39)
106	$P4_2bc$	(3), (5)			C_{4v}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(2,3,4,4,4,4,4,4,4,4)
107	$I4mm$	(3), (5)	✓		C_{4v}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,6,9,12,15,18,21,24,27,30)
108	$I4cm$	(3), (5)			C_{4v}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,5,6,7,8,9,10,11,12,13)
109	$I4_1md$	(3), (5)			C_{4v}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(2,3,5,7,9,11,13,15,17,19)
110	$I4_1cd$	(3), (5)			C_{4v}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(2,2,2,2,2,2,2,2,2,2)
111	$P\bar{4}2m$	(3), (5)	✓	✓	D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
112	$P\bar{4}2c$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,7,11,15,19,23,27,31,35,39)
113	$P\bar{4}2_1m$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
114	$P\bar{4}2_1c$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(2,3,4,4,4,4,4,4,4,4)
115	$P\bar{4}m2$	(3), (5)	✓	✓	D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(4,9,15,21,27,33,39,45,51,57)
116	$P\bar{4}c2$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,6,8,10,12,14,16,18,20,22)
117	$P\bar{4}b2$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
118	$P\bar{4}n2$	(3), (5)			D_{2d}	Dih_4	8:2	$\mathbb{Z}_2$	d_1, I	d_1, I	(3,5,7,9,11,13,15,17,19,21)
119	$I\bar{4}m2$	(3), (5)	✓	✓	D_{2d}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,6,10,14,18,22,26,30,34,38)
120	$I\bar{4}c2$	(3), (5)			D_{2d}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,5,6,8,10,12,14,16,18,20)
121	$I\bar{4}2m$	(3), (5)	✓		D_{2d}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(3,6,9,12,15,18,21,24,27,30)
122	$I\bar{4}2d$	(3), (5)			D_{2d}	Dih_4	8:8	Dih_4	q_4, d_3	q_5, d_6	(2,3,4,4,4,4,4,4,4,4)
123	$P4/mmm$	(3), (5), (9)	✓	✓	D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(5,14,29,50,77,110,149,194,245,302)
124	$P4/mcc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,9,15,21,27,33,39,45,51,57)
125	$P4/nbm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,9,15,21,27,33,39,45,51,57)
126	$P4/nnc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,6,9,12,15,18,21,24,27,30)
127	$P4/mbm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,9,16,25,36,49,64,81,100,121)
128	$P4/mnc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,6,10,14,18,22,26,30,34,38)
129	$P4/nmm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,9,15,21,27,33,39,45,51,57)
130	$P4/ncc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,5,6,7,8,9,10,11,12,13)
131	$P4_2/mmc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,11,22,37,56,79,106,137,172,211)
132	$P4_2/mcm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,10,18,28,40,54,70,88,108,130)
133	$P4_2/nbc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,6,9,12,15,18,21,24,27,30)
134	$P4_2/nnm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(4,9,15,21,27,33,39,45,51,57)
135	$P4_2/mbc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,6,9,12,15,18,21,24,27,30)
136	$P4_2/mnm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,7,13,21,31,43,57,73,91,111)
137	$P4_2/nmc$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,6,9,12,15,18,21,24,27,30)
138	$P4_2/ncm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:2	$\mathbb{Z}_2$	d_1, I, I	d_1, I, I	(3,7,11,15,19,23,27,31,35,39)
139	$I4/mmm$	(3), (5), (9)	✓		D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:8	Dih_4	q_4, d_3, I	q_5, d_6, I	(4,10,19,31,46,64,85,109,136,166)
140	$I4/mcm$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:8	Dih_4	q_4, d_3, I	q_5, d_6, I	(4,9,15,22,30,39,49,60,72,85)
141	$I4_1/amd$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:8	Dih_4	q_4, d_3, I	q_5, d_6, I	(3,6,10,14,18,22,26,30,34,38)
142	$I4_1/acd$	(3), (5), (9)			D_{4h}	$Dih_4\times\mathbb{Z}_2$	16:8	Dih_4	q_4, d_3, I	q_5, d_6, I	(3,5,6,7,8,9,10,11,12,13)
143	$P3$	(2)	✓	✓	C_3	$\mathbb{Z}_3$	3:3	$\mathbb{Z}_3$	t_2	t_{2r}	(1,1,1,0,0,0,0,0,0,0)
144	$P3_1$	(2)		✓	C_3	$\mathbb{Z}_3$	3:3	$\mathbb{Z}_3$	t_2	t_{2r}	(1,1,1,0,0,0,0,0,0,0)
145	$P3_2$	(2)		✓	C_3	$\mathbb{Z}_3$	3:3	$\mathbb{Z}_3$	t_2	t_{2r}	(1,1,1,0,0,0,0,0,0,0)
146	$R3$	(2)	✓	✓	C_3	$\mathbb{Z}_3$	3:3	$\mathbb{Z}_3$	t_3	t_4	(1,1,1,0,0,0,0,0,0,0)

147	$P\bar{3}$	$(2),(4)$	✓	✓	S_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
148	$R\bar{3}$	$(2),(4)$	✓	✓	S_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_3, I	t_4, I	(2,3,4,4,4,4,4,4,4,4)
149	$P312$	$(2),(4)$	✓	✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
150	$P321$	$(2),(4)$	✓	✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
151	$P3_112$	$(2),(4)$		✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
152	$P3_121$	$(2),(4)$		✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
153	$P3_212$	$(2),(4)$		✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
154	$P3_221$	$(2),(4)$		✓	D_3	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
155	$R32$	$(2),(4)$	✓	✓	D_3	Dih_3	6:6	Dih_3	t_3, d_4	t_4, d_5	(2,3,4,4,4,4,4,4,4,4)
156	$P3m1$	$(2),(4)$	✓	✓	C_{3v}	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
157	$P31m$	$(2),(4)$	✓	✓	C_{3v}	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
158	$P3c1$	$(2),(4)$			C_{3v}	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(1,1,1,0,0,0,0,0,0,0)
159	$P31c$	$(2),(4)$			C_{3v}	Dih_3	6:6	Dih_3	t_2, d_1	t_{2r}, d_1	(1,1,1,0,0,0,0,0,0,0)
160	$R3m$	$(2),(4)$	✓	✓	C_{3v}	Dih_3	6:6	Dih_3	t_3, d_4	t_4, d_5	(2,3,4,4,4,4,4,4,4,4)
161	$R3c$	$(2),(4)$			C_{3v}	Dih_3	6:6	Dih_3	t_3, d_4	t_4, d_5	(1,1,1,0,0,0,0,0,0,0)
162	$P\bar{3}1m$	$(2)(7),(4)$	✓	✓	D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
163	$P\bar{3}1c$	$(2)(7),(4)$			D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
164	$P\bar{3}m1$	$(2)(7),(4)$	✓	✓	D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
165	$P\bar{3}c1$	$(2)(7),(4)$			D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
166	$R\bar{3}m$	$(2)(7),(4)$	✓	✓	D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_3, d_4	t_4, d_5	(3,6,10,14,18,22,26,30,34,38)
167	$R\bar{3}c$	$(2)(7),(4)$			D_{3d}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_3, d_4	t_4, d_5	(2,3,4,4,4,4,4,4,4,4)
168	$P6$	$(2),(4)$	✓	✓	C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
169	$P6_1$	$(2),(4)$			C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(1,1,1,0,0,0,0,0,0,0)
170	$P6_5$	$(2),(4)$			C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(1,1,1,0,0,0,0,0,0,0)
171	$P6_2$	$(2),(4)$		✓	C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
172	$P6_4$	$(2),(4)$		✓	C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
173	$P6_3$	$(2),(4)$			C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(1,1,1,0,0,0,0,0,0,0)
174	$P\bar{6}$	$(2),(4)$	✓	✓	C_{3h}	$\mathbb{Z}_3 \times \mathbb{Z}_2$	6:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
175	$P6/m$	$(2)(4),(7)$	✓	✓	C_{6h}	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	12:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(3,6,10,14,18,22,26,30,34,38)
176	$P6_3/m$	$(2)(4),(7)$			C_{6h}	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	12:3	$\mathbb{Z}_3$	t_2, I	t_{2r}, I	(2,3,4,4,4,4,4,4,4,4)
177	$P622$	$(2)(4),(7)$	✓	✓	D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
178	$P6_122$	$(2)(4),(7)$			D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
179	$P6_522$	$(2)(4),(7)$			D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
180	$P6_222$	$(2)(4),(7)$		✓	D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
181	$P6_422$	$(2)(4),(7)$		✓	D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
182	$P6_322$	$(2)(4),(7)$			D_6	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
183	$P6mm$	$(2)(4),(7)$	✓	✓	C_{6v}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
184	$P6cc$	$(2)(4),(7)$			C_{6v}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
185	$P6_3cm$	$(2)(4),(7)$			C_{6v}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
186	$P6_3mc$	$(2)(4),(7)$			C_{6v}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
187	$P\bar{6}m2$	$(2)(4),(7)$	✓	✓	D_{3h}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
188	$P\bar{6}c2$	$(2)(4),(7)$			D_{3h}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
189	$P\bar{6}2m$	$(2)(4),(7)$	✓	✓	D_{3h}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(3,6,10,14,18,22,26,30,34,38)
190	$P\bar{6}2c$	$(2)(4),(7)$			D_{3h}	$Dih_3 \times \mathbb{Z}_2$	12:6	Dih_3	t_2, d_1	t_{2r}, d_1	(2,3,4,4,4,4,4,4,4,4)
191	$P6/mmm$	$(2)(4),(7),(13)$	✓	✓	D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	24:6	Dih_3	t_2, d_1, I	t_{2r}, d_1, I	(4,10,20,34,52,74,100,130,164,202)
192	$P6/mcc$	$(2)(4),(7),(13)$			D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	24:6	Dih_3	t_2, d_1, I	t_{2r}, d_1, I	(3,6,10,14,18,22,26,30,34,38)
193	$P6_3/mcm$	$(2)(4),(7),(13)$			D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	24:6	Dih_3	t_2, d_1, I	t_{2r}, d_1, I	(3,6,10,14,18,22,26,30,34,38)
194	$P6_3/mmc$	$(2)(4),(7),(13)$			D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	24:6	Dih_3	t_2, d_1, I	t_{2r}, d_1, I	(3,6,10,14,18,22,26,30,34,38)
195	$P23$	$(2),(5)$	✓	✓	T	A_4	12:3	$\mathbb{Z}_3$	I, t_1	I, t_1	(1,4,8,8,12,16,16,20,24,24)
196	$F23$	$(2),(5)$	✓	✓	T	A_4	12:12	A_4	d_5, t_1	d_4, t_1	(0,3,6,2,6,10,6,10,14,10)
197	$I23$	$(2),(5)$	✓	✓	T	A_4	12:12	A_4	d_4, t_1	d_5, t_1	(1,2,4,4,6,8,8,10,12,12)
198	$P2_13$	$(2),(5)$			T	A_4	12:3	$\mathbb{Z}_3$	I, t_1	I, t_1	(0,0,1,0,0,0,0,0,0,0)
199	$I2_13$	$(2),(5)$			T	A_4	12:12	A_4	d_4, t_1	d_5, t_1	(1,1,2,2,2,2,2,2,2,2)
200	$Pm\bar{3}$	$(2),(5)(13)$	✓	✓	T_h	$A_4 \times \mathbb{Z}_2$	24:3	$\mathbb{Z}_3$	t_1, I	t_1, I	(2,6,14,22,34,50,66,86,110,134)
201	$Pn\bar{3}$	$(2),(5)(13)$			T_h	$A_4 \times \mathbb{Z}_2$	24:3	$\mathbb{Z}_3$	t_1, I	t_1, I	(2,4,6,6,8,10,10,12,14,14)
202	$Fm\bar{3}$	$(2),(5)(13)$	✓		T_h	$A_4 \times \mathbb{Z}_2$	24:12	A_4	t_1, d_5	t_1, d_4	(1,4,9,10,15,22,25,32,41,46)
203	$Fd\bar{3}$	$(2),(5)(13)$			T_h	$A_4 \times \mathbb{Z}_2$	24:12	A_4	t_1, d_5	t_1, d_4	(1,3,5,3,5,7,5,7,9,7)
204	$Im\bar{3}$	$(2),(5)(13)$	✓	✓	T_h	$A_4 \times \mathbb{Z}_2$	24:12	A_4	t_1, d_4	t_1, d_5	(2,4,8,12,18,26,34,44,56,68)
205	$Pa\bar{3}$	$(2),(5)(13)$			T_h	$A_4 \times \mathbb{Z}_2$	24:3	$\mathbb{Z}_3$	t_1, I	t_1, I	(1,1,2,2,2,2,2,2,2,2)

206	$Ia\bar{3}$	$(2),(5)(13)$			T_h	$A_4 \times \mathbb{Z}_2$	24:12	A_4	t_1, d_4	t_1, d_5	(2,3,4,4,4,4,4,4,4,4)
207	$P432$	$(5),(13)$	✓	✓	O	S_4	24:6	Dih_3	t_1, d_1	t_1, d_1	(2,5,9,11,15,19,21,25,29,31)
208	$P4_232$	$(5),(13)$		✓	O	S_4	24:6	Dih_3	t_1, d_1	t_1, d_1	(2,5,9,11,15,19,21,25,29,31)
209	$F432$	$(5),(13)$	✓	✓	O	S_4	24:24	S_4	t_1, d_7	t_1, d_8	(1,4,8,8,12,16,16,20,24,24)
210	$F4_132$	$(5),(13)$			O	S_4	24:24	S_4	t_1, d_7	t_1, d_8	(1,3,5,3,5,7,5,7,9,7)
211	$I432$	$(5),(13)$	✓	✓	O	S_4	24:24	S_4	t_1, d_8	t_1, d_7	(2,4,7,9,12,15,17,20,23,25)
212	$P4_332$	$(5),(13)$			O	S_4	24:6	Dih_3	t_1, d_1	t_1, d_1	(1,1,2,2,2,2,2,2,2,2)
213	$P4_132$	$(5),(13)$			O	S_4	24:6	Dih_3	t_1, d_1	t_1, d_1	(1,1,2,2,2,2,2,2,2,2)
214	$I4_132$	$(5),(13)$			O	S_4	24:24	S_4	t_1, d_8	t_1, d_7	(2,3,5,7,9,11,13,15,17,19)
215	$P\bar{4}3m$	$(2)(13),(5)$	✓	✓	T_d	S_4	24:6	Dih_3	d_1, t_1	d_1, t_1	(2,5,9,11,15,19,21,25,29,31)
216	$F\bar{4}3m$	$(2)(13),(5)$	✓	✓	T_d	S_4	24:24	S_4	d_7, t_1	d_8, t_1	(1,4,8,8,12,16,16,20,24,24)
217	$I\bar{4}3m$	$(2)(13),(5)$	✓		T_d	S_4	24:24	S_4	d_8, t_1	d_7, t_1	(2,4,6,7,9,11,12,14,16,17)
218	$P\bar{4}3n$	$(2)(13),(5)$			T_d	S_4	24:6	Dih_3	d_1, t_1	d_1, t_1	(1,3,5,5,7,9,9,11,13,13)
219	$F\bar{4}3c$	$(2)(13),(5)$			T_d	S_4	24:24	S_4	d_7, t_1	d_8, t_1	(1,3,4,2,4,6,4,6,8,6)
220	$I\bar{4}3d$	$(2)(13),(5)$			T_d	S_4	24:24	S_4	d_8, t_1	d_7, t_1	(1,1,2,2,2,2,2,2,2,2)
221	$Pm\bar{3}m$	$(13),(5)(25)$	✓	✓	O_h	$S_4 \times \mathbb{Z}_2$	48:6	Dih_3	t_1, d_1	t_1, d_1	(3,8,17,28,43,62,83,108,137,168)
222	$Pn\bar{3}n$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:6	Dih_3	t_1, d_1	t_1, d_1	(2,4,6,7,9,11,12,14,16,17)
223	$Pm\bar{3}n$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:6	Dih_3	t_1, d_1	t_1, d_1	(2,5,10,15,22,31,40,51,64,77)
224	$Pn\bar{3}m$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:6	Dih_3	t_1, d_1	t_1, d_1	(3,7,12,16,21,26,30,35,40,44)
225	$Fm\bar{3}m$	$(13),(5)(25)$	✓		O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_7	t_1, d_8	(2,6,13,20,31,45,59,77,98,119)
226	$Fm\bar{3}c$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_7	t_1, d_8	(2,5,9,11,15,20,23,28,34,38)
227	$Fd\bar{3}m$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_7	t_1, d_8	(2,5,9,11,15,19,21,25,29,31)
228	$Fd\bar{3}c$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_7	t_1, d_8	(2,4,5,4,5,6,5,6,7,6)
229	$Im\bar{3}m$	$(13),(5)(25)$	✓		O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_8	t_1, d_7	(3,7,13,20,29,40,52,66,82,99)
230	$Ia\bar{3}d$	$(13),(5)(25)$			O_h	$S_4 \times \mathbb{Z}_2$	48:24	S_4	t_1, d_8	t_1, d_7	(2,3,4,5,6,7,8,9,10,11)

For those point groups that are 2-groups (all the elements have orders that aer powers of 2), we can use the following code to check the first and 2nd cohomology that appear in $E_2^{1,1}$, $E_2^{1,2}$, $E_2^{2,1}$, $E_2^{2,2}$. The basic code looks like (for the example of point group= $\mathbb{Z}_2^2 = V$): (see `https://math.stackexchange.com/questions/2611736/how-to-compute-group-cohomology-h2`

```
V:=Image(IsomorphismPcpGroup(SmallGroup(4,2)));

d3t:=[[0,1,1],[0,1,0],[1,1,0]]*Z(2);

d4t:=[[0,1,1],[1,0,1],[0,0,1]]*Z(2);

d5t:=[[0,1,0],[1,0,0],[1,1,1]]*Z(2);

d6t:=[[0,0,1],[1,1,1],[1,0,0]]*Z(2);

cr1:=CRRecordByMats(V,[d5t,d6t]);

cr2:=CRRecordByMats(V,[d4t,d3t]);

TwoCohomologyCR(cr1);

TwoCohomologyCR(cr2);

OneCohomologyCR(cr1);

OneCohomologyCR(cr2);
```

Importantly, note that here the input is the transpose of the matrices $d_{3,4,5,6}$. This is because one needs to use the basis change matrices, rather than the coordinate transformation matrices (which is the one used in our Mathematica code). To read the result, simply look at the `rels` part, where e.g. $[2,2]$ means $\mathbb{Z}_2^2$. See `https://docs.gap-system.org/pkg/polycyclic/doc/manual.pdf` for more detail. The "`rels`" output of `TwoCohomologyCR(cr1); TwoCohomologyCR(cr2);` gives $[2,2,2,2]$ and $[2,2]$, respectively, meaning that the cohomology groups are $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^2$, respectively; similarly, we check that the one cohomology gives $\mathbb{Z}_2^3$ and $\mathbb{Z}_2$, respectively, which match previous table.

