RMT and path integral approaches to the SYK model and its entanglement entropy

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Outline

1. SYK as a RMT model
2. SYK from path integral
3. Entanglement entropy of SYK from RMT
4. Entanglement entropy from path integral

SYK as a RMT model

Sachdev-Ye-Kitaev (SYK) Models

- Let $2 \leq q \leq N$, q and N both even. The SYK model is defined by the following Hamiltonian (Hermitian operator) on a $2^{N/2}$ Hilbert space ($\left(\mathbb{C}^2\right)^{\otimes N/2}$ vector space)

$$H_q = i^{q/2} \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq N} J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_q, \tag{1}$$

$J_{i_1 i_2 \ldots i_q}$ are real i.i.d. Gaussian random variables with

$$\overline{J_{i_1 i_2 \ldots i_q}} = 0, \qquad \overline{J^2_{i_1 i_2 \ldots i_q}} = \frac{(q-1)! J^2}{N^{q-1}}. \tag{2}$$

- Here χ_i are Majorana fermions that obey the Clifford algebra

$$\{\chi_i, \chi_j\} \equiv \chi_i \chi_j + \chi_j \chi_i = 2\delta_{ij}, \tag{3}$$

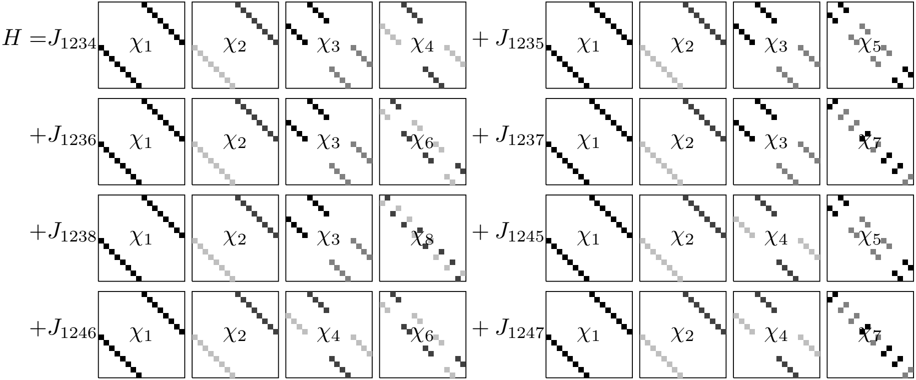
mathematically, they are representation of the Clifford algebra: define Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (the Jordan-Wigner transformation)

$$\begin{aligned} \chi_1 &= \sigma^1 \otimes \sigma^0 \otimes \cdots \otimes \sigma^0, & \chi_2 &= \sigma^2 \otimes \sigma^0 \otimes \cdots \otimes \sigma^0, \\ \chi_3 &= \sigma^3 \otimes \sigma^1 \otimes \cdots \otimes \sigma^0, & \chi_4 &= \sigma^3 \otimes \sigma^2 \otimes \cdots \otimes \sigma^0, \\ &\cdots & &\cdots \\ \chi_{N-1} &= \sigma^3 \otimes \sigma^3 \otimes \cdots \otimes \sigma^1, & \chi_N &= \sigma^3 \otimes \sigma^3 \otimes \cdots \otimes \sigma^2. \end{aligned} \tag{4}$$

Exmaple: $\mathrm{SYK}_{q=4}$, $N = 8$

$$H_{q=4} = \sum_{1\leq i<j<k<l\leq N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l \tag{5}$$

Matrix size= $2^{N/2}$-by-$2^{N/2}$ = 16-by-16; Number of terms = $\binom{N}{q} = \binom{8}{4} = 70$.



$+$ the rest 62 terms

(6)

Sachdev-Ye-Kitaev Models $q = 2$ and $q = 4$

$$\begin{aligned} H_{q=2} &= \sum_{i,j=1}^{N} iJ_{ij}\chi_i\chi_j = \boldsymbol{\chi}^T \mathbf{J} \boldsymbol{\chi}, \\ H_{q=4} &= \sum_{1\le i<j<k<l\le N} J_{ijkl}\,\chi_i\chi_j\chi_k\chi_l. \end{aligned} \tag{7}$$

- $q = 2$:
  - Diagonalization of $\mathbf{J}$ (an N-by$N$ matrix):

    $$\mathcal{E} = \pm\{\epsilon_1, ..., \epsilon_{N/2}\} \tag{8}$$

    this is the excitation energy for *quasiparticles*. – Semi-circle.
  - Diagonalization of $H_{q=2}$ (a $2^{N/2}$-by-$2^{N/2}$ matrix):

    $$E = \{\text{sum}(\mathcal{E}_i) \text{ for } \mathcal{E}_i \text{ in } 2^{\mathcal{E}}\} \tag{9}$$

    this is the many-body energy levels. – Gaussian.
  - This is a usual metal. – *Fermi liquid*.
- $q \geq 4$:
  - No quasiparticles
  - Manybody energies: how does the spectrum look like?
  - describes a strongly correlated metal. – *non-Fermi liquid*.