On the other hand, let's try calculating the cohomology of $\mathbb{Z}_2^2$ with coefficient $\mathbb{Z}$: the code is

```
V:=Image(IsomorphismPcpGroup(SmallGroup(4,2)));

d3tmod0:=[[0,-1,-1],[0,1,0],[-1,-1,0]];

d4tmod0:=[[0,-1,-1],[-1,0,-1],[0,0,1]];
```

```
d5tmod0:=[[0,1,0],[1,0,0],[-1,-1,-1]];
d6tmod0:=[[0,0,1],[-1,-1,-1],[1,0,0]];
cr1mod0:=CRRecordByMats(V,[d5tmod0,d6tmod0]);
cr2mod0:=CRRecordByMats(V,[d4tmod0,d3tmod0]);
TwoCohomologyCR(cr1mod0);
TwoCohomologyCR(cr2mod0);
OneCohomologyCR(cr1mod0);
OneCohomologyCR(cr2mod0);
```

Again looking at the "`rels`" we get that the second cohomology groups are both $\mathbb{Z}_2$, while the first cohomology groups are $\mathbb{Z}_2^2$ and trivial, respectively.

Lieb-Schultz-Mattis Theorem and the Filling Constraint:

`https://arxiv.org/abs/2104.09561`

Next, cohomology with $\mathbb{Z}$ as coefficient: we checked that $\mathcal{H}^0(SG,\mathbb{Z})=\mathbb{Z}$ and $\mathcal{H}^1(SG,\mathbb{Z})=\mathbb{Z}_1=\{0\}$ for all 230 space groups.

28 SPT cohomology

Some cohomology with U(1) or $\mathbb{Z}$ coefficients: we have (easy to check in GAP)

$$H^n(\mathbb{Z}^d,\mathbb{Z})=\mathbb{Z}^{\binom{d}{n}}, \tag{211}$$

This of course agrees with Eq. (24). This means that

$$H^n(\mathbb{Z}^d,U(1))=\mathbb{Z}^{\binom{d}{n+1}}, \tag{212}$$

So For $T=\mathbb{Z}^3$, $H^1(T,U(1))=\mathbb{Z}^3$, $H^2(T,U(1))=\mathbb{Z}$,

Using $0\to T\to SG\to PG\to 0$, we have

$$\begin{array}{c|cccccc}
\vdots & \vdots & \vdots & \vdots & & \vdots & \iddots \\
q=3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
q=2 & \mathbb{Z}^{PG} & H^1(PG,\mathbb{Z}) & H^2(PG,\mathbb{Z}) & H^3(PG,\mathbb{Z}) & H^4(PG,\mathbb{Z}) & \cdots \\
q=1 & (\mathbb{Z}^3)^{PG} & H^1(PG,\mathbb{Z}^3) & H^2(PG,\mathbb{Z}^3) & H^3(PG,\mathbb{Z}^3) & H^4(PG,\mathbb{Z}^3) & \cdots \\
q=0 & U(1)^{PG} & H^1(PG,U(1)) & H^2(PG,U(1)) & H^3(PG,U(1)) & H^4(PG,U(1)) & \cdots \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & \cdots\cdots
\end{array} \tag{213}$$

But a more serious question is what types of cohomology classifies SPT.

We know the difference between a finite cyclic group and an infinite cyclic group is that the former has a free resolution that is infinitely long while the latter has a two-nonzero-term resolution. This directly tells us that the cohomology for the infinite cyclic group, no matter what the coefficient is, is finite length. In fact the relevant concept here is cohomology dimension. A good place to look at this is the (excellent and concise!!) lecture notes `http://www.ltcc.ac.uk/media/london-taught-course-centre/documents/Complete-Course-Notes.pdf`.

Let us start from the most basic things. We know that (see appendix), for the infinite cyclic group $G=\langle x\rangle\cong\mathbb{Z}$, we have $H^0(G,A)=A^G$, $H^1(G,A)=A/DA$, $H^{n\geq 2}(G,A)=0$. Taking $A=\mathbb{Z}$, this means that for trivial action, we have $H^{0,1}(\mathbb{Z},\mathbb{Z})=\mathbb{Z}$, in agreement with (211).

Now, if the action is the magnetic translation for the Neel order (denote the action as "neel"), meaning $x^m.n=(-1)^m n$ for $m\in\mathbb{Z}$ and $n\in A=\mathbb{Z}$, then $DA=2A$, so $H_{neel}^{n=0\text{ or }n\geq 2}=0$ and $H^1_{neel}(G,A)=\mathbb{Z}_2$. Note that for H^1 we can compute Z^1 and B^1 separately by definition: note that $Z^1=\{f\colon G\to A|f(xy)=x.f(y)+f(x)\}$, which for us is $f(x^{m+n})=(-1)^m f(x^n)+f(x^m)$. A little algebra shows that we must have $f(x^{2n})=0$ and $f(x^{2n+1})=l$ for any $n,l\in\mathbb{Z}$, so that $Z^1\cong\mathbb{Z}$. The denominator is $B^1=\{f\colon G\to A|f(x^n)=x^n.l-l\}=\{f|f(x^n)=((-1)^n-1)l\}\cong 2\mathbb{Z}$, so that $H^1=Z^1/B^1=\mathbb{Z}/2\mathbb{Z}=\mathbb{Z}_2$, with the explicit representative cochain for the cocycle

$$f(x^n)\equiv n \mod 2. \tag{214}$$

We are attempted to conclude that $H^n(\mathbb{Z}^2,\mathbb{Z})! =! \oplus_{p+q=n}\left(H^p(\mathbb{Z},\mathbb{Z})\otimes H^q(\mathbb{Z},\mathbb{Z})\right)! =!\mathbb{Z}_2\delta_{n=2}.$, but remeber that this Kunneth is wrong! In fact the Kunneth formula for nontrivial action is more complicated than this, and in fact we have not presented this form. So, actually the safest way is again to use spectral sequence: $0 \to T_x \to T \to T_y \to 0$, with $Ep, q_2 = H^p(T_y, H^q(T_x, \mathbb{Z}_{\text{neel}}))$. From above we know that only the $q = 1$ row is nonzero. We have $E_2^{0,1} = H^1(T_x, \mathbb{Z}_{\text{neel}})^{T_y}$, and since $(y.f)(x^n)) = y.(f(y^{-1}x^n y)) = y.f(x^n) = y.n = -n \equiv n \mod 2$, so we see that T_y has trivial action on $H^1(T_x, \mathbb{Z}_{\text{neel}})$, and $H^1(T_x, \mathbb{Z}_{\text{neel}})^{T_y} = H^1(T_x, \mathbb{Z}_{\text{neel}}) = \mathbb{Z}_2$. Using the result of Eq. (24) for $n = 1$, we see that $H^1(T_y, H^1(T_x, \mathbb{Z}_{\text{neel}})) = \mathbb{Z}_2$ while $H^{p\geq 2}(T_y, H^1(T_x, \mathbb{Z}_{\text{neel}})) = 0$. Since $E_2^{p,q}$ already collapses, we have

$$H^n_{neel}(T = \mathbb{Z}^2, \mathbb{Z}) = \left\{ \begin{array}{ll} \mathbb{Z}_2, & n = 1, 2, \\ 0, & \text{other } n. \end{array} \right. , \tag{215}$$

with representative cochains for the cocycles

$$C^1(T, \mathbb{Z}) \ni f(T_x^x T_y^y) \equiv x + y \mod 2, \quad C^2(T, \mathbb{Z}) \ni \omega(T_x^{x_1} T_y^{y_1}, T_x^{x_2} T_y^{y_2}) \equiv (x_1 \mod 2)(y_2 \mod 2). \tag{216}$$

Next let's use this to compute $H^*_{neel}(p4m, \mathbb{Z})$. The relation $C_4 T_1 C_4^{-1} = T_2$, $C_4 T_2 C_4^{-1} = T_1^{-1}$, $M T_1 M = T_1^{-1}$, $M T_2 M = T_2$, $M C_4 M = C_4^{-1}$ and the requirement that C_4 and M acts trivially on the $\mathbb{Z}$ coefficient means that $H^{1,2}_{neel}(T, \mathbb{Z})^{PG} = H^{1,2}_{neel}(T, \mathbb{Z})$, where $PG = \langle C_4, M \rangle = D_4$ with eight elements. Then fir this spectral sequence we have $E_2^{p,q} = H^p(D_4, H^q_{neel}(T, \mathbb{Z}))$, the only nonzero rows are $q = 1, 2$; we have $E_2^{0,2} = \mathbb{Z}_2$, $E_2^{0,1} = \mathbb{Z}_2$, $E_2^{1,1} = \mathbb{Z}_2^2$, $E_2^{1,0} = 0$. Now we need to see if the $d_2 \colon E_2^{0,2} \to E_2^{2,1}$ is nontrivial or not. Note that $E_2^{2,1} = \mathbb{Z}_2^3$.

On the other hand, let's directly calculate $H^1_{neel}(p4m, U(1))$: the relevant terms in $E_2^{p,q} = H^p(D_4, H^q_{neel}(T, U(1)))$ is $E_2^{1,0} = H^1(D_4, U(1)^T) = H^1(D_4, \mathbb{Z}_2) = \mathbb{Z}_2^2$, and $E_2^{0,1} = H^1_{neel}(T, U(1))^{D_4}$, here $H^1_{neel}(T, U(1)) = H^2_{neel}(T, \mathbb{Z}) = \mathbb{Z}_2$. So essentially the spectral sequence $E_2^{p,q}$ is related to the one with the $\mathbb{Z}$ coefficient with $E_2^{p,q+1}$, as they should.

Now we need to look at the d_2 map. Since we do not know how $d_2 \colon E_2^{0,2} \to E_2^{2,1}$ looks like, we cannot work with the spectral sequence of $H^*_{neel}(p4m, \mathbb{Z})$; we have to work with $H^*_{neel}(p4m, U(1))$, and look at $d_2 \colon E_2^{0,1} \to E_2^{2,0}$. First, note that $H^2_{neel}(D_4, U(1)^T) = H^2_{neel}(D_4, \mathbb{Z}_2) = \mathbb{Z}_2^3$, and $H^1_{neel}(T, U(1))^{D_4} = H^1_{neel}(T, U(1)) \cong H^2_{neel}(T, \mathbb{Z}) = \mathbb{Z}_2$, where, using the fact that $H^1(T, \mathbb{Z}_2) \xrightarrow{\tilde{i}} H^1_{neel}(T, U(1))$ is surjective (see Weicheng Ye's paper), we can find the generator of $H^1_{neel}(T, U(1))$. Note that $f(T_x^x T_y^y) = e^{i\pi(x+y)}$ is a coboundary in the case of antiunitary translations (although it was a cocycle in $H^1(T, \mathbb{Z}_2)$: $((T_x^x T_y^y).e^{i\theta})e^{-i\theta} = e^{i((-1)^{x+y}-1)\theta} = e^{-2i(x+y)\theta}$, where we have used the fact that $(-1)^b - 1 = -2b$ for $b = 0, 1$. We see that setting $\theta = \pi/2$ gives $f(T_x^x T_y^y) = e^{i\pi(x+y)}$, meaning that it is a coboundary in $H^1(T, U(1))$. Since we also know that $H^1_{neel}(p4m, \mathbb{Z}_2) \to H^1_{neel}(p4m, U(1))$ is surjective, we know that $H^1_{neel}(p4m, U(1)) \subset \mathbb{Z}_2^2$, coming from the point group. But we know these two copies are true cocycles, so we have

$$H^1_{neel}(p4m, U(1)) = \mathbb{Z}_2^2. \tag{217}$$

A side note: Note that $p4m$ is a "symmorphic" wallpaper group: we know from the wallpaper lecture notes of Patrick J. Morandi that $H^2(D_4, T) = \mathbb{Z}_2$ where the trivial element is $p4m$, which splits (i.e. is a semidirect product $T \rtimes D_4$), while the nontrivial element is $p4g$. We know that for semidirect product, there is an isomorphic copy of the point group D_4, meaning that $(g_1 g_2)^{*-1} g_1^* g_2^* = 1$. Therefore we see that $d_2 f$ is trivial. But in our particular case of antiunitary translation action, we already have $E_2^{0,1} = 0$, so we do not even need this information.

More generally: about the differential maps? See the papers by Siegel:`https://www.sciencedirect.com/science/article/pii/0022404995000208`, `https://www.ams.org/journals/tran/1997-349-04/S0002-9947-97-01747-9/S0002-9947-pdf`, which make use of the theorem by Charlap and Vasquez, `https://www.jstor.org/stable/1994432?seq=2#metadata_info_tab_contents`.

A simple thing that is relevant for the paper: denote point group as P, then for the spectral sequence for $H^1(p4m, U(1))$ at $E_2^{1,0}$ we are calculating $H^1(P, U(1))$ or $H^1(P, U(1)^{T_{mag}}) = H^1(P, \mathbb{Z}_2)$, where the former is for nonmagnetic translation while for the latter is magnetic translation so that $U(1)^{T_{mag}} = \mathbb{Z}_2$. Then going from the first to the second is really a change of ring: $U(1) \xrightarrow{p} \mathbb{Z}_2 = \{1, e^{i\pi}\}$. We know if $P = \mathbb{Z}_4 = \langle C_4 \rangle$ then the cohomology map is $\mathbb{Z}_4 \to \mathbb{Z}_2$, so we must have $\{1, i, -1, -i\} \to \{1, -1\}$, where $i \mapsto -1$.

(Written 2023 May) It seems eventually we will want to calculate the cohomology ring with $\mathbb{Z}$ coefficient. This should contain all the information for finite coefficient of type $\mathbb{Z}_n$ according to the universal coefficient theorem. But is of course a very ambitious goal.

A few recent references: two useful papers by Adem and others `https://arxiv.org/abs/math/0406130` `https://arxiv.org/abs/0704.1823`, the second one raises the question of whether symmorphic group spectral sequence has a terminating E_2 page; the famosu paper who gives conterexamples was Totaro's 1996 paper "Cohomology of Semidirect Product Groups" (which can be easily find online). See also a paper by Petrosyan, "COHOMOLOGY OF SPLIT GROUP EXTENSIONS AND CHARACTERISTIC CLASSES", see `https://arxiv.org/abs/0707.3526`. They also cite a string theory paper that touches on crystallographic group cohomology without explicit mentioning `https://arxiv.org/abs/hep-th/0103170`; the Propitius thesis (which contans many useful things and a bit of spectral sequence calculation) `https://arxiv.org/abs/hep-th/9511195`. The Adem-Pan-et al conjecture was addressed negatively in `https://reader.`

`elsevier.com/reader/sd/pii/S0022404911002684?token=B7B9D92FB9E61BB8DA52CB91BE462A6B733887BEFB358650E171E2A8DF` `originRegion=us-east-1&originCreation=20230515235649`.

The paper `https://arxiv.org/abs/math/0406130` can help further understand the spectral sequence calculation. This paper discusses the cases where a symmorphic crystallographic group in dimension d should have terminating E_2 page. The examples in this paper is helpful. In particular, it states that for coefficient $\mathbb{Z}$, one should get that $H^q(PG, H^p(\mathbb{Z}^d, \mathbb{Z}))$ and $H^{d-q}(PG, H^p(\mathbb{Z}^d, \mathbb{Z}))$ are isomorphic; and this works for any coefficient module M provided it satisfies $M \cong M^*$, which is true for $M = \mathbb{Z}^d$. Here M^* is the dual module of M. [Recall: $M = \mathbb{Z}^d$ is a $\mathbb{Z}G$ lattice where $G := PG$ is the point group. See footnote 1 of the above paper: if G is a finite group, a $\mathbb{Z}G$-lattice is a $\mathbb{Z}G$-module which happens to be a free abelian group of finite rank.

28.1 A special version of universal coefficient theorem

According to Manjunath's question, `https://mathoverflow.net/questions/316499/is-the-following-variant-of-the-unive` we have a version that works for our case

$$H^n(G, \mathbb{Z}_2) = H^n(G, \mathbb{Z}^{\text{or}}) \otimes \mathbb{Z}_2 \times \text{Tor}_1^{\mathbb{Z}}(H^{n+1}(G, \mathbb{Z}^{\text{or}}), \mathbb{Z}_2), \tag{218}$$

28.2 pyrochlore

First, let's calculate the class $H^2(O_{\text{h}}, T)$, using $C_2^{-1} T_1^x T_2^y T_3^z C_2 = T_1^y T_2^x T_3^{-x-y-z}$, $C_2'^{-1} T_1^x T_2^y T_3^z C_2' = T_1^z T_2^{-x-y-z} T_3^x$, $C_3^{-1} T_1^x T_2^y T_3^z C_3 = T_1^y T_2^z T_3^x$, $M^{-1} T_1^x T_2^y T_3^z M = T_1^y T_2^x T_3^z$, $P^{-1} T_1^x T_2^y T_3^z P = T_1^{-x} T_2^{-y} T_3^{-z}$: so we have Denote $\boldsymbol{r} = (x, y, z)^T$, then we have

$$C_2 \colon \boldsymbol{r} \to \begin{pmatrix} & 1 & \\ 1 & & \\ -1 & -1 & -1 \end{pmatrix} \boldsymbol{r}, \quad C_2' \colon \boldsymbol{r} \to \begin{pmatrix} & & 1 \\ -1 & -1 & -1 \\ 1 & & \end{pmatrix} \boldsymbol{r}, \qquad C_3 \colon \boldsymbol{r} \to \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \boldsymbol{r}, \quad M \colon \boldsymbol{r} \to \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \boldsymbol{r}, \tag{219}$$

and $P \colon \boldsymbol{r} \to -1_{3\times 3} \boldsymbol{r}$. Note that $(34) = (12)(34)(12) = C_2 M = \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & -1 & -1 \end{pmatrix}$.

The code is

```
Oh:=Group((1,2,3),(3,4),(5,6));
ZZZ:=GL(3,Integers);;
Z3:=GroupHomomorphismByImages(Oh,ZZZ,[(1,2,3),(3,4),(5,6)],
[[[0,1,0],[0,0,1],[1,0,0]],[[1,0,0],[0,1,0],[-1,-1,-1]],[[-1,0,0],[0,-1,0],[0,0,-1]]]);
R:=ResolutionFiniteGroup(Oh,11);;
C:=HomToIntegralModule(R,Z3);
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

We get

$$H^{0,1,2,3,4,5,6,7,8,9,10}(O_{\text{h}}, T) = \mathbb{Z}_2^{0,1,2,3,5,7,9,12,15,18,22}$$

, we see that $H^2(O_{\text{h}}, T) = \mathbb{Z}_2^2$.

We can also calculate $H^2(S_4, T)$. The only change in code is to instead of using `Z3` use `Zz3:=GroupHomomorphismByImages(S4,ZZZ,[(1,2,3),(3,4)],[[[0,1,0],[0,0,1],[1,0,0]],[[1,0,0],[0,1,0],[-1,-1,-` and `R:=ResolutionFiniteGroup(S4,11);`. We get $H^{0,1,2,3,4,5,6,7,8,9,10}(S_4, T) = \mathbb{Z}_1, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^3, \mathbb{Z}_2^2 \times \mathbb{Z}_4, \mathbb{Z}_2^4$.

Using the code

```
G:=SpaceGroupIT(3,227);
R:=ResolutionSpaceGroup(G,16);
ZZ:=GL(1,Integers);;
Zor:= GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)]]);;
```

```
C:=HomToIntegralModule(R,Zor);;
for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15] do
Print("H^",n,"=",Cohomology(C,n),";;");
> od;
```

we get $H^n(Fd\bar{3}m, \mathbb{Z}^{\text{or}})$ for $n = 0, 1, ..., 15$ to be

$$\mathbb{Z}_1, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z} \times \mathbb{Z}_2^4, \quad \mathbb{Z}_2^4, \quad \mathbb{Z}_2^7, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^{10} \times \mathbb{Z}_{12}, \quad \mathbb{Z}_2^{10},$$

$$\mathbb{Z}_2^{15}, \quad \mathbb{Z}_2^{14}, \quad \mathbb{Z}_2^{16} \times \mathbb{Z}_{12}, \quad \mathbb{Z}_2^{18}, \quad \mathbb{Z}_2^{21}, \quad \mathbb{Z}_2^{20}, \quad \mathbb{Z}_2^{24} \times \mathbb{Z}_{12}.$$

Using

```
Oh:=Group((1,2,3),(3,4),(5,6));
rho:=IrreducibleRepresentations(Oh)[4];
R:=ResolutionFiniteGroup(Oh,11);;
C:=HomToIntegralModule(R,rho);;
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

or alternatively using

```
rho:=GroupHomomorphismByFunction(Oh,ZZ,x->[[SignPerm(x)]]);
```

instead of defiining `rho` using irreducible representation would give the same result.

Result: we get, for $p = 0, 1, 2, ..., 10$, $E_2^{p,0} = H^p(O_\text{h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^{0,1,1,3,4,6,8,11,13,17,20}$.

Next look at $E_2^{p,1}$ and $E_2^{p,2}$. Using $C_2^{-1} T_1^x T_2^y T_3^z C_2 = T_1^y T_2^x T_3^{-x-y-z}$, $C_2'^{-1} T_1^x T_2^y T_3^z C_2' = T_1^z T_2^{-x-y-z} T_3^x$, $C_3^{-1} T_1^x T_2^y T_3^z C_3 = T_1^y T_2^z T_3^x$, $M^{-1} T_1^x T_2^y T_3^z M = T_1^y T_2^x T_3^z$, $P^{-1} T_1^x T_2^y T_3^z P = T_1^{-x} T_2^{-y} T_3^{-z}$, therefore

$$(C_2 \cdot \chi_1)(T_1^x T_2^y T_3^z) = C_2.(\chi_1(C_2^{-1} T_1^x T_2^y T_3^z C_2)) = C_2.(\chi_1(T_1^y T_2^x T_3^{-x-y-z})) = C_2.(y) = y = \chi_2(T_1^x T_2^y T_3^z),$$

$$(C_2 \cdot \chi_2)(T_1^x T_2^y T_3^z) = C_2.(\chi_2(C_2^{-1} T_1^x T_2^y T_3^z C_2)) = C_2.(\chi_2(T_1^y T_2^x T_3^{-x-y-z})) = C_2.(x) = x = \chi_1(T_1^x T_2^y T_3^z),$$

$$(C_2 \cdot \chi_3)(T_1^x T_2^y T_3^z) = C_2.(\chi_2(C_2^{-1} T_1^x T_2^y T_3^z C_2)) = C_2.(\chi_3(T_1^y T_2^x T_3^{-x-y-z})) = C_2.(-x-y-z) = -(\chi_1 + \chi_2 + \chi_3)(T_1^x T_2^y T_3^z),$$

$$(C_2' \cdot \chi_1)(T_1^x T_2^y T_3^z) = \chi_1(T_1^z T_2^{-x-y-z} T_3^x) = z = \chi_3(T_1^x T_2^y T_3^z),$$

$$(C_2' \cdot \chi_2)(T_1^x T_2^y T_3^z) = \chi_2(T_1^z T_2^{-x-y-z} T_3^x) = -x-y-z = -(\chi_1 + \chi_2 + \chi_3)(T_1^x T_2^y T_3^z),$$

$$(C_2' \cdot \chi_3)(T_1^x T_2^y T_3^z) = \chi_3(T_1^z T_2^{-x-y-z} T_3^x) = x = \chi_1(T_1^x T_2^y T_3^z),$$

$$(C_3 \cdot \chi_1)(T_1^x T_2^y T_3^z) = \chi_1(T_1^y T_2^z T_3^x) = y = \chi_2(T_1^x T_2^y T_3^z),$$

$$(C_3 \cdot \chi_1)(T_1^x T_2^y T_3^z) = \chi_1(T_1^y T_2^z T_3^x) = y = \chi_2(T_1^x T_2^y T_3^z),$$

Therefore for $q = 1$, we have

$$C_2\colon \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_2'\colon \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad C_3\colon \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M\colon \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P\colon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Note that $C_2 = (12)(34)$, $C_2' = (13)(24)$, $M = (12)$, $C_3 = (123)$. The corresponding code to compute $E_2^{p,1}$ is

```
Oh:=Group((1,2,3),(3,4),(5,6));
ZZZ:=GL(3,Integers);;
ZZZor:=GroupHomomorphismByImages(Oh,ZZZ,[(1,2,3),(3,4),(5,6)],
[[[0,0,1],[1,0,0],[0,1,0]],[[-1,0,1],[0,-1,1],[0,0,1]],[[1,0,0],[0,1,0],[0,0,1]]]);
R:=ResolutionFiniteGroup(Oh,11);;
C:=HomToIntegralModule(R,ZZZor);
```

```
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

We get $E_2^{p,1}$ for $p = 0, 1, ..., 10$ to be $\mathbb{Z}_2^{0,0,1,2,3,5,7,9,12,15,18}$.

Next, action on $b_{1,2,3}$. Recall that $b_1(g_1, g_2) = x_1y_2$, $b_2(g_1, g_2) = y_1z_2$, and $b_3(g_1, g_2) = x_1z_2$. We have

$$(C_2.b_1)(g_1, g_2) = y_1x_2 = -b_1(g_1, g_2) + \text{coboundary},$$

$$(C_2.b_2)(g_1, g_2) = x_1(-x_2 - y_2 - z_2) = -(b_1 + b_3)(g_1, g_2)\text{coboundary},$$
$$(C_2.b_3)(g_1, g_2) = y_1(-x_2 - y_2 - z_2) = (b_1 - b_2)(g_1, g_2)\text{coboundary},$$
$$(C_2'.b_1)(g_1, g_2) = -z_1(x_2 + y_2 + z_2) = (b_2 + b_3)(g_1, g_2) + \text{coboundary},$$
$$(C_2'.b_2)(g_1, g_2) = -(x_1 + y_1 + z_1)x_2 = (b_1 + b_3)(g_1, g_2) + \text{coboundary},$$
$$(C_2'.b_3)(g_1, g_2) = z_1x_2 = -b_3(g_1, g_2) + \text{coboundary},$$
$$(C_3.b_1)(g_1, g_2) = y_1z_2 = b_2(g_1, g_2) + \text{coboundary},$$
$$(C_3.b_2)(g_1, g_2) = z_1x_2 = -b_3(g_1, g_2) + \text{coboundary},$$
$$(C_3.b_3)(g_1, g_2) = y_1x_2 = -b_1(g_1, g_2) + \text{coboundary},$$
$$(M.b_1)(g_1, g_2) = y_1x_2 = -b_1(g_1, g_2) + \text{coboundary},$$
$$(M.b_2)(g_1, g_2) = x_1z_2 = b_3(g_1, g_2) + \text{coboundary},$$
$$(M.b_3)(g_1, g_2) = y_1z_2 = b_2(g_1, g_2) + \text{coboundary},$$

Therefore, for $q = 2$, we have

$$\widetilde{C}_2\colon \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \widetilde{C'}_2\colon \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad \widetilde{C}_3\colon \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \widetilde{M}\colon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \widetilde{P}\colon \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Here instead of `ZZZor` we use

```
ZZZZor:=GroupHomomorphismByImages(Oh,ZZZ,[(1,2,3),(3,4),(5,6)],
[[[0,0,-1],[1,0,0],[0,-1,0]],[[-1,-1,1],[0,1,0],[0,0,1]],[[-1,0,0],[0,-1,0],[0,0,-1]]]);
CC:=HomToIntegralModule(R,ZZZZor);
```

We get $E_2^{p,2}$ for $p = 0, 1, ..., 10$ to be $\mathbb{Z}_2^{0,1,2,3,5,7,9,12,15,18,22}$.

Finally, we look at $E_0^{p,3}$. We have $\tau(g_1, g_2, g_3) = x_1y_2z_3$. $(C_2.\tau)(g_1, g_2, g_3) = C_2.(\tau(C_2g_1C_2, C_2g_2C_2, C_2g_3C_2)) = -y_1x_2(x_3+y_3+z_3) = \tau(g_1, g_2, g_3)$, $(C_2'.\tau)(g_1, g_2, g_3) = -z_1(x_2+y_2+z_2)x_3 = \tau(g_1, g_2, g_3)$, $(C_3.\tau)(g_1, g_2, g_3) = \tau(g_1, g_2, g_3)$, $(M.\tau)(g_1, g_2, g_3) = M.(\tau(Mg_1M, Mg_2M, Mg_3M)) = -(y_1x_2z_3) = \tau(g_1, g_2, g_3)$, $(P.\tau)(g_1, g_2, g_3) = \tau(g_1, g_2, g_3)$. We see that for the $q = 3$ row, the group O_h has trivial action on $H^3(T, \mathbb{Z})$, and hence we can simply use the following to get the cohomology

```
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",GroupCohomology(Oh,n),";;");
od;
```

We get

$$\begin{gathered} E_2^{0,3} = \mathbb{Z}, \quad E_2^{1,3} = \mathbb{Z}_1, \quad E_2^{2,3} = E_2^{3,3} = \mathbb{Z}_2^2, \quad E_2^{4,3} = \mathbb{Z}_2^4 \times \mathbb{Z}_{12}, \quad E_2^{5,3} = \mathbb{Z}_2^5, \\ E_2^{6,3} = \mathbb{Z}_2^9, \quad E_2^{7,3} = \mathbb{Z}_2^{10}, \quad E_2^{8,3} = \mathbb{Z}_2^{13} \times \mathbb{Z}_{12}, \quad E_2^{9,3} = \mathbb{Z}_2^{16}, \quad E_2^{10,3} = \mathbb{Z}_2^{21}. \end{gathered} \tag{220}$$

$$\begin{array}{c|ccccccccccc}
q=3 & \mathbb{Z} & 0 & 2 & 2 & 4, \mathbb{Z}_{12} & 5 & 9 & 10 & 13, \mathbb{Z}_{12} & 16 & 21 \\
q=2 & 0 & 1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 & 22 \\
q=1 & 0 & 0 & 1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 \\
q=0 & 0 & 1 & 1 & 3 & 4 & 6 & 8 & 11 & 13 & 17 & 20 \\
\hline
E_2^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10 \cdots\cdots
\end{array} \tag{221}$$

Conjectured E_3 page

$$\begin{array}{c|ccccccccccc}
q=3 & 2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}_{12} & 0 & 0 & 0 & \mathbb{Z}_{12} & 0 & 0 \\
q=2 & 0 & 1 & 1 & 3 & 3 & 5 & 5 & 7 & 6 & 8 & 9 \\
q=1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q=0 & 0 & 1 & 1 & 3 & 3 & 4 & 5 & 6 & 6 & 8 & 8 \\
\hline
E_3^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10\cdots\cdots
\end{array} \tag{222}$$

Conjectured E_4 page

$$\begin{array}{c|ccccccccccc}
q=3 & 2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}_{12} & 0 & 0 & 0 & \mathbb{Z}_{12} & 0 & 0 \\
q=2 & 0 & 1 & 1 & 3 & 3 & 4 & 5 & 7 & 6 & 8 & 9 \\
q=1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q=0 & 0 & 1 & 1 & 3 & 3 & 4 & 5 & 6 & 5 & 8 & 8 \\
\hline
E_3^{p,q} & p=0 & p=1 & p=2 & p=3 & p=4 & p=5 & p=6 & p=7 & p=8 & p=9 & p=10\cdots\cdots
\end{array} \tag{223}$$

which we further conjecture to be equal to the E_∞ page, hence it should agree with

$$\mathbb{Z}_1, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z}\times\mathbb{Z}_2^4, \quad \mathbb{Z}_2^4, \quad \mathbb{Z}_2^7, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^{10}\times\mathbb{Z}_{12}, \quad \mathbb{Z}_2^{10},$$

$$\mathbb{Z}_2^{15}, \quad \mathbb{Z}_2^{14}, \quad \mathbb{Z}_2^{16}\times\mathbb{Z}_{12}, \quad \mathbb{Z}_2^{18}, \quad \mathbb{Z}_2^{21}, \quad \mathbb{Z}_2^{20}, \quad \mathbb{Z}_2^{24}\times\mathbb{Z}_{12}.$$

According to the special version of universal coefficient theorem that applies to our case, from the short exact sequence

$$0\to\mathbb{Z}^{\text{or}}\xrightarrow{i=\times 2}\mathbb{Z}^{\text{or}}\xrightarrow{p=\text{mod }2}\mathbb{Z}_2\to 0, \tag{224}$$

we have the long exact sequence

$$\cdots\to H^2(\mathbb{Z}_2)\xrightarrow{\beta_2}H^3(\mathbb{Z}^{\text{or}})\xrightarrow{i_3^*=\times 2}H^3(\mathbb{Z}^{\text{or}})\xrightarrow{p_3^*=\text{mod }2}H^3(\mathbb{Z}_2)\xrightarrow{\beta_3}H^4(\mathbb{Z}^{\text{or}})\xrightarrow{i_4^*=\times 2}H^4(\mathbb{Z}^{\text{or}})\xrightarrow{p_4^*=\text{mod }2}H^4(\mathbb{Z}_2)\to\cdots, \tag{225}$$

where we have suppressed the group $Fd\bar{3}m$ of the cohomology and only write out the coefficient (either $\mathbb{Z}_2$ or $\mathbb{Z}^{\text{or}}$). where all $i_{1,2,\ldots}^*$ are multiplication by 2 for coycles and all $p_{1,2,\ldots}^*$ are mod 2 for cocycles. We have

$$\ker\beta_3=\text{im}p_3^*=\mathbb{Z}_2^5, \qquad \text{im}\beta_3=\ker i_4^*=\mathbb{Z}_2^4, \quad \beta_3\text{ is surjective}, \tag{226}$$

the map $p_3^*\colon H^3(\mathbb{Z}^{\text{or}})\to H^3(\mathbb{Z}_2)$ is

$$p_3^*\colon\mathbb{Z}\times\mathbb{Z}_2^4\xrightarrow{p_3^*=\text{mod }2}\mathbb{Z}_2^5, \tag{227}$$

we see that in the universal coefficient theorem, this is stated in the form of

$$(\mathbb{Z}\times\mathbb{Z}_2^4)\to(\mathbb{Z}\times\mathbb{Z}_2^4)\otimes\mathbb{Z}_2=(\mathbb{Z}\otimes\mathbb{Z}_2)\times(\mathbb{Z}_2^4)\otimes\mathbb{Z}_2=\mathbb{Z}_2^5, \tag{228}$$

where "×" means direct product. We have worked out the map in Mathematica. Recall that

$$H^3(\mathbb{Z}_2)=\langle x\beta, x_1\beta, xy, x^3, x^2x_1, xx_1^2, x_1^3, c, \tau\rangle, \quad x_1y=0, \tag{229}$$

where c is the degree-3 generator in $H^3(S_4,\mathbb{Z}_2)$, and τ is another degree-3 generator of $H^3(Fd\bar{3}m,\mathbb{Z}_2)$. We have showed in Mathematica, that

$$\text{im}p_3^*\colon\mathbb{Z}_2^4\times\mathbb{Z}\to\langle x_1\beta, xx_1^2+x_1^3, x^3+x^2x_1, c, \tau+x_1^3\rangle, \tag{230}$$

where we have that $\mathbb{Z}$ is mapped to τ (reflected in the no-solution of the relevant equation if we just use the eight cocycles $x\beta, x_1\beta, xy, x^3, x^2x_1, x_1^2x, x_1^3, c$ that we have explicit form–see Mathematica).

Using this we can also work out the Bockstein homomorphism. It is defined by $\beta\colon H^3(\mathbb{Z}_2)\to H^3(\mathbb{Z}^{\text{or}})\xrightarrow{d}H^4(\mathbb{Z}^{\text{or}})\xrightarrow{p_4^*}H^4(\mathbb{Z}_2)$, or $\beta\colon H^3(\mathbb{Z}_2)\xrightarrow{\beta_3}H^4(\mathbb{Z}_2)\xrightarrow{p_4^*}H^4(\mathbb{Z}_2)$.

28.3 Breathing pyrochlore (No. 216, F$\bar{4}$3m)

Using the code

```
G:=SpaceGroupIT(3,216);

R:=ResolutionSpaceGroup(G,16);

ZZ:=GL(1,Integers);;
```

```
Zor:= GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)]]);;
C:=HomToIntegralModule(R,Zor);;
for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15] do
Print("H^",n,"=",Cohomology(C,n),";;");
> od;
```

we get $H^n(F\bar{4}3m, \mathbb{Z}^{\text{or}})$ for $n = 0, 1, ..., 15$ to be

$$\mathbb{Z}_1, \quad \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}\times\mathbb{Z}_2^3\times\mathbb{Z}_4, \quad \mathbb{Z}_2^3, \quad \mathbb{Z}_2^5, \quad \mathbb{Z}_2^6\times\mathbb{Z}_6, \quad \mathbb{Z}_2^7\times\mathbb{Z}_4\times\mathbb{Z}_{12}, \quad \mathbb{Z}_2^7,$$
$$\mathbb{Z}_2^{13}, \quad \mathbb{Z}_2^{10}\times\mathbb{Z}_6, \quad \mathbb{Z}_2^{11}\times\mathbb{Z}_4\times\mathbb{Z}_{12}, \quad \mathbb{Z}_2^{15}, \quad \mathbb{Z}_2^{17}, \quad \mathbb{Z}_2^{14}\times\mathbb{Z}_6, \quad \mathbb{Z}_2^{19}\times\mathbb{Z}_4\times\mathbb{Z}_{12}.$$

Then, using

```
S4:=Group((1,2,3),(3,4));
ZZ:=GL(1,Integers);;
rho:=GroupHomomorphismByFunction(S4,ZZ,x->[[SignPerm(x)]]);
R:=ResolutionFiniteGroup(S4,11);;
C:=HomToIntegralModule(R,rho);;
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

Result: we get, for $p = 0, 1, 2, ..., 10$, $E_2^{p,0} = H^p(S_4, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2^2, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2\times\mathbb{Z}_6, \mathbb{Z}_2^3, \mathbb{Z}_2^2, \mathbb{Z}_2^4, \mathbb{Z}_2^2\times\mathbb{Z}_6$.

The corresponding code to compute $E_2^{p,1}$ is

```
S4:=Group((1,2,3),(3,4));
ZZZ:=GL(3,Integers);;
ZZZor:=GroupHomomorphismByImages(S4,ZZZ,[(1,2,3),(3,4)],
[[[0,0,1],[1,0,0],[0,1,0]],[[-1,0,1],[0,-1,1],[0,0,1]]]);
R:=ResolutionFiniteGroup(S4,11);;
C:=HomToIntegralModule(R,ZZZor);
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

We get $E_2^{p,1}$ for $p = 0, 1, ..., 10$ to be $\mathbb{Z}_2^{0,0,1,1,1,2,2,2,3,3,3}$.

For $E_2^{p,2}$: instead of `ZZZor` we use

```
ZZZZor:=GroupHomomorphismByImages(S4,ZZZ,[(1,2,3),(3,4)],
[[[0,0,-1],[1,0,0],[0,-1,0]],[[-1,-1,1],[0,1,0],[0,0,1]]]);
CC:=HomToIntegralModule(R,ZZZZor);
```

We get $E_2^{p,2}$ for $p = 0, 1, ..., 10$ to be $\mathbb{Z}_1, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2\times\mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^3, \mathbb{Z}_2^2\times\mathbb{Z}_4, \mathbb{Z}_2^4$.

Finally, for $E_2^{p,3}$: we use

```
for n in [0,1,2,3,4,5,6,7,8,9,10] do
Print("H^",n,"=",GroupCohomology(S4,n),";;");
od;
```

we get $E_3^{p,3}$ for $p = 0, 1, ..., 10$ to be $\mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2^3, \mathbb{Z}_2^2, \mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_3, \mathbb{Z}_2^3, \mathbb{Z}_2^4$.

The simple diagonal sum on E_2 page gives $\mathbb{Z}_1$ for $n = 0$, $\mathbb{Z}_2$ for $n = 1$, $\mathbb{Z}_3$ for $n = 2$, $\mathbb{Z}_2^3\times\mathbb{Z}_4\times\mathbb{Z}$ for $n = 3$, $\mathbb{Z}_2^3$ for $n = 4$, $\mathbb{Z}_2^5$ for $n = 5$, $\mathbb{Z}_2^6\times\mathbb{Z}_6$ for $n = 6$, and so on. We see that out of these checked cases, $E_2 = E_\infty$.

Note that for $n = 3$, No. 227 has a $2\mathbb{Z}$, while for $n = 3$, No. 216 has $\mathbb{Z}$.

28.4 Point Group SPT

We look at polyhedral point groups: T, T_h, O, T_d, O_h. Note that $T \cong A_4$, $T_h \cong A_4 \times \mathbb{Z}_2$, $O \cong T_d \cong S_4$, $O_h \cong S_4 \times \mathbb{Z}_2$.

For S_4: we use the standard cohomology ring $H^*(S_4, \mathbb{Z}_2) = \mathbb{F}_2[\sigma_1, \sigma_2, \sigma_3]/(\sigma_1\sigma_3)$. We will use the following reference for Stiefel Whitney classes: `http://dr.iiserpune.ac.in:8080/xmlui/bitstream/handle/123456789/4836/20151115_thesis.pdf?sequence=3&isAllowed=y`, which lists w_1 and w_2 for any representation of the Klein four group (Theorem 18), Dihedral group (Theorem 19) and symmetric group.

- C_1: trivial.
- C_i: we have $H^*(C_i, \mathbb{Z}_2) = \mathbb{F}_2[\iota]$, and as it is orientation reversing (see below) we have $H^4(C_i, U(1)^{\text{or}}) = \langle \iota \rangle \cong \mathbb{Z}_2$.
- C_2: we have $H^*(C_2, \mathbb{Z}_2) = \mathbb{F}_2[\rho]$ and $H^{2i}(C_2, \mathbb{Z}) = \langle \rho^{2i} \rangle$ while $H^{2i}(C_2, U(1)) = H^{2i+1}(C_2, \mathbb{Z}) = \mathbb{Z}_1$.
- C_s: we have $H^*(C_s, \mathbb{Z}_2) = \mathbb{F}_2[\sigma']$ and $H^{2i}(C_s, \mathbb{Z}) = \mathbb{Z}_1$ while $H^{2i}(C_s, U(1)^{\text{or}}) = \langle \sigma'^{2i} \rangle \cong H^{2i+1}(C_s, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2$.
- C_{2h}: we have $H^*(C_{2h}, \mathbb{Z}_2) = \mathbb{F}_2[\rho, \sigma']$, the total Stiefel–Whitney class is $(1+\rho)^2(1+\sigma')$, i.e. $w_1 = \sigma'$, $w_2 = \rho^2$, $w_3 = \rho^2\sigma'$. Since the action on the $\mathbb{Z}$ coefficient is the same as C_{2v} (see below) the cohomology expression is also the same: we have $H^4(C_{2h}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^3\sigma', \rho^2\sigma'^2, \rho\sigma'^3, \sigma'^4 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma') \rangle} \cong \mathbb{Z}_2^3$.
- D_2: we have $H^*(D_2, \mathbb{Z}_2) = \mathbb{F}_2[\rho', \rho]$, there is no orientation reversing elements, and $H^1(D_2, U(1)) = \langle \rho, \rho' \rangle = \mathbb{Z}_2^2$, $H^2(D_2, U(1)) = \langle \rho\rho' \rangle = \mathbb{Z}_2$, $H^3(D_2, U(1)) = \frac{\langle \rho^3, \rho^2\rho', \rho\rho'^2, \rho'^3 \rangle}{\langle \rho\rho'(\rho+\rho') \rangle} = \mathbb{Z}_2^3$, $H^4(D_2, U(1)) = \langle \rho^3\rho', \rho\rho'^3 \rangle \cong \mathbb{Z}_2^2$. This has also been verified using the free resolution produced by GAP. Note the total Witefel Whitney class is $(1+\rho)(1+\rho')(1+\rho+\rho')$, therefore $w_1 = 0$, $w_2 = \rho^2 + \rho\rho' + \rho'^2$, and $w_3 = \rho\rho'(\rho+\rho')$.
- C_{2v}: we have $H^*(C_{2v}, \mathbb{Z}_2) = \mathbb{F}_2[\rho, \sigma]$. The total Stiefel–Whitney class is $(1+\rho)(1+\sigma+\rho)$, i.e. $w_1 = \sigma$ and $w_2 = \rho(\rho+\sigma)$, and $w_3 = 0$. Using the resolution produced from GAP we obtain $H^1(C_{2v}, \mathbb{Z}^{\text{or}}) = \langle \sigma \rangle = \mathbb{Z}_2$, $H^2(C_{2v}, \mathbb{Z}^{\text{or}}) = \langle \rho(\rho+\sigma) \rangle = \mathbb{Z}_2$, $H^3(C_{2v}, \mathbb{Z}^{\text{or}}) = \langle \rho^2\sigma, \sigma^3 \rangle = \mathbb{Z}_2^2$, $H^4(C_{2v}, \mathbb{Z}^{\text{or}}) = \langle \rho^3(\rho+\sigma), \rho\sigma^2(\rho+\sigma) \rangle \cong \mathbb{Z}_2^2$, therefore $H^4(C_{2v}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^3\sigma, \rho^2\sigma^2, \rho\sigma^3, \sigma^4 \rangle}{\langle \rho^3(\rho+\sigma), \rho\sigma^2(\rho+\sigma) \rangle} \cong \mathbb{Z}_2^3$.
- D_{2h}: we have $H^*(D_{2h}, \mathbb{Z}_2) = \mathbb{F}_2[\rho, \rho', \sigma'] = \mathbb{F}_2[\rho, \rho', \iota]$. Using the resolution produced in GAP we get $H^1(D_{2h}, \mathbb{Z}^{\text{or}}) = \langle \sigma' \rangle \cong \mathbb{Z}_2$, $H^2(D_{2h}, \mathbb{Z}^{\text{or}}) = \langle \rho(\rho+\sigma'), \rho'(\rho'+\sigma') \rangle \cong \mathbb{Z}_2^2$, $H^3(D_{2h}, \mathbb{Z}^{\text{or}}) = \langle \rho(\rho+\rho')(\rho'+\sigma'), \rho'^2\sigma', \rho\rho'(\rho+\rho'+\sigma'), \sigma'^3 \rangle \cong \mathbb{Z}_2^4$, $H^4(D_{2h}, \mathbb{Z}^{\text{or}}) = \langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma'), \rho\rho'(\rho+\rho')\sigma', \rho'\sigma'^2(\rho'+\sigma'), \rho\rho'^2(\rho+\sigma'), \rho'^3(\rho'+\sigma') \rangle \cong \mathbb{Z}_2^6$, therefore $H^4(D_{2h}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^2\rho'^2, \rho'^4, \rho^3\sigma', \rho^2\rho'\sigma', \rho\rho'^2\sigma', \rho'^4\sigma', \rho^2\sigma'^2, \rho'^2\sigma'^2, \rho\sigma'^3, \rho'\sigma'^3 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma'), \rho\rho'(\rho+\rho')\sigma', \rho'\sigma'^2(\rho'+\sigma'), \rho\rho'^2(\rho+\sigma'), \rho'^3(\rho'+\sigma') \rangle} \times \langle \sigma'^4, \sigma'^2\rho\rho', \rho^3\rho', \rho\rho'^3 \rangle \cong \mathbb{Z}_2^9$. We can write the total Stiefel–Whitney class which is $(1+\rho)(1+\rho+\rho')(1+\rho'+\sigma')$ which gives $w_1 = \sigma'$, $w_2 = \rho^2+\rho\rho'+\rho'^2+\rho'\sigma'$, $w_3 = \rho(\rho+\rho')(\rho'+\sigma')$; alternatively using ι we have the total class $(1+\rho+\iota)(1+\rho'+\rho+\iota)(1+\rho'+\iota)$ we get $w_1 = \iota$, $w_2 = \iota^2 + \rho^2 + \rho\rho' + \rho'^2$, $w_3 = \iota^3 + \iota(\rho^2 + \rho\rho' + \rho'^2) + \rho\rho'(\rho+\rho')$,
- C_4: $H^*(C_4, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\rho}]$ but $H^4(C_4, U(1)) = \mathbb{Z}_1$.
- S_4: $H^*(C_4, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\iota}]$, where $\widetilde{\iota}$ is the character of $S_4 \cong \mathbb{Z}_4$ with $\widetilde{\iota}(C_4) = -1$. We have $H^4(S_4, U(1)^{\text{or}}) = \langle \widetilde{\iota}^4 \rangle$ ($H^{2i}(S_4, U(1)^{\text{or}}) = \langle \widetilde{\iota}^{2i} \rangle$ while we get trivial result in odd dimension). Therefore we have $w_1 = \widetilde{\iota}$, $w_2 = 0$, and $w_3 = \widetilde{\iota}^3$.
- C_{4h}: $H^*(C_{4h}, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\rho}, \sigma']$, we have $H^5(C_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^3$, result should be the same as C_{2h}, therefore we have $H^4(C_{4h}, U(1)^{\text{or}}) = \frac{\langle \widetilde{\rho}^4, \widetilde{\rho}^3\sigma', \widetilde{\rho}^2\sigma'^2, \widetilde{\rho}\sigma'^3, \sigma'^4 \rangle}{\langle \widetilde{\rho}^3(\widetilde{\rho}+\sigma'), \widetilde{\rho}\sigma'^2(\widetilde{\rho}+\sigma') \rangle} \cong \mathbb{Z}_2^3$. We should have $w_1 = \sigma'$, $w_2 = \widetilde{\rho}^2$ and $w_3 = \widetilde{\rho}^2\sigma'$.
- D_4: $H^*(D_4, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\rho}, \rho', \theta]/(\widetilde{\rho}(\widetilde{\rho}+\rho'))$, where using the free resolution produced by GAP we get $H^1(D_4, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_1$, $H^2(D_4, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^2, \rho'^2 \rangle \cong \mathbb{Z}_2^2$, $H^3(D_4, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^3 + \theta\rho' \rangle \cong \mathbb{Z}_2$, $H^4(D_4, \mathbb{Z}^{\text{or}}) \otimes \mathbb{Z}_2 = \langle \rho^4, \rho'^4, \theta^2 \rangle\rangle = \mathbb{Z}_2^3$ (note that there is some ambiguity in choosing θ) and $H^5(D_4, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^2$, therefore we have $H^4(D_4, U(1)) = \langle \theta\widetilde{\rho}^2, \theta\rho'^2 \rangle \cong \mathbb{Z}_2^2$, and $w_1 = 0$, $w_2 = \rho'^2 + \theta(?)$, $w_3 = \widetilde{\rho}^3 + \theta\rho'(?)$.
- C_{4v}: we have $H^*(C_{4v}, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\rho}, \sigma, \theta']/(\widetilde{\rho}(\widetilde{\rho}+\sigma))$. $C_{4v} \cong D_4$ but lives in a different representation. We use GAP: not ethat D_4 can be reached via `D:=SmallGroup(8,3)`. The three generators `f1,f2,f3` correspond to the C_4C_2', C_2', and C_4^2 elements in D_4. Now we are in C_{4v} so that C_2' is replaced with some mirror and C_4C_2' also some mirror, therefore the $\mathbb{Z}^{\text{or}}$ must be reversed by both `f1` and `f2`, hence the fourthe irrep, `IrreducibleRepresentations(D)[4];`. We get $H^1(C_{4v}, \mathbb{Z}^{\text{or}}) = \langle \sigma \rangle \cong \mathbb{Z}_2$, $H^2(C_{4v}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^2 + \theta' \rangle \cong \mathbb{Z}_4$, $H^3(C_{4v}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^3, \sigma^3 \rangle \cong \mathbb{Z}_2^2$, $H^4(C_{4v}, \mathbb{Z}^{\text{or}}) = \langle \theta'(\widetilde{\rho}^2 + \sigma^2), \widetilde{\rho}^2(\theta' + \widetilde{\rho}^2) \rangle \cong \mathbb{Z}_2^2$, $H^5(C_{4v}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^3$, we have $H^4(C_{4v}, U(1)^{\text{or}}) = \frac{\langle \widetilde{\rho}^4, \sigma^4, \widetilde{\rho}^2\theta', \sigma^2\theta', \theta'^2 \rangle}{\langle \theta'(\widetilde{\rho}^2+\sigma^2), \widetilde{\rho}^2(\theta'+\widetilde{\rho}^2) \rangle} \cong \mathbb{Z}_2^3$.
- D_{2d}: we have $H^*(D_{2d}, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\iota}, \rho', \theta'']/(\widetilde{\iota}(\widetilde{\iota} + \rho'))$. $D_{2d} \cong D_4$ but lives in the representation where `f2` preserves orientation but `f1` reverses orientation. This is `IrreducibleRepresentations(D)[2];` We have $H^1(D_{2d}, \mathbb{Z}^{\text{or}}) = \langle \rho' \rangle \cong \mathbb{Z}_2$, $H^2(D_{2d}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\iota}^2 + \rho'^2 \rangle \cong \mathbb{Z}_2$, $H^3(D_{2d}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\iota}^3, \theta''(\widetilde{\iota}+\rho') \rangle \cong \mathbb{Z}_2^2$, $H^4(D_{2d}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\iota}^4 + \rho'^4, \widetilde{\iota}^2(\theta''+\widetilde{\iota}^2) \rangle \cong \mathbb{Z}_2^2$, $H^5(D_{2d}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^3$, $H^4(D_{2d}, U(1)^{\text{or}}) = \frac{\langle \widetilde{\iota}^4, \rho'^4, \widetilde{\iota}^2\theta'', \rho'^2\theta'', \theta''^2 \rangle}{\langle \widetilde{\iota}^4+\rho'^4, \widetilde{\iota}^2(\theta''+\widetilde{\iota}^2) \rangle} \cong \mathbb{Z}_2^3$.

- D_{4h} is isomorphic to $D_4 \times \mathbb{Z}_2$, with mod-2 cohomology $H^*(D_{4h}, \mathbb{Z}_2) = \mathbb{F}_2[\widetilde{\rho}, \rho', \theta, \sigma' \sim \iota]/(\widetilde{\rho}(\widetilde{\rho} + \rho'))$. This is `SmallGroup(16,11)`, generated by `f1,f2,f3,f4` and `f3` commutes with all other three generators, therefore naturally we choose `f3` to be either the horizontal mirror plane or inversion. We are looking at the fifth irrep, `IrreducibleRepresentations(D4h)[5];`, we have $H^1(D_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2$, $H^2(D_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^2$, $H^3(D_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^4$, $H^4(D_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^6$, $H^5(D_{4h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^9$. The guess is that the result is identical to that of D_{2h}. Using the free resolution produced in GAP we obtain $H^1(D_{4h}, \mathbb{Z}^{\text{or}}) = \langle \sigma' \rangle \cong \mathbb{Z}_2$, $H^2(D_{4h}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}(\widetilde{\rho} + \sigma'), \rho'(\rho' + \sigma') \rangle \cong \mathbb{Z}_2^2$, $H^3(D_{4h}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^3 + (\rho' + \sigma')\theta, \widetilde{\rho}^2\sigma', \rho'^2\sigma', \sigma'^3 \rangle \cong \mathbb{Z}_2^4$, $H^4(D_{4h}, \mathbb{Z}^{\text{or}}) = \langle \widetilde{\rho}^3(\widetilde{\rho} + \sigma'), \rho'^3(\rho' + \sigma'), (\widetilde{\rho}^3 + \rho'\theta)\sigma', (\widetilde{\rho} + \sigma')\widetilde{\rho}\sigma'^2, (\rho'+\sigma')\rho'\sigma'^2, (\widetilde{\rho}^2+\theta)\widetilde{\rho}\sigma' \rangle$, therefore $H^4(D_{4h}, U(1)^{\text{or}}) = \frac{\langle \widetilde{\rho}^4, \rho'^4, \theta\widetilde{\rho}^2, \theta\rho'^2, \sigma'\widetilde{\rho}^3, \sigma'\rho'^3, \sigma'\widetilde{\rho}\theta, \sigma'\rho'\theta, \sigma'^2\widetilde{\rho}^2, \sigma'^2\rho'^2, \sigma'^3\widetilde{\rho}, \sigma'^3\rho' \rangle}{\langle \widetilde{\rho}^3(\widetilde{\rho}+\sigma'), \rho'^3(\rho'+\sigma'), (\widetilde{\rho}^3+\rho'\theta)\sigma', (\widetilde{\rho}+\sigma')\widetilde{\rho}\sigma'^2, (\rho'+\sigma')\rho'\sigma'^2, (\widetilde{\rho}^2+\theta)\widetilde{\rho}\sigma' \rangle} \times \langle \sigma'^4, \sigma'^2\theta, \theta^2 \rangle \cong \mathbb{Z}_2^9$.

- C_3: both mod-2 cohomology and $\mathbb{Z}$ cohomolgoy is trivial, hence $H^4(C_{3i}, U(1)^{\text{or}}) = \mathbb{Z}_1$.

- S_6 (another notation is C_{3i}), same as C_i: $H^4(C_{3i}, U(1)^{\text{or}}) = \langle \iota \rangle \cong \mathbb{Z}_2$.

- D_3: we have $H^*(D_3, \mathbb{Z}_2) = \mathbb{F}_2[\rho']$, there is no orientation reversing elements, and $H^4(D_3, U(1)) = H^5(D_3, \mathbb{Z}) = \mathbb{Z}_1$.

- C_{3v}: we have $H^*(C_{3v}, \mathbb{Z}_2) = \mathbb{F}_2[\sigma]$, $H^5(C_{3v}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2$ therefore $H^4(C_{3v}, U(1)^{\text{or}}) = \langle \sigma \rangle \cong \mathbb{Z}_2$.

- D_{3d}: we have $H^*(D_{3d}, \mathbb{Z}_2) = \mathbb{F}_2[\rho', \iota]$, the group $D_{3d} \cong D_3 \times \mathbb{Z}_2$ where the second summand can be assigned to inversion. The representation on the coefficient $\mathbb{Z}^{\text{or}}$ is the same as the case of D_{3h} (see below), hence we can use the result there so long as we rewrite the generators: $H^4(D_{3d}, U(1)^{\text{or}}) = \frac{\langle \rho'^4, \rho'^3\iota, \rho'^2\iota^2, \rho'\iota^3, \iota^4 \rangle}{\langle \rho'^3(\rho'+\iota), \rho'\iota^2(\rho'+\iota) \rangle} \cong \mathbb{Z}_2^3$.

- C_6: result is identical to C_2: $H^4(C_6, U(1)^{\text{or}}) \cong H^5(C_6, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_1$.

- C_{3h}: result is identical C_s: $H^4(C_{3h}, U(1)^{\text{or}}) = \langle \sigma'^4 \rangle \cong \mathbb{Z}_2$.

- C_{6h}: we have $H^*(C_{2h}, \mathbb{Z}_2) = \mathbb{F}_2[\rho, \sigma']$, the result should be the same as C_{2h}, which is $H^4(C_{6h}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^3\sigma', \rho^2\sigma'^2, \rho\sigma'^3, \sigma'^4 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma') \rangle} \cong \mathbb{Z}_2^3$.

- D_6: we have $H^*(D_6, \mathbb{Z}_2) = \mathbb{F}_2[\rho', \rho]$, there is no orientation reversing elements, and $H^4(D_6, U(1)) = \langle \rho'\rho^3, \rho'^3\rho \rangle \cong \mathbb{Z}_2^2$.

- C_{6v}: we have $H^*(C_{6v}, \mathbb{Z}_2) = \mathbb{F}_2[\rho, \sigma]$. Note that abstrctly $D_6 \cong C_{6v} \cong D_{3h} \cong Dih_3 \times \mathbb{Z}_2$. This group is `SmallGroup(12,4)` in gap, and we can look at its irreducible representations using `IrreducibleRepresentations(SmallGroup` we see that out of the generators `f1,f2,f3`, `f3` is the threefold rotation while `f2` belongs to the direct summand $\mathbb{Z}_2$, which is the two fold rotation in C_6. We see that the action of C_{6v} on $\mathbb{Z}$ is the second irrep listed in GAP, which gives `Pcgs([ f1, f2, f3 ]) -> [ [ [ -1 ] ], [ [ 1 ] ], [ [ 1 ] ] ]`. Then we can obtain $H^5(C_{6v}, \mathbb{Z}^{\text{or}})$ using the following command

  ```
  D:=SmallGroup(12,4);

  rho:=IrreducibleRepresentations(D)[2];

  R:=ResolutionFiniteGroup(D,7);;

  C:=HomToIntegralModule(R,rho);;

  Cohomology(C,5);
  ```

  which gives $H^5(C_{6v}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^3$. As $H^*(C_{6v}, \mathbb{Z}_2) \cong H^*(C_{2v}, \mathbb{Z}_2)$, we can copy the result of C_{2v} and get $H^4(C_{2v}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^3\sigma, \rho^2\sigma^2, \rho\sigma^3, \sigma^4 \rangle}{\langle \rho^3(\rho+\sigma), \rho\sigma^2(\rho+\sigma) \rangle} \cong \mathbb{Z}_2^3$.

- D_{3h}: we have $H^*(D_{3h}, \mathbb{Z}_2) = \mathbb{F}_2[\rho', \sigma']$. In this group the $\mathbb{Z}_2$ comes from the horizontal mirror, therefore changing the representation code to `rho:=IrreducibleRepresentations(D)[3];`, which gives the third irrep irrep listed in GAP, which gives `Pcgs([ f1, f2, f3 ]) -> [ [ [ 1 ] ], [ [ -1 ] ], [ [ 1 ] ] ]`, we get $H^5(D_{3h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^3$. Since ρ', σ' can be identified with ρ, σ in C_{2v} (by rotating the principle axis to horizontal plane, we should have $H^4(D_{3h}, U(1)^{\text{or}}) = \frac{\langle \rho'^4, \rho'^3\sigma', \rho'^2\sigma'^2, \rho'\sigma'^3, \sigma'^4 \rangle}{\langle \rho'^3(\rho'+\sigma'), \rho'\sigma'^2(\rho'+\sigma') \rangle} \cong \mathbb{Z}_2^3$.

- D_{6h}: we have $H^*(D_{6h}, \mathbb{Z}_2) = \mathbb{F}_2[\rho', \rho, \sigma'] = \mathbb{F}_2[\rho', \rho, \iota = \rho+\sigma']$. Again we can use GAP: the group is `D6h:=SmallGroup(24,14);` which has foru generators `f1,f2,f3,f4` where `f2,f3` are the ones that commute with the rest, i.e. they stay in $\mathbb{Z}_2^2$. Choosing say `f_2` to be the twofold rotation along principal axis and `f_3` to be the horizontal mirror/inversion, we choose the fifth irrep which gives `Pcgs([ f1, f2, f3, f4 ]) -> [ [ [ 1 ] ], [ [ 1 ] ], [ [ -1 ] ], [ [ 1 ] ] ]`, we get $H^4(D_{6h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^6$ and $H^5(D_{6h}, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^9$, which should be completely inherited from D_{2h}, hence $H^4(D_{6h}, U(1)^{\text{or}}) = \frac{\langle \rho^4, \rho^2\rho'^2, \rho'^4, \rho^3\sigma', \rho^2\rho'\sigma', \rho\rho'^2\sigma', \rho'^4\sigma', \rho^2\sigma'^2, \rho'^2\sigma'^2, \rho\sigma'^3, \rho'\sigma'^3 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma'), \rho\rho'(\rho+\rho')\sigma', \rho'\sigma'^2(\rho'+\sigma'), \rho\rho'^2(\rho+\sigma'), \rho'^3(\rho'+\sigma') \rangle} \times \langle \sigma'^4, \sigma'^2\rho\rho', \rho^3\rho', \rho\rho'^3 \rangle \cong \mathbb{Z}_2^9$.

- T: this acts on the vertices of a tetrahedron and can be easily identified with A_4. For later use we write $H^*(T,\mathbb{Z}_2)=\mathbb{F}_2[\rho,\rho']^{\mathbb{Z}_3}$ where $\mathbb{Z}_3$ action on ρ,ρ' is defined as $\rho\mapsto\rho'$, $\rho'\mapsto\rho+\rho'$. Then we have $H^1(T,\mathbb{Z}_2)=\mathbb{Z}_1$, $H^2(T,\mathbb{Z}_2)=\langle\delta=\rho^2+\rho\rho'+\rho'^2\rangle\cong\mathbb{Z}_2$, $H^3(T,\mathbb{Z}_2)=\langle\psi'=\rho^3+\rho'^3+\rho^2\rho',\psi=\rho\rho'(\rho+\rho')\rangle\cong\mathbb{Z}_2^2$, $H^4(T,\mathbb{Z}_2)=\langle\delta^2=\rho^4+\rho^2\rho'^2+\rho'^4\rangle=\mathbb{Z}_2$. We also have $w_1=0$, $w_2=\delta$, $w_3=\psi$. There is no orientation reversing operation, and we can just look at $H^5(A_4,\mathbb{Z})=\mathbb{Z}_1$, meaning that $H^4(T,U(1))=\mathbb{Z}_1$.

- T_h: can be viewed as T product with inversion. Using the following code

```
Th:=Group((1,2,3),(1,2)(3,4),(5,6));

ZZ:=GL(1,Integers);;

rho:=GroupHomomorphismByFunction(Th,ZZ,x->[[SignPerm(x)]]);

R:=ResolutionFiniteGroup(Th,6);;

C:=HomToIntegralModule(R,rho);;

for n in [0,1,2,3,4,5] do

Print("H^",n,"=",Cohomology(C,n),";;");

od;
```

  we get results $H^{0,1,2,3,4,5}(T_h,\mathbb{Z}^{\text{or}})=\mathbb{Z}_1,\mathbb{Z}_2,\mathbb{Z}_1,\mathbb{Z}_2^2,\mathbb{Z}_2^2,\mathbb{Z}_2^3$. Note that $H^*(T_h,\mathbb{Z}_2)=\mathbb{F}_2[\rho,\rho']^{\mathbb{Z}_3}\otimes\mathbb{F}_2[\iota]$, and $H^4(T_h,\mathbb{Z}_2)=\langle\iota^4,\delta\iota^2,\psi\iota,\psi'\iota,\delta^2\rangle\cong\mathbb{Z}_2^5$, considering the result for O_h, the only consistent possibility (which can be easily verified) is $w_1=\iota$, $w_2=\iota^2+\delta$, $w_3=\iota^3+\iota\delta+\psi$, and that $H^4(T_h,U(1)^{\text{or}})=\langle\iota^4,\iota^2\delta,\delta^2\rangle\cong\mathbb{Z}_2^3$.

- T_d: this acts on the vertices of a tetrahedron and can be easily identified with S_4. It transforms spatial coordinates in the T_2 irrep, corresponding to the $(3,1)$ irrep of S_4. We already have identified $V_1=\langle(12),(34)\rangle=\langle M,C_2M\rangle=C_{2v}$ and $V_2=\langle(12)(34),(14)(23)\rangle=\langle C_2,C_2'\rangle=D_2$. σ is the character of $M=(12)$ and if we denote ρ,ρ' as the character of C_2 and C_2' then we can think of $\delta=\rho^2+\rho\rho'+\rho'^2$ and $\psi=\rho\rho'(\rho+\rho')$. We have $w_1=\sigma$, $w_2=\delta$, $w_3=e(T_2)=\psi$. The corresponding SPTs are classified by $H^4(T_d,U(1)^{\text{or}})=\langle\sigma^4,\delta^2\rangle=\mathbb{Z}_2^2$.

- O: we have $O\cong S_4$ with mod-2 cohomology $\mathbb{F}_2[\rho'',\delta,\psi]/(\rho''\psi)$. Its group action is always described in terms of an octehdron, which of course has a higher symmetry O_h (It seems hard to find a polygon with just O symmetry and not O_h). Consequently one describes its action on a cardinality-6 set: denote the top and botton as 1 and 6, and the middle ones as 2,3,4,5 according to the right-hand rules, then the minimal generators are $\widetilde{C}_3=(123)(456)$, $\widetilde{C}_4=(5432)$ (note the orientation of C_4!). Here we are using the tilded symbol to denote the equivalent of the untilded symbol in T_d, i.e. $g\in T_d\mapsto\widetilde{g}\in O$. Then we have $\widetilde{C}_2\widetilde{C}_2'=C_4^2=(24)(35)$ and $\widetilde{C}_2=C_3^{-1}\widetilde{C}_2\widetilde{C}_2'C_3=(16)(35)$, $\widetilde{C}_2'=(16)(24)$, The equivalence of M above, denoted as $\widetilde{M}$, is $\widetilde{M}=\widetilde{C}_2C_4C_3^2=(15)(24)(36)$, which is a disallowed operation on octehedron. O transforms spatial coordinates in its T_1 irrep, corresponding to the three-row $(2,1,1)$ irrep of S_4. Note that operation-wise we have $\widetilde{C}_4P=S_4\in T_d$. We have $w_1=0$, $w_2=\delta+\rho''^2$, and $w_3=e(T_1)=\rho''\delta+\psi$. The corresponding SPTs are classified by (note that there there is no orientation reversing) $H^1(O,\mathbb{Z})=\mathbb{Z}_1$, $H^2(O,\mathbb{Z})=\langle\rho''^2\rangle\cong\mathbb{Z}_2$, $H^3(O,\mathbb{Z})=\langle\rho''\delta+\psi\rangle\cong\mathbb{Z}_2$, $H^4(O,\mathbb{Z})=\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_3$, and $H^4(O,\mathbb{Z})\otimes\mathbb{Z}_2=\langle\rho''^4,\delta^2\rangle\cong\mathbb{Z}_2^2$, $H^5(O,\mathbb{Z})\cong\mathbb{Z}_2$, $H^4(O,U(1))=\langle\rho''^2\delta\rangle\cong\mathbb{Z}_2$.

- O_h: for te Stiefel-Whitney classes, the result must match that of D_{2h} upon setting $\sigma=0$. Recall that that when restricting to D_{2h} we have $\delta=\rho^2+\rho\rho'+\rho'^2$ and $\psi=\rho\rho'(\rho+\rho')$, then using the results there we get where we get $w_1=\sigma+\iota$, $w_2=\iota^2+\delta$, $w_3=(\iota+\sigma)\iota^2+\iota\delta+\psi$, and $H^1(O_h,\mathbb{Z}^{\text{or}})=\langle\sigma+\iota\rangle\cong\mathbb{Z}_2$, $H^2(O_h,\mathbb{Z}^{\text{or}})=\langle\iota\sigma\rangle\cong\mathbb{Z}_2$, $H^3(O_h,\mathbb{Z}^{\text{or}})=\langle\iota^2(\iota+\sigma),\sigma^2(\iota+\sigma),\delta\iota+\psi\rangle\cong\mathbb{Z}_2^3$, $H^4(O_h,\mathbb{Z}^{\text{or}})=\langle\iota^3\sigma,\iota\sigma^3,\sigma\delta(\iota+\sigma),\iota\psi\rangle\cong\mathbb{Z}_2^4$, $H^5(O_h,\mathbb{Z}^{\text{or}})=\mathbb{Z}_2^6$, $H^4(O_h,U(1)^{\text{or}})=\frac{\langle\sigma^4,\sigma^2\iota^2,\iota^4,\sigma^2\delta,\sigma\iota\delta,\iota^2\delta,\delta^2\rangle}{\langle(\sigma+\iota)\sigma\delta\rangle}\cong\mathbb{Z}_2^6$.

The result is summarized in Table 8.

29 Understanding LSM

29.1 Lattice homotopy in 2D and 3D

Main references are Mike's PRL paper `https://arxiv.org/pdf/1703.06882.pdf` and Weicheng Ye's Scipost paper`https://scipost.org/SciPostPhys.13.3.066/pdf`. Note that for both 2D and 3D, we have checked that Eq. (218) holds, i.e. the relation $H^2(G,\mathbb{Z}_2)=(H^2(G,\mathbb{Z}^{\text{or}})\otimes\mathbb{Z}_2)\times(H^3(G,\mathbb{Z}^{\text{or}})\otimes\mathbb{Z}_2)$ holds for all the 17 wallpaper groups, and $H^3(G,\mathbb{Z}_2)=(H^3(G,\mathbb{Z}^{\text{or}})\otimes\mathbb{Z}_2)\times(H^4(G,\mathbb{Z}^{\text{or}})\otimes\mathbb{Z}_2)$ all the 230 space groups. According to Weicheng's paper, it is the kernel of the map $H^2(G,\mathbb{Z}_2)\to H^2(G,U(1)^{\text{or}})\cong H^3(G,\mathbb{Z}^{\text{or}})$, which is exactly $H^2(G,\mathbb{Z}^{\text{or}})\otimes\mathbb{Z}_2$, that gives the LSM

Table 8: Point Group SPT. Generators: ρ (ρ') denotes the character for a twofold rotation along principal (horizontal) axis; $\widetilde{\rho}$ ($\widetilde{\iota}$) denote the character for a fourfold rotation (rotoinversion) along principal axis; σ (σ') denotes the character for a vertical (horizontal) mirror plane; ι denotes the character for inversion. $\delta, \theta, \theta', \theta''$ are 2-cocycles and ψ is a 3-cocycle. Restricting to subgroup D_2 δ and ψ reduce to $\delta = \rho^2 + \rho\rho' + \rho'^2$ and $\psi = \rho\rho'(\rho + \rho')$; we have used the notation δ and ψ whenever possible to reduce the length of an expression. Sometimes it is possible to write the mod-2 cohomology ring using either the horizontal mirror σ' or inversion ι, and whenever this happens we write "$\sigma' \sim \iota$". We denoted $P_4(\rho, \rho', \sigma') \equiv \frac{\langle \rho^4, \rho^2\rho'^2, \rho'^4, \rho^3\sigma', \rho^2\rho'\sigma', \rho\rho'^2\sigma', \rho'^4\sigma', \rho^2\sigma'^2, \rho'^2\sigma'^2, \rho\sigma'^3, \rho'\sigma'^3 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma'), \rho\rho'(\rho+\rho')\sigma', \rho'\sigma'^2(\rho'+\sigma'), \rho\rho'^2(\rho+\sigma'), \rho'^3(\rho'+\sigma') \rangle} \times \langle \sigma'^4, \sigma'^2\rho\rho', \rho^3\rho', \rho\rho'^3 \rangle$ and $Q_4(\widetilde{\rho}, \rho', \theta, \sigma') \equiv \frac{\langle \widetilde{\rho}^4, \rho'^4, \theta\widetilde{\rho}^2, \theta\rho'^2, \sigma'\widetilde{\rho}^3, \sigma'\rho'^3, \sigma'\widetilde{\rho}\theta, \sigma'\rho'\theta, \sigma'^2\widetilde{\rho}^2, \sigma'^2\rho'^2, \sigma'^3\widetilde{\rho}, \sigma'^3\rho' \rangle}{\langle \widetilde{\rho}^3(\widetilde{\rho}+\sigma'), \rho'^3(\rho'+\sigma'), (\widetilde{\rho}^3+\rho'\theta)\sigma', (\widetilde{\rho}+\sigma')\widetilde{\rho}\sigma'^2, (\rho'+\sigma')\rho'\sigma'^2, (\widetilde{\rho}^2+\theta)\widetilde{\rho}\sigma' \rangle} \times \langle \sigma'^4, \sigma'^2\theta, \theta^2 \rangle$. $\mathcal{C} \cong H^4(PG, U(1)^{\mathrm{or}})$ and $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ gives its decomposition following Huang, Song, Huang, and Hermele [PRB.96.205106]; since all of them only have $\mathbb{Z}_2$ summands we only list their $\mathbb{Z}_2$ dimension $\dim(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$. The last column lists the third Stiefel–Whitney class.

Schönflies	Abstract	Mod-2 Coh. Ring	$H^4(PG, U(1)^{\mathrm{or}})$	$\mathcal{C}$	$\dim(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$	w_3
C_1	Trivial	Trivial	Trivial	Trivial	$(1,0,0)$	0
C_i	$\mathbb{Z}_2$	$\mathbb{F}_2[\iota]$	$\langle \iota^4 \rangle$	$\mathbb{Z}_2$	$(1,0,0)$	ι^3
C_2	$\mathbb{Z}_2$	$\mathbb{F}_2[\rho]$	Trivial	Trivial	$(0,0,0)$	0
C_s	$\mathbb{Z}_2$	$\mathbb{F}_2[\sigma']$	$\langle \sigma'^4 \rangle$	$\mathbb{Z}_2$	$(0,0,1)$	0
C_{2h}	$\mathbb{Z}_2^2$	$\mathbb{F}_2[\rho, \sigma' \sim \iota]$	$\frac{\langle \rho^4, \rho^3\sigma', \rho^2\sigma'^2, \rho\sigma'^3, \sigma'^4 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma') \rangle}$	$\mathbb{Z}_2^3$	$(2,0,1)$	$\rho^2\sigma'$
D_2	$\mathbb{Z}_2^2$	$\mathbb{F}_2[\rho, \rho']$	$\langle \rho^3\rho', \rho\rho'^3 \rangle$	$\mathbb{Z}_2^2$	$(2,0,0)$	ψ
C_{2v}	$\mathbb{Z}_2^2$	$\mathbb{F}_2[\rho, \sigma]$	$\frac{\langle \rho^4, \rho^3\sigma, \rho^2\sigma^2, \rho\sigma^3, \sigma^4 \rangle}{\langle \rho^3(\rho+\sigma), \rho\sigma^2(\rho+\sigma) \rangle}$	$\mathbb{Z}_2^3$	$(0,1,2)$	0
D_{2h}	$\mathbb{Z}_2^3$	$\mathbb{F}_2[\rho, \rho', \sigma' \sim \iota]$	$P_4(\rho, \rho', \sigma')$	$\mathbb{Z}_2^9$	$(3,3,3)$	$\iota^3 + \iota\delta + \psi$
C_4	$\mathbb{Z}_4$	$\mathbb{F}_2[\widetilde{\rho}]$	Trivial	Trivial	$(0,0,0)$	0
S_4	$\mathbb{Z}_4$	$\mathbb{F}_2[\widetilde{\iota}]$	$\langle \widetilde{\iota}^4 \rangle$	$\mathbb{Z}_2$	$(1,0,0)$	$\widetilde{\iota}^3$
C_{4h}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{F}_2[\widetilde{\rho}, \sigma' \sim \iota]$	$\frac{\langle \widetilde{\rho}^4, \widetilde{\rho}^3\sigma', \widetilde{\rho}^2\sigma'^2, \widetilde{\rho}\sigma'^3, \sigma'^4 \rangle}{\langle \widetilde{\rho}^3(\widetilde{\rho}+\sigma'), \widetilde{\rho}\sigma'^2(\widetilde{\rho}+\sigma') \rangle}$	$\mathbb{Z}_2^3$	$(2,0,1)$	$\widetilde{\rho}^2\sigma'$
D_4	Dih_4	$\mathbb{F}_2[\widetilde{\rho}, \rho', \theta]/(\widetilde{\rho}(\widetilde{\rho}+\rho'))$	$\langle \theta\widetilde{\rho}^2, \theta\rho'^2 \rangle$	$\mathbb{Z}_2^2$	$(2,0,0)$	$\widetilde{\rho}^3 + \rho'\theta$
C_{4v}	Dih_4	$\mathbb{F}_2[\widetilde{\rho}, \sigma, \theta']/(\widetilde{\rho}(\widetilde{\rho}+\sigma)$	$\frac{\langle \widetilde{\rho}^4, \sigma^4, \widetilde{\rho}^2\theta', \sigma^2\theta', \theta'^2 \rangle}{\langle \theta'(\widetilde{\rho}^2+\sigma^2), \widetilde{\rho}^2(\theta'+\widetilde{\rho}^2) \rangle}$	$\mathbb{Z}_2^3$	$(0,1,2)$	0
D_{2d}	Dih_4	$\mathbb{F}_2[\widetilde{\iota}, \rho', \theta'']/(\widetilde{\iota}(\widetilde{\iota}+\rho'))$	$\frac{\langle \widetilde{\iota}^4, \rho'^4, \widetilde{\iota}^2\theta'', \rho'^2\theta'', \theta''^2 \rangle}{\langle \widetilde{\iota}^4+\rho'^4, \widetilde{\iota}^2(\theta''+\widetilde{\iota}^2) \rangle}$	$\mathbb{Z}_2^3$	$(1,1,1)$	$\widetilde{\iota}^3 + (\widetilde{\iota} + \rho')\theta''$
D_{4h}	$Dih_4 \times \mathbb{Z}_2$	$\mathbb{F}_2[\widetilde{\rho}, \rho', \theta, \sigma' \sim \iota]/(\widetilde{\rho}(\widetilde{\rho}+\rho'))$	$Q_4(\widetilde{\rho}, \rho', \theta, \sigma')$	$\mathbb{Z}_2^9$	$(3,3,3)$	$\widetilde{\rho}^3 + (\rho' + \sigma')\theta$
C_3	$\mathbb{Z}_3$	Trivial	Trivial	Trivial	$(0,0,0)$	0
S_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\iota]$	$\langle \iota^4 \rangle$	$\mathbb{Z}_2$	$(1,0,0)$	ι^3
D_3	Dih_3	$\mathbb{F}_2[\rho']$	Trivial	Trivial	$(0,0,0)$	0
C_{3v}	Dih_3	$\mathbb{F}_2[\sigma]$	$\langle \sigma^4 \rangle$	$\mathbb{Z}_2$	$(0,0,1)$	0
D_{3d}	$Dih_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho', \iota]$	$\frac{\langle \rho'^4, \rho'^3\iota, \rho'^2\iota^2, \rho'\iota^3, \iota^4 \rangle}{\langle \rho'^3(\rho'+\iota), \rho'\iota^2(\rho'+\iota) \rangle}$	$\mathbb{Z}_2^3$	$(2,0,1)$	$\iota^3 + \rho'^2\iota$
C_6	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho]$	Trivial	Trivial	$(0,0,0)$	0
C_{3h}	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\sigma']$	$\langle \sigma'^4 \rangle$	$\mathbb{Z}_2$	$(0,0,1)$	0
C_{6h}	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$\mathbb{F}_2[\rho, \sigma']$	$\frac{\langle \rho^4, \rho^3\sigma', \rho^2\sigma'^2, \rho\sigma'^3, \sigma'^4 \rangle}{\langle \rho^3(\rho+\sigma'), \rho\sigma'^2(\rho+\sigma') \rangle}$	$\mathbb{Z}_2^3$	$(2,0,1)$	$\rho^2\sigma'$
D_6	$Dih_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho, \rho']$	$\langle \rho'\rho^3, \rho'^3\rho \rangle$	$\mathbb{Z}_2^2$	$(2,0,0)$	ψ
C_{6v}	$Dih_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho, \sigma]$	$\frac{\langle \rho^4, \rho^3\sigma, \rho^2\sigma^2, \rho\sigma^3, \sigma^4 \rangle}{\langle \rho^3(\rho+\sigma), \rho\sigma^2(\rho+\sigma) \rangle}$	$\mathbb{Z}_2^3$	$(0,1,2)$	0
D_{3h}	$Dih_3 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho', \sigma']$	$\frac{\langle \rho'^4, \rho'^3\sigma', \rho'^2\sigma'^2, \rho'\sigma'^3, \sigma'^4 \rangle}{\langle \rho'^3(\rho'+\sigma'), \rho'\sigma'^2(\rho'+\sigma') \rangle}$	$\mathbb{Z}_2^3$	$(0,1,2)$	0
D_{6h}	$Dih_3 \times \mathbb{Z}_2^2$	$\mathbb{F}_2[\rho, \rho', \sigma' \sim \iota]$	$P_4(\rho, \rho', \sigma')$	$\mathbb{Z}_2^9$	$(3,3,3)$	$\iota^3 + \iota\delta + \psi$
T	A_4	$\mathbb{F}_2[\rho, \rho']^{\mathbb{Z}_3}$	Trivial	Trivial	$(0,0,0)$	ψ
T_h	$A_4 \times \mathbb{Z}_2$	$\mathbb{F}_2[\rho, \rho']^{\mathbb{Z}_3} \otimes \mathbb{F}_2[\iota]$	$\langle \iota^4, \iota^2\delta, \delta^2 \rangle$	$\mathbb{Z}_2^3$	$(1,1,1)$	$\iota^3 + \iota\delta + \psi$
O	S_4	$\mathbb{F}_2[\rho'', \delta, \psi]/(\rho''\psi)$	$\langle \rho''^2\delta \rangle$	$\mathbb{Z}_2$	$(1,0,0)$	$\rho''\delta + \psi$
T_d	S_4	$\mathbb{F}_2[\sigma, \delta, \psi]/(\sigma\psi)$	$\langle \sigma^4, \delta^2 \rangle$	$\mathbb{Z}_2^2$	$(0,1,1)$	ψ
O_h	$S_4 \times \mathbb{Z}_2$	$\mathbb{F}_2[\sigma, \iota, \delta, \psi]/(\sigma\psi)$	$\frac{\langle \sigma^4, \sigma^2\iota^2, \iota^4, \sigma^2\delta, \sigma\iota\delta, \iota^2\delta, \delta^2 \rangle}{\langle (\sigma+\iota)\sigma\delta \rangle}$	$\mathbb{Z}_2^6$	$(2,2,2)$	$(\iota + \sigma)\iota^2 + \iota\delta + \psi$

constraints. Indeed, we have checked all the 17 wallpaper groups and find that $H^2(G,\mathbb{Z}^{\rm or})\otimes\mathbb{Z}_n$ actually gives exactly TABLE II of Mike's paper `https://arxiv.org/pdf/1703.06882.pdf`. We also checked that the number of elementary abelian groups here exactly agree with the number of irreducible Wyckoff positions (IWP) given in ITC.

However, in 3D it seems the number of elementary abelian grous of $H^3(G,\mathbb{Z}^{\rm or})\otimes\mathbb{Z}_2$ (restricting to $SO(3)$ spin) does not agree with the number of IWPs. For example, No. 216 and 227 each have four IWPs, but both have $H^3(G,\mathbb{Z}^{\rm or})\otimes\mathbb{Z}_2=\mathbb{Z}_2^5$.

On the other hand, let's compare No. 196 (F23) and No. 216 (F-43m). Note that the high symmetry Wyckoff positions of them are exactly the same: 4a,4b,4c,4d,16e,24f,24g. But $H^3(F23,\mathbb{Z}^{\rm or})\otimes\mathbb{Z}_2=\mathbb{Z}_2^4$ and $H^3(F-43m,\mathbb{Z}^{\rm or})\otimes\mathbb{Z}_2=\mathbb{Z}_2^5$.

29.2 Else and Thorngren

The main reference is Else, Thorngren, `https://journals.aps.org/prb/pdf/10.1103/PhysRevB.101.224437`.

- Input: a global symmetry group G, spatial dimension is d. We define $X=\mathbb{R}^d$. We will mostly require $G=G_{\rm int}\times G_{\rm spatial}$ with $\mathcal{H}^2(G_{\rm int},U(1))=\mathbb{Z}_2$.
- Anomalous texture: an anomalous texture is an element $[\omega]\in\mathcal{H}^2(G,Z_0(\mathbb{R}^d,U(1)))$. For our interest, $Z_0(\mathbb{R}^d,U(1))=\Lambda$ is the lattice.
- defect network:
- The Traditional LSM theorem is the statement that: The ground state is noninvertible (i.e. has topological order) iff the equivariant push foward map

  $$\mathcal{H}^2(G,\Lambda=Z_0(\mathbb{R}^d,U(1)))\to H^{d+2}(G,U(1))=H^G_{-2}(X=\mathbb{R}^d,U(1)),[\omega]\mapsto[\mu] \tag{231}$$

  has nontrivial image, i.e. $[\mu]$ is a genuine cocycle and not a coboundary. In words, $[\omega]\mapsto[0]$ means that the (anomaly of the) anomalous texture can be canceled by (the anomaly of some) invertible-substrate defect network, while $[\omega]\mapsto[\mu]\neq[0]$ means that the (anomaly of the) anomalous texture cannot be canceled by (the anomaly of any) invertible-substrate defect network, and hence the group state has an anomaly (and hence is invertible/has topological order).
- In principle, it can happen that the map $[\omega]\mapsto[0]\in H^G_{-2}(X,U(1))$ is trivial (i.e. no traditional LSM), but the map $[\omega]\mapsto[\mu_1]\neq[0]\in H^G_{-2}(X_1,U(1))$ is nontrivial (in the sense of lattice homotopy). The main result of the Else, Thorngren paper is that, by explict checking, they find that for specific forms of G, which are $G=SO(3)\times G_{\rm spatial}$ (and two other cases),

  $$[\omega]\mapsto[\mu]=[0]\Leftrightarrow[\omega]\mapsto[\mu_1]=[0], \tag{232}$$

  hence lattice homotopy can serve as the equivalent criterion of LSM criterion, i.e. having a nontrivial lattice homotopy is equivalent to having a noninvertible ground state/has topological order.
- The map $[\omega]\mapsto[\mu]$ is called equivariant pushforward. This map is defined as follows:

```
LoadPackage(``HAP'');

Fd3m:=SpaceGroupIT(3,227);

wy:=WyckoffPositions(Fd3m);

Size(WyckoffStabilizer(wy[1]));
```

This size is the order of the stabilizer group which is 24. It turns out this one (the point $(1/8,1/8,1/8)$) is one of the highest symmetry points (the other one is $1/8,1/8,5/8$).

```
wyorb:=WyckoffOrbit(wy[1]);

List(wyorb,WyckoffTranslation);
```

The outputs `[ [ 1/8, 1/8, 5/8 ], [ 3/8, 3/8, 3/8 ] ]`.

```
TranslationBasis(Fd3m);
```

This outputs `[ [ 1/2, 0, 1/2 ], [ 0, 1/2, 1/2 ], [ 0, 0, 1 ] ]`.

Note in Dominic's code, in his `good_atom_locations(d,G, L=None)` he has `L=[1]*d`; this `L` is basically T_1, T_2 and T_3 which he will use to translate the highest symmetry points to get the other vortices of the Wigner-Seitz cell.

In GAP's code, however, we are searching for the lowest symmetry points, see `https://github.com/gap-packages/hap/blob/c130fb61b0c6aad88e3894a96a90a146442643ab/lib/Polymake/cryst.gi`. Let's work that out:

```
LoadPackage(‘‘HAP’’);

G:=SpaceGroupIT(3,227);

G:=StandardAffineCrystGroup(G);

AG:=AffineCrystGroupOnRight(GeneratorsOfGroup(G));;

SAG:=StandardAffineCrystGroup(AG);;

F:=FundamentalDomainStandardSpaceGroup([1/31,1/37,1/41],SAG);;

V:=Polymake(F,"VERTICES");
```

Note here $[1/31, 1/37, 1/41]$ is the dafault lowest-symmetry points that GAP will choose, and we are writing it here explicitly. The last command outputs the vertices of the polytope:

```
[ [ -1/4, -1/4, 22093/50824 ], [ 0, -1/2, 1/2 ], [ -1/4, -1/4, 5/8 ],

[ 1/4, 1/4, -1/8 ], [ 0, 0, 0 ], [ 1/4, -9387/25412, 9387/50824 ] ]
```

Call them $V_{1,2,3,4,5,6}$. One can easily check that $V_{1,2,4,5}$ coplanar, $V_{1,2,3,6}$ coplanar, and $V_{3,4,5,6}$ coplanar. The volume then can ca calculated as $\frac{1}{6}|\boldsymbol{v}_{61}\cdot\boldsymbol{v}_{21}\times\boldsymbol{v}_{41})| + \frac{1}{6}|\boldsymbol{v}_{63}\cdot\boldsymbol{v}_{23}\times\boldsymbol{v}_{43})| + \frac{1}{6}|\boldsymbol{v}_{25}\cdot\boldsymbol{v}_{35}\times\boldsymbol{v}_{45})| = \frac{1}{48}$.

Good places to understand G-equivariant cohomology: Definition of Dirichlet-Voloroi region: P130.

P218-219: If G acts on X fixed point-free, then $C_*(X)$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and the space $X(=\mathbb{R}^d)$ is in fact the space $X = \widetilde{K(\mathcal{P})}$: P122: a presentation of G can be used to construct a free G-CW-space $\widetilde{K} = \widetilde{K(x|r)}$ whose 1-skeleton $\widetilde{K}^1 = \Gamma_{G,x}$ is the Cayley graph of G with respect to the generating set $x \in G$ and whose fundamental group $\pi_1(\widetilde{K})$ is trivial. We say the quotient CW-space $K = \widetilde{K}/G$ is the 2-complex of the presentation $\langle x|r\rangle$. $\widetilde{K}$ is the universal covering space of K.

Some code related note: note that in my mac upgrading python package can be done using e.g.

```
python3.11 -m pip install --upgrade cython
```

To access (upgrade/install) modules for the python3.11 that is used in sage, use something like

```
/private/var/tmp/sage-10.0-current/local/var/lib/sage/venv-python3.11.1/bin/pip install JuPyMake
```

One might have to compilre these modules using amd64 architecture, in which case we need to use the following

```
ARCHFLAGS="-arch arm64" /private/var/tmp/sage-10.0-current/local/var/lib/sage/venv-python3.11.1/bin/pip

install JuPyMake --compile --no-cache-dir
```

see this post `https://apple.stackexchange.com/questions/436801/m1-mac-mach-o-file-but-is-an-incompatible-archite`

There are numerous places where we had to modify Dominic Else's code to make it run on python3 on MacM1. E.g. we had to make the change `xrange` → `range`, and (in utils.py) `self.coeffs.items()` → `iter(self.coeffs.items())`, in resolutions.py to make `itertools.izip` → `zip`, and note that while `magnaconv` is imported in linalg.py, the functions in `magmaconv_cython` was never used. After opening sage, we import the lsm module using

```
import equichain.lsm
```

and then executing

```
print(equichain.lsm.check_space_group(227))
```

which gives ([2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]), whose dimensions are 44,13,41,41. It seems wierd that the later pages have larger dimensions; but this actually happens for many space groups (e.g. see also 166 and 196.

If for 227 group we use `best = wyckoff_positions[4]`, which is unit cell that we used in Mathematica, then we get ([2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]).

Understanding the spectral sequence code (see `https://github.com/dominicelse/equichain/blob/9294373a3d10481dbdd4ed` `equichain/__init__.py#L588`): the main calculation of the $E_{0,1,2,3}$ pages of the relevant entries of the equivariant spectral sequence are listed in `__init__.py`. Denote d as the boundary map for the simplicial homology, and δ as the coboundary map for group cohomology (of the point group G_{pt}). We now have the 0th page $(E^0)^p_q = \mathcal{C}^p(G_{\mathrm{pt}}, C_q(T^d, R))$, where the spectral sequence is located in the fourth quadrant, with $p = 0, 1, ...$ the row index and $q = 0, 1...$ the column index (note that this is flipped from the LHS case). The E^1 page is taking homology for the map δ: $(E^1)^p_q = \ker(\delta\colon (E^0)^p_q \to (E^0)^{p+1}_q)/\mathrm{im}(\delta\colon (E^0)^{p-1}_q \to (E^0)^p_q)$. In the no-spin–orbit coupling case we are interested in $p = q = 0$ with the coefficient $R = \mathbb{Z}_2$. Then, $(E^0)^0_0$ is several copies of $\mathbb{Z}_2$, which denotes the anomalous textures put on the high symmetry points: the naive ways of putting anomalous textures (spin one-half's) at these high symmetry points $(C_0(T^d, \mathbb{Z}))$ gives the abelian group $C_0(T^d, \mathbb{Z}) \otimes \mathbb{Z}_2$. Then, taking δ (δ_0 in the code) gives the truly inequivalent ways of putting them on $C_0(T^d)$, here "truly inequivalent" is the same as taking group cohomology, and this brings us to the E^1 page. This is done implicitly in the code by selecting the subspace of $(E^0)^0_0$ that lives in the kernal of δ_0. Then, in going to E^2 we need to through the elements of $(E^0)^0_0$ that gets trivialized by adding $C_1(T^d)$: they are the image of $d_1\colon (E^1)^0_1 \to (E^0)^0_0$ where $(E^1)^0_1 := \ker(\delta_1\colon (E^0)^0_1 \to (E^0)^1_1)$. Therefore, viewing d_1 and δ_1 as matrices, we need to get the vector space $V_1 = \{d_1\boldsymbol{a} | \delta_1\boldsymbol{a} = 0, \boldsymbol{a} \in (E^0)^0_1\}$, and we have $(E^1)^0_0 = (E^0)^0_0/V_1$. Next, we need to find V_2 s.t. $(E^2)^0_0 = (E^0)^0_0/V_2$. It is not hard to see that $V_2 = \{d_1\boldsymbol{a} | \boldsymbol{a} \in (E^0)^0_1, \delta_1\boldsymbol{a} = d_2\boldsymbol{b}$ for some $\boldsymbol{b} \in (E^0)^1_2$ s.t. $\delta_2\boldsymbol{b} = 0\}$, where this $\boldsymbol{b}$ satisfying $\delta_2\boldsymbol{b} = 0$ is simplying saying that $\boldsymbol{b}$ is an element of $(E^1)^1_2$. We must have $(E^3)^0_0 = (E^2)^0_0$ as argued in the paper so that the final LSM anomaly is isomorphic to $(E^\infty)^0_0 = (E^3)^0_0 = (E^0)^0_0/V_2$; here what is realy means is an element $\omega_0 + \omega_1 + \omega_2 + \omega_3$: an element $\omega_0 \in (E^0)^0_0$ that survives in the final cohomology will pick up elements ω_i along the diagonal at the E^i page.

Discuss with weicheng: it seems LHS SS for $\mathbb{Z}^d \to SG \to PG$ is just a special case of equivariant SS, viewing the translation groups as torus. This means there should be a way to see detailed matching.

We say LHS is preferred because there is the cup product, which appears in topological partition function (crystalline topological response). But equivariant is preferred in the sense of physical picture (decorating lower dimensional SPT). Real question: what really does the crystalline topological response mean?

Let's look at the one dimensional space group $p1m = \langle T, M\rangle$ with $TM = MT^{-1}$. For equivariant homology, let's decompose the one-dimensional torus $\mathbb{T} = \mathbb{T}^1$ into points A, B and edges a, b, with $\partial a = A - B$ and $\partial b = B - A$.

Group cohomology: $E_2^{p,q} = \mathcal{H}^p(\langle M\rangle, \mathcal{H}^q(\mathbb{Z}, \mathbb{Z}_2)) = \mathcal{H}^p(\langle M\rangle, \delta_{q=0,1}\mathbb{Z}_2) = \mathbb{Z}_2\delta_{q=0,1}$. Using the fact that $p1m = T \rtimes M$ one can immediately conclude that $d_2 = 0$, therefore the LHS spectral sequence stabilizes at E_2.

Equivariant homology: $E_0^{p,q} = \mathcal{C}^p(G_{\mathrm{pt}}, C_{d-q}(\mathbb{T}, \mathbb{Z}_2))$, therefore we have $E_0^{0,0} = C_{1-0}(\mathbb{T}, \mathbb{Z}_2) = \mathbb{Z}_2^2$, taking differential δ gives $E_1^{0,0} = C_{1-0}(\mathbb{T}, \mathbb{Z}_2)^{\langle M\rangle} = \langle a + b\rangle \cong \mathbb{Z}_2$, $E_0^{0,1} = C_{1-1}(\mathbb{T}, \mathbb{Z}_2)^{\langle M\rangle} = \langle A, B\rangle = \mathbb{Z}_2^2$, $E_0^{1,0} = \{f(M) | f(M) = a$ or b or $a + b\} = \langle f_a, f_b\rangle = \mathbb{Z}_2^2$, and $E_0^{1,1} = \langle f_A, f_B\rangle = \mathbb{Z}_2^2$. Now, take δ, We get $E_1^{0,1} = \langle A, B\rangle = \mathbb{Z}_2^2$, and $\partial E_0^{0,0} = \langle \partial(a + b)\rangle = 0$, meaning that all the elements of $E_1^{0,1}$ will not be trivialized, therefore $E_\infty^{0,1} = \mathbb{Z}_2^2$. On the other hand, when taking δ we have $\delta f = 0 \Rightarrow (\delta f)(g_1, g_2) = 0 \Rightarrow g_1.f(g_2) + f(g_1g_2) - f(g_1) = 0$, if f is f_a, then setting $g_1 = g_2 = M$ shows f_a is not a cocycle; similarly f_b is not a cocycle. But $f := f_a + f_b$ satisfies $f(M) = a + b = M.a - a$ therefore f is actually a coboundary, meaning that $E_1^{1,0} = 0$. Therefore the equivariant homology is concentrated at the $(p, q) = (0, 1)$ corner.

30 Fermionic SPT and AHSS

AHSS:

$$w_1(\xi) \in H^1(BG, \mathbb{Z}_2)$$
$$w_2(\xi) \in H^2(BG, \mathbb{Z}_2)$$

$$n_1 \equiv w_1(\xi)$$
$$\lambda \equiv w_2(\xi)$$

$$\mathbb{Z}_2^F \to G_f \to G_b$$

$$H^p(BG_b, \Omega^q_{spin}(\star))$$

$$\Rightarrow \Omega^D_{spin}(BG, \xi)$$
$$\Omega^q_{spin}(\star) = \Omega^q_{spin}(\star, U(1))$$

$$\Omega^{-1}_{spin}(\star)) = \mathbb{Z}$$

$$\Omega^0_{spin}(\star)) = U(1)$$

$$\Omega^1_{spin}(\star)) = \mathbb{Z}_2$$

$$\Omega^2_{spin}(\star)) = \mathbb{Z}_2$$

$$D = 1 + 1 = 2$$

$$H^{D+1}(BG_b, \Omega^{-1}_{spin}(\star)) = H^{D+1}(BG_b, \mathbb{Z}^{\xi})$$

$$H^{D}(BG_b, \Omega^{0}_{spin}(\star)) = H^{D}(BG_b, U(1)^{\xi})$$

$$H^{D-1}(BG_b, \Omega^{1}_{spin}(\star)) = H^{D-1}(BG_b, \mathbb{Z}_2)$$

$$\nu_1 \quad \nu_0 \quad \nu_{-1}$$

Example: we know that $H^*(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 = \langle t \rangle$. Now consider two cases: the action of $\mathbb{Z}_2$ on $\mathbb{Z}$ is trivial (e.g. unitary) or orientation-reversing (e.g. anti-unitary).

- Trivial action: in this case $Sq^1(1) = 0$, $Sq^1(t) = t^2$. I.e. w have $Sq^{2n+1}(t) = t^{2n+2}$ and $Sq^{2n}(t) = 0$.
- Nontrivial action: in this case we define the new Steenrod square as Sq^1_{or}: the notation for this in Weicheng's paper is $\mathcal{SQ}$ and the notation for this in Ryan's paper is Sq^1_{ξ}. We have $Sq^1_{\text{or}}(1) = t$, $Sq^1_{\text{or}}(t^{2n}) = t^{2n+1}$ and $Sq^1_{\text{or}}(t^{2n+1}) = 0$.

Note that generally we have (see Lemma A.1. of weicheng's paper, `https://arxiv.org/pdf/2111.12097.pdf`, Eq. (A33))

$$Sq^1_{\text{or}}(x) = \frac{1}{2} dx \tag{233}$$

and

$$Sq^1_{\text{or}}(x) = Sq^1_{\text{or}}(1) \cup x + Sq^1(x) = \frac{1}{2}(-2w_1 \cup x + dx), \tag{234}$$

So now finally we understand the meaning of the fraction in the front (which I read for the first time in Ryan and Dominic's paper and it bewildered me for a long time)! But note that the crucial thing is that x must be a cocycle in the cohomolgy with coefficient $\mathbb{Z}$. This is hard to obtain.

Note that we can view w_1, the first Stiefel-Whitney class that captures the orientation reversing effect, as $w_1 = Sq^1_{\text{or}}(1)$.

First, an example, for the wallpaper group $p4m$:

```
G:=SpaceGroupIT(2,11);
R:=ResolutionSpaceGroup(G,15);
ZZ:=GL(1,Integers);;
Zor:= GroupHomomorphismByFunction(G,ZZ,x->[[Determinant(x)]]);;
C:=HomToIntegralModule(R,Zor);;
for n in [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14] do
Print("H^",n,"=",Cohomology(C,n),";;");
od;
```

In particular we get

$$H^3_{p4m}(\mathbb{R}^2, U(1)) = H^4_{p4m}(\mathbb{R}^2, \mathbb{Z}) = \mathcal{H}^4(p4m, \mathbb{Z}^{\text{or}}) = \mathbb{Z}_2^6.$$

for p4m: the point group according to ITC is C_{4v} (denoted as 4mm), which of course is isomorphic to Dih_4, or D_4.

There are three IWPs: the IWP $(0,0)$ has little group C_{4v}, the IWP $(1/2,1/2)$ has little group C_{4v}, and the IWP $(1/2,0),(0,1/2)$ having little group C_{2v}, isomorphic to Dih_2.

31 Perturbing method to get resolution

For $N \to G \to Q$, the general way to perturb the product $R_*^N \otimes_{\mathbb{Z}} R_*^Q$ constructed out of a free resolution for N, R^N, and the free resolution for Q, R^Q, is given in Graham's book, see section 3.3.1 on P267. Here we first consider a simpler case, where $G = N \rtimes Q$, following Lemma 3.3.7. In this case, it is not too hard to obatin a $\mathbb{Z}G$ resolution.

Translation resolution:

$$\mathbb{Z}T[e_3] \xrightarrow{\partial_3} \mathbb{Z}T[e_2^1, e_2^2, e_2^3] \xrightarrow{\partial_2} \mathbb{Z}T[e_1^1, e_1^2, e_1^3] \xrightarrow{\partial_1} \mathbb{Z}T \xrightarrow{\partial_0} \mathbb{Z} \to 0,$$

where $\partial_3(e_3) = (T_1 - 1)e_2^1 + (T_2 - 1)e_2^2 + (T_3 - 1)e_2^3$, and $\partial_2(e_2^i) = (T_{i+1} - 1)e_1^{i-1} - (T_{i-1} - 1)e_1^{i+1}$, and $\partial_1(e_1^i) = (T_i - 1)$, and $\partial_0(T_1^{r_1}T_2^{r_2}T_3^{r_3}) = r_1 + r_2 + r_3$. As one can easily check, we have $\partial_{k-1}\partial_k = 0$.

Following P269 of Graham, for $a \in R_k^N$, we define the action of $q \in Q$ on a, denoted as $q.a$, by $\partial_k(q.a) := q.(\partial_k(a))$. Concretely: we should have $\partial_1(q.e_1^i) := q.(\partial_1(e_1^i)) = q.(T_i - 1) = q.T_i - 1$. Generally we must have $q.e_1^i = t_{i1}e_1^1 + t_{i2}e_1^2 + t_{i3}e_1^3$ for some $t_{ij} \in T$. Note that $\partial_1(t_1e_1^1 + t_2e_1^2 + t_3e_1^3) = t_1(T_1 - 1) + t_2(T_2 - 1) + t_3(T_3 - 1)$. In order for it to agree with $q.T_i - 1 \equiv T_1^{r_1}T_2^{r_2}T_3^{r_3} - 1$, we set $t_3 = 1$, so we'd like to have $t_1(T_1 - 1) + t_2(T_2 - 1) + T_3 = T_1^{r_1}T_2^{r_2}T_3^{r_3}$. Note that for the specific case of No. 216 we always have $r_{1,2,3} \in \{0, +1, -1\}$, so we can always find $t_1, t_2, t_3 \in T$ that solves the above equation. Then, after finding $q.e_1^i$, we'd like to find $q.e_2^i$ by using $\partial_2(q.e_2^i) = q.(\partial_2(e_2^i)) = (q.T_{i+1} - 1)q.e_1^{i-1} - (q.T_{i-1} - 1)q.e_1^{i+1}$, where we have used the fact (P269) that $q.(na) = (q.n)(q.a)$. We have seen that we always have $q.e_1^i = t_{i1}e_1^1 + t_{i2}e_1^2 + t_{i3}e_1^3$ where each $t_{ij} = T_1^{b_{ij1}}T_2^{b_{ij2}}T_3^{b_{ij3}}$ defines some $b_{ijk} \in \{0, \pm 1\}$. Again write the general form $q.e_2^i = \tau_{i1}e_2^1 + \tau_{i2}e_2^2 + \tau_{i3}e_2^3$, Then we have $\partial_2(q.e_2^i) = \tau_{i1}\partial_2(e_2^1) + \tau_{i2}\partial_2(e_2^2) + \tau_{i3}\partial_2(e_2^3) = \tau_{i1}(T_2 - 1)e_1^3 - \tau_{i1}(T_3 - 1)e_1^2 + \tau_{i2}(T_3 - 1)e_1^1 - \tau_{i2}(T_1 - 1)e_1^3 + \tau_{i3}(T_1 - 1)e_1^2 - \tau_{i3}(T_2 - 1)e_1^1$, which must be equal to $(q.T_{i+1} - 1)(t_{(i-1)1}e_1^1 + t_{(i-1)2}e_1^2 + t_{(i-1)3}e_1^3) - (q.T_{i-1} - 1)(t_{(i+1)1}e_1^1 + t_{(i+1)2}e_1^2 + t_{(i+1)3}e_1^3)$. Therefore we have

$$\tau_{i2}T_3 - \tau_{i2} - \tau_{i3}T_2 + \tau_{i3} = (q.T_{i+1} - 1)t_{(i-1)1} - (q.T_{i-1} - 1)t_{(i+1)1}, \tag{235a}$$
$$\tau_{i3}T_1 - \tau_{i3} - \tau_{i1}T_3 + \tau_{i1} = (q.T_{i+1} - 1)t_{(i-1)2} - (q.T_{i-1} - 1)t_{(i+1)2}, \tag{235b}$$
$$\tau_{i1}T_2 - \tau_{i1} - \tau_{i2}T_1 + \tau_{i2} = (q.T_{i+1} - 1)t_{(i-1)3} - (q.T_{i-1} - 1)t_{(i+1)3}, \tag{235c}$$

from which we can solve for $\tau_{ij} \in T$. Finally, we set $q.e_3 = \Delta e_3$, and we must have $\partial_3(q.e_3) := q.(\partial_3(e_3))$: the former equals $\partial_3(\Delta e_3) = \Delta(T_1 - 1)e_2^1 + \Delta(T_2 - 1)e_2^2 + \Delta(T_3 - 1)e_2^3$, whereas the latter equals $(q.T_1 - 1)q.e_2^1 + (q.T_2 - 1)q.e_2^2 + (q.T_3 - 1)q.e_2^3 = (q.T_1 - 1)(\tau_{11}e_2^1 + \tau_{12}e_2^2 + \tau_{13}e_2^3) + (q.T_2 - 1)(\tau_{21}e_2^1 + \tau_{22}e_2^2 + \tau_{23}e_2^3) + (q.T_3 - 1)(\tau_{31}e_2^1 + \tau_{32}e_2^2 + \tau_{33}e_2^3)$. They should be equal, so

$$\Delta(T_1 - 1) = (q.T_1 - 1)\tau_{11} + (q.T_2 - 1)\tau_{21} + (q.T_3 - 1)\tau_{31}, \tag{236a}$$
$$\Delta(T_2 - 1) = (q.T_1 - 1)\tau_{12} + (q.T_2 - 1)\tau_{22} + (q.T_3 - 1)\tau_{32}, \tag{236b}$$
$$\Delta(T_3 - 1) = (q.T_1 - 1)\tau_{13} + (q.T_2 - 1)\tau_{23} + (q.T_3 - 1)\tau_{33}, \tag{236c}$$

A Preliminary homological algebra

Rotman Corollary 9.55: If G is a free group then $H_n(G, A) = \{0\} = H^n(G, A)$ for all $n \geq 2$ and all G-modules A.

Proof: The sequence $0 \to \mathcal{G} \xrightarrow{\epsilon} \mathbb{Z}G \to \mathbb{Z} \to 0$ is a G-free resolution of $\mathbb{Z}$, see Prop 9.23 pn P520: here $\mathbb{Z}G$ is the integral group ring whose elements are $\sum_{x \in G} m_x x$, where $x \in G$ and $m_x \in \mathbb{Z}$. The map ϵ, defined by $\epsilon \colon \sum m_x x \mapsto \sum m_x$, is called the augmentation, and the augmentation ideal $\mathcal{G}$ is defined by $\mathcal{G} \equiv \ker\epsilon$, so that the above sequence is exact. Note by definition, $\ker\epsilon = \mathcal{G}$ is a two-sided idea in $\mathbb{Z}G$. [About ideal: the canonical example is $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$.] Then we have to explain what is a G-free resolution. We have to start with the definition of a projective resolution: a projective resolution of $A \in \text{obj}(\mathcal{A})$, where $\mathcal{A}$ is an abelian category, is an exact sequence

$$\mathbf{P} = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

in which each P_n is projective. If $\mathcal{A}$ is ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$, then a free resolution of a module A is a projective resolution in which each P_n is free.

We introduce free module: in some sense this is the most fundamental concept in algebra, and we can say the whole idea of homological algebra is built on it. See Page xiv of Rotman for an excellent introduction. In short, a free module is a module that has a basis (see wikipedia). The canonical example:

- a free abelian group is precisely a free module over $\mathbb{Z}$;
- $\mathbb{Z}G$ is free abelian group with basis $\{x \in G\}$.

Other examples: for any integer $n > 0$, R^n, the cartesian product of n copies of R as a left R-module, is free; a projective module over a local ring is free [Kaplansky].

A G-free resolution of $\mathbb{Z}$ means it is a free resolution with well-defined G action on each places (i.e. each free module in the free resolution is also a G-module).

The following two cases, we can easily write down a G-free resolution of $\mathbb{Z}$:

- When G is a finite cyclic group [Lemma 9.26 of Rotman]: if $G = \langle x \rangle$ is a finite cyclic group of order k, then define D and N of $\mathbb{Z}G$ by $D = x - 1$ and $N = 1 + x + \cdots + x^{k-1}$, then we have a G-free resolution of $\mathbb{Z}$
$$\cdots \mathbb{Z}G \xrightarrow{D} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{D} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{237}$$
where the maps alternate beling multiplication by D and multiplication by N.

- When G is a free group [Corollary 9.55 of Rotman]: Then we have the Gruenberg resolution: the sequence
$$0 \to \mathcal{G} \to \mathbb{Z}G \to \mathbb{Z} \to 0 \tag{238}$$
is a G-free resolution of $\mathbb{Z}$. To show this one just have to show that $\mathcal{G}$ is a free G-module. This is exactly Prop 9.54 of Rotman: if G is a free group with basis X, then its augmentation ideal $\mathcal{G}$ is a free G-module with basis $X - 1 = \{x - 1 \colon x \in X\}$.

- For any group G [Gruenberg, Theorem 9.59 of Rotman], if $1 \to R \to F \to G \to 1$ is an exact sequence of groups, where F is free with basis X and R is free with basis Y, then there is a G-free resolution of $\mathbb{Z}$: $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$, see Rotman's P544 for the definition of the maps d_k.

In the above, the free-resolutions of the finite cyclic group and free group are provided (Eq. (237) and (238)). A resolution is arguably the most important thing in homological algebra. Writing down a (projective) resolution means every cohomology group can be computed.

The cohomology groups $H^n(G, A)$ are obtained by applying the functor $\mathrm{Hom}(\square, A)$ to the bar resolution $\mathbf{B}(G)$, obtaining $H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$., where $\mathbb{Z}$ is viewed as a trivial G-module [P520 of Rotman]. The homology groups are obtained by applying $\square \otimes_G A$ to $\mathbf{B}(G)$ [see Ex. 9.3 of Rotman to see the tensor is well-defined], obtaining $H_n(G, A) = \mathrm{Tor}^{\mathbb{Z}G}_n(\mathbb{Z}, A)$, where $\mathbb{Z}$ is viewed a a trivial G-module.

Recall that $\mathrm{Ext}(\mathbb{Z}, \square)$ is computed with a G-projective resolution of $\mathbb{Z}$. The $\mathrm{Ext}(\mathbb{Z}, \square)$ are the right derived functors of $\mathrm{Fix}^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \square)$.

In the above we have introduced that a projective resolution for an object A in the abelian category is an exact sequence in which each place P_n is projective. The definition of projective module [P99 of Rotman]: A left R-module P is projective if, whenever p is surjective and h is any map, ∃ a lifting g. I.e. ∃ a map g making the diagram commute:

$$\begin{tikzcd}
& P \arrow[dl, dashed, "g"] \arrow[d, "h"] & \\
A \arrow[r, "p"'] & A'' \arrow[r] & 0.
\end{tikzcd}$$

The covariant Hom functor $\mathrm{Hom}_R(P, \square)$ are left exact [Thm 2.38 of Rotman], that is, for any module P, applying $\mathrm{Hom}_R(P, \square)$ to an exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{p} A''$ gives an exact sequence $0 \to \mathrm{Hom}_R(P, A') \xrightarrow{i_*} \mathrm{Hom}_R(P, A) \xrightarrow{p_*} \mathrm{Hom}_R(P, A'')$. The essence of projective module is the following result [Prop. 3.2 of Rotman]: P is a left R-module, then
$$P \text{ is projective} \Leftrightarrow \mathrm{Hom}_R(P, \square) \text{ is an exact functor.} \tag{239}$$

One then wonders what would be the statement for the other, contravariant, Hom functor, $\mathrm{Hom}_R(\square, E)$. Recall that $\mathrm{Hom}_R(\square, E)$ is also left exact, but in the contravariant sense: applying $\mathrm{Hom}_R(\square, E)$ to an exact sequence $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ gives an exact sequence $0 \to \mathrm{Hom}_R(C, E) \xrightarrow{p^*} \mathrm{Hom}_R(B, E) \xrightarrow{i^*} \mathrm{Hom}_R(A, E)$.

A left R-module E is injective if, whenever i is an injection, there ∃ a dashed arrow g making the diagram commute:

$$\begin{tikzcd}
& E & \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B. \arrow[ul, dashed, "g"']
\end{tikzcd}$$

We have: if E is a left R-module, then
$$E \text{ is injective} \Leftrightarrow \mathrm{Hom}_R(\square, E) \text{ is an exact functor.} \tag{240}$$

Note in the above statements about Hom and modules we assumes R is commutative. This because for a generic R, $\mathrm{Hom}_R(A, \Box)$ or $\mathrm{Hom}_R(\Box, B)$ are only functors ${}_R\mathbf{Mod} \to \mathbf{Ab}$, and ${}_R\mathbf{Mod} \to {}_{Z(R)}\mathbf{Mod}$, and become functors of ${}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ when R is commutative.

Graham's book gives many practical ways to understand the somewhat abstract concepts in homological algebra. See below.

Note that for a group G, so we have the group ring $\mathbb{Z}G$, a free $\mathbb{Z}G$-resolution is a chain complex

$$R_*^G\colon \quad \cdots \xrightarrow{\partial_{k+1}} R_k^G \xrightarrow{\partial_k} R_{k-1}^G \xrightarrow{\partial_{k-1}} R_{k-2}^G \xrightarrow{\partial_{k-2}} \cdots \xrightarrow{\partial_2} R_1^G \xrightarrow{\partial_1} R_0^G \tag{241}$$

that is exact except for the last place, each R_k^G is a free $\mathbb{Z}G$-module, and that the last palce $R_0^G \cong \mathbb{Z}G$ in the module sense. More concretely, if we further append $\mathbb{Z}$ to the right then we get the following exact sequence

$$\cdots \xrightarrow{\partial_{k+1}} R_k^G \xrightarrow{\partial_k} R_{k-1}^G \xrightarrow{\partial_{k-1}} R_{k-2}^G \xrightarrow{\partial_{k-2}} \cdots \xrightarrow{\partial_2} R_1^G \xrightarrow{\partial_1} R_0^G \xrightarrow{\partial_0} \mathbb{Z} \to 0, \tag{242}$$

where ∂_0 is defined by $\sum n_i g_i \mapsto \sum n_i$. Nte that each R_k^G is a free $\mathbb{Z}G$-module means that it consists of several copies of the ring $\mathbb{Z}G$, labeled by a set of basis (this is the structural theorem for free R-module: a free R-module is direct product of R, labeled by a basis of the module). Therefore for a free $\mathbb{Z}G$-resolution R_*^G, the complete data is the rank of each R_k^G (i.e. the number n_k of copies of $\mathbb{Z}G$ as in $R_k^G \cong \oplus_{n_k}\mathbb{Z}G$) and the differentials ∂_k. Note that the homology of R_*^G is $H_k(R_*^G) = \mathrm{ker}\partial_k/\mathrm{im}\partial_{k+1} = 0$ for $k \geq 1$. For $k = 0$, we have $H_0(R_*^G) = R_0^G/\mathrm{im}\partial_1$, note that from the exact sequence (242) above we have $\mathrm{im}\partial_0 = \mathbb{Z}$, and $\mathrm{im}\partial_1 = \mathrm{ker}\partial_0 = R_0^G/\mathrm{im}\partial_0 = R_0^G/\mathbb{Z}$, so that $H_0(R_*^G) = \mathbb{Z}$. Therefore we see that R_*^G (appending $\to 0$ on the far right) has homology $H_k(R_*^H) = 0$ for $k \geq 1$ and $\mathbb{Z}$ for $k = 0$.

Note that, according to the correct definition, R_*^G is called the deleted free resolution while (242) is the free resolution (a resolution is, by definition, an exact sequence). But we do not lose information when deleting the $\mathbb{Z}$.

It is important to have a good feeling of the free resolution for the following groups:

- finite abelian group: all free resolutions are infinite.
- free abelian group: free resolutions are finite, length depends on the rank of the free abelian group (i.e. the number of copies of $\mathbb{Z}$).
- symmetric group;
- point group;
- crystallographic group.

"Since P is finite, any resolution for P will have infinite length and thus so too will the resolution for G", see Graham's comment on `https://github.com/gap-packages/hap/issues/69`.

From Rotman, P518: if Q is a group and K a Q-module, then we have

$$H^1(Q, K) = Z^1(Q, K)/B^1(Q, K) = \mathrm{Der}(Q, K)/\mathrm{PDer}(Q, K) = \mathrm{Stab}(Q, K)/\mathrm{Inn}(Q, K), \tag{243}$$

where $\mathrm{PDer}(Q, K) = \{d_0\colon Q \to K | d_0(x) = x.a_0 - a_0 \text{ for some } a_0 \in K\}$, $\mathrm{Der}(Q, K) = \{d\colon Q \to K | d(1) = 0, d(xy) = x.d(y) + d(x) \forall x, y \in Q\}$.

Similarly,

$$H^2(Q, K) = Z^2(Q, K)/B^2(Q, K), \tag{244}$$

where $Z^2(Q, K) = \{f\colon Q \times Q \to K | f(1, y) = 0 = f(x, 1), f(x, y) + f(xy, z) = x.f(y, z) + f(x, yz), \forall x, y, z \in Q\}$, called the factor set (set of cocycles), is all the function $f\colon Q \times Q \to K$ that satisfies the above criterion. Of course it is also an Abelian group under addition $f + f'$ for $f, f' \in Z(Q, K)$. Then, $B^2(Q, K) = \{g\colon Q \times Q \to K | \exists h\colon Q \to K, h(1) = 0, s.t. g(x, y) = x.h(y) - h(xy) + h(x) \forall x, y \in Q\}$, here $g\colon Q \times Q \to K$ and $h\colon Q \to K$ are both functions, and the above condition is the coboundary condition.

Now we know that the starting point of computing cohomology is to build a (quite often, free) resolution for the group. This is precisely what Rotman's 9.3 is about. The notation $[x_1|x_2|...|x_n]$ is simply an element of G^n, which is a basis of $R_n^G \subset (\mathbb{Z}G)^n$. This gives a free resolution of $\mathbb{Z}$ with $\mathbb{Z}G$ action, and the map is the differential map defined in the usual way. Then, we know that we have to apply the functor $\mathrm{Hom}(\Box, K)$ (where K is the G-module) to the resolution, and the element $\mathrm{Hom}(R_n^G, K)$ coincides when $n = 1, 2$ with the factor set elements and derivation set elements defined above. It is important to note that, in the above, it is required that $f(1, y) = f(x, 1) = 0$ for $f \in Z^2(Q, K)$, and $d(1) = 0$ for $d \in Z^1(Q, K) = \mathrm{Der}(Q, K)$. The corresponding resolution that automatically gives this condition is called the normalized bar resolution [Rotman, P543]. On P543 it is mentioned that $[x_1|\cdots|x_n]^\star = 0$ if some $x_i = 1$, meaning that after taking Hom to get an element of $f\colon G^n \to K$, f must satisfy $f(x_1, ..., x_n) = 0$ if some $x_i = 1$.

With this in mind, let us examine the situation when $G = \mathbb{Z}/2\mathbb{Z} \equiv \mathbb{Z}_2$, namely we want to compute $H^*(\mathbb{Z}_2, A)$ for some module A. Turns out it is simple. Call the generator of $G = \mathbb{Z}_2$ x, i.e. $G = \{x, e\}$ with $e = x^2$. According to the above, when computing $H^n(G,A) = Z^n(G,A)/B^n(G,A)$ with $G = \mathbb{Z}_2$, the only data we need about a $f \in Z^n(G,A)$ is $f(x,x,...,x)$, since $f = 0$ if any of its arguments is the identity e. We know that $f \in Z^n(G,A)$ satisfies

$$0 = x_1.f(x_2,...,x_{n+1}) + \sum_{i=1}^{n-1}(-1)^i f(x_1,...,x_{i-1},x_ix_{i+1},x_{i+2},...,x_{n+1}) + (-1)^n f(x_1,...,x_{n-1},x_nx_{n+1}) + (-1)^{n+1} f(x_1,...,x_n),$$

setting all $x_{1,...,n+1} = x$, then we see that all the terms that contain the the product of two x's vanish, and we are only left with the first and the last term $x.f(x,...,x) = (-1)^{n+1} f(x,...,x)$, if we further have that $K = \mathbb{Z}_2^m$ for some m, i.e. K is m copies of $\mathbb{Z}_2$, which is always the case for the places in a spectral sequence when we calculate the mod-2 cohomology, the $(-1)^{n+1}$ does not matter, so we have $x.f(x,...,x) = f(x,...,x)$, i.e. x must have trivial action on K. In this case, we identify the element f of $Z^n(G = \mathbb{Z}_2, K = \mathbb{Z}_2^m)$ with its value $f(x,...,x)$, which is further identified with an element in K that is stabilized by x, namely

$$Z^n(G = \mathbb{Z}_2, K = \mathbb{Z}_2^m) = K^G,$$

furthermore, since an element $g \in B^n(G,A)$ satisfies

$$g(x_1,...,x_n) = x_1.h(x_2,...,x_n) + \sum_{i=1}^{n-2}(-1)^i h(x_1,...,x_{i-1},x_ix_{i+1},x_{i+2},...,x_n) + (-1)^{n-1} h(x_1,...,x_{n-2},x_{n-1}x_n) + (-1)^n h(x_1,...,x_{n-1})$$

$$\text{for some function } h\colon G^{n-1} \to K \text{ s.t. } h = 0 \text{ of any of its arguments is identity,}$$

and for $G = \mathbb{Z}_2$ this is drastically simplified

$$g(x,...,x) = (x + (-1)^n).h(x,...,x),$$

so really, any $g \in B^n(G,A)$ is identified with $g(x,...,x)$, and is further identified with the image of the action $x - 1$ on the the element of $h(x,...,x) \in K$, and is exactly $x.K - K$. So, in summary, we have

$$\text{For } G = \mathbb{Z}_2, K = \mathbb{Z}_2^m \text{ with possiblly nontrivial action, } H^{n\geq 1}(G = \mathbb{Z}_2, K = \mathbb{Z}_2^m) = K^G/(x.K - K), \tag{245}$$

In the actually calculation, the detailed contain of $\{x.K - K\}$ is important: any element in $x.K - K$ produces a relation to the ring element (see the concrete examples for an understanding).

About the term

$$H^1(N,A)^{G/N},$$

see `https://mathoverflow.net/questions/212636/the-term-h1n-ag-n-in-the-inflation-restriction-exact-sequenc`

Let's consider the group cohomology for a finite cyclic group with order k, and for an infinite cyclic group. First, consider the finite cyclide group, $G = \langle x \rangle$. This is given in P521-522 of Rotman, so let's repeat it. The complete information needed is summarized in the following diagram

$$\begin{tikzcd}
{} \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "D"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z}G \arrow[r, "N"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z}G \arrow[r, "D"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Hom}"] & 0 \\
{} & A \arrow[l, "d^4=N^*"'] & A \arrow[l, "d^3=D^*"'] & A \arrow[l, "d^2=N^*"'] & A \arrow[l, "d^1=D^*"'] & A \arrow[l, "d^0=\epsilon^*"'] &
\end{tikzcd} \tag{246}$$

where D is multiplying by $x - 1$ and N is multiplying by $1 + x + \cdots + x^{k-1}$; the first line is the projective (free) resolution for $\mathbb{Z}$; and for the last line we should identify all the A appearing at the bottom. We have $\mathrm{ker} d^{2n-1} = \mathrm{ker} D^* = \{g\colon \mathbb{Z}G \to A | g(x^m) = f(D(x^m)) = 0 \text{ for } \forall m \text{ and } f\colon \mathbb{Z}G \to A\} = \{g\colon \mathbb{Z}G \to A | g(x^m) = f(D(x^m)), f(x^{m+1}) = f(x^m) \text{ for } f\colon \mathbb{Z}G \to A\} = \{g\colon \mathbb{Z}G \to A | g(x) = f(D(x)), f(x) = a, x.a = x\} \cong \{a \in A | x.a = a\} = A^G$. Similarly, we have $\mathrm{ker} N^* = {}_NA = \{a \in A\colon Na = 0\}$, and $\mathrm{im} N^* = NA$ and $\mathrm{im} D^* = DA$. Then we have $H^0(G,A) = A^G$, $H^{2n-1}(G,A) = {}_NA/DA$, and $H^{2n}(G,A) = A^G/NA$.

For infinite cyclic group, things are simplified, and we have

$$\begin{tikzcd}
0 \arrow[r, "N=0"] & \mathbb{Z}G \arrow[r, "D"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "\mathrm{Hom}"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Hom}"] & 0 \\
{} & A \arrow[l, "d^2=N^*=0"'] & A \arrow[l, "d^1=D^*"'] & A \arrow[l, "d^0=\epsilon^*"'] &
\end{tikzcd} \tag{247}$$

We see that from above we simply have $NA = 0$, ${}_NA = A$, and $d^{n\geq 2} = 0$, so we have $H^0(G,A) = A^G$, $H^1(G,A) = A/DA$, and $H^{n\geq 2}(G,A) = 0$.