Spectrum of SYK, result

- Complete results: (almost sure convergence)
  - $q^2/N \to 0$: Gaussian distribution
  - $q^2/N \to$ finite: $\eta = e^{-2q^2/N}$, $E \in [-\frac{2}{\sqrt{1-\eta}}, \frac{2}{\sqrt{1-\eta}}]$,

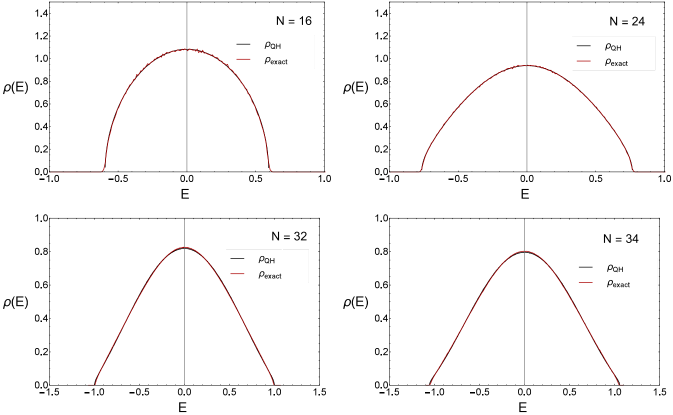
$$\rho_{N\to\infty}(E) = c_N \sqrt{1 - \frac{E^2}{E_0^2}} \prod_{p=1}^{\infty} \left[1 - 4\frac{E^2}{E_0^2} \frac{1}{2 + \eta^p + \eta^{-p}}\right]. \tag{10}$$

  - $q^2/N \to \infty$: the semicircle law

$$\rho_{N\to\infty}(E) = \frac{1}{2\pi}\sqrt{4 - E^2}. \tag{11}$$

- kth moment $\overline{J^k_{i_1 i_2,\ldots,q_n}}$ for $k > 2$ can be nonzero but uniformly bounded: result is the same.

Cotler et al. 2017; **García-García and Verbaarschot, 2017**; **Feng, Tian and Wei, 2019**; **Parisi 1994**; **Erdős, Schröder, 2014**



Asymptotic expression for bulk and edge at large N:

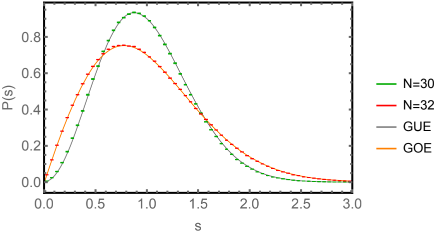
$$\rho_{\text{Asymp., bulk}} \propto \exp\left[\frac{2\arcsin^2(E/E_0)}{\log\eta}\right], \tag{12}$$

$$\rho_{\text{Asymp., edge}} \propto \exp\left[\frac{\pi^2}{2\log\eta}\right]\sinh\left[\frac{2\pi\sqrt{2}\sqrt{1-E/E_0}}{-\log\eta}\right]. \tag{13}$$

García-García and Verbaarschot, 2017

Quantum Chaos: level statistics of SYK

- Quantum Chaos = study of energy level statistics (fine grained structure)
- Quantum chaotic = (many-body) level repulsion
- SYK spectrum exhibits level repulsion factor.
- The eight-fold way: particle-hole symmetry of SYK determines that the statistics are that of GUE ($N \bmod 8 = 2$ or 6), GOE ($N \bmod 8 = 0$), and GSE ($N \bmod 8 = 4$).



You, Ludwig, Xu, 2017; **Cotler et al. 2017**

Spectral form factor

Partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$ and its analytical continuation

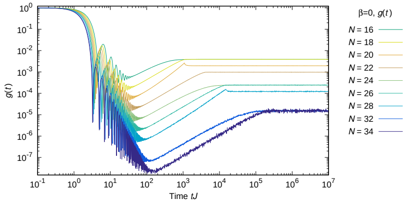
$$Z(\beta, t) \equiv \mathrm{Tr}(e^{-(\beta+it)H}) \tag{14}$$

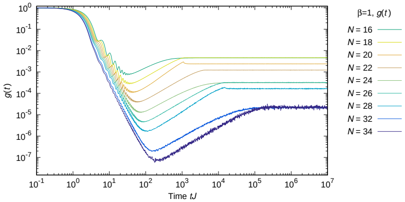
The spectral form factor

$$g(t, \beta) \equiv \langle |Z(\beta, t)|^2 \rangle \tag{15}$$

probes the discreteness of the spectrum at late times.

- Spectral form factor exhibits late time ramp and plateau.
- The eight-fold way also reflected in the ensemble:
  - In GUE the ramp and plateau connect at a sharp corner
  - in GOE they connect smoothly
  - in GSE they connect at a kink

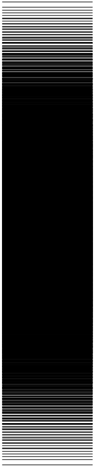




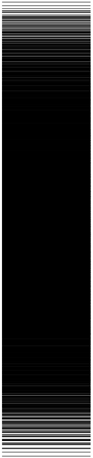
Understanding of ramp and plateau **Cotler et al. 2017**

Summarize: SYK as a RMT model

- SYK model is a certain type of sparse random matrix with correlated randomness in the entries.
- Possesses level repulsion; bulk spectrum Gaussian; edge spectrum rescaled.
- Physical interest lies in the edge properties (the "low energy" sector)
  - *Fermi liquid vs non-Fermi liquid*: $1/N$ vs e^{-Ns_0} at spectrum edge. (quasiparticle vs intrisic many-body excitation)
  - Emergent (0+1d) conformal symmetry; reparameterization mode described by Schwarzian action; characteristic of black holes in Einstein gravity in 2D



SYK_2 Fermi-liquid



SYK_4 (non-Fermi-liquid)

Cotler et al. 2017; **Talks by Sachdev; Maldacena, Stanford, 2016; Rosenhaus, 2019.**

SYK from path integral

Coherent state path integral for fermions

Creation and annihilation operators:

- Define $a_j = \frac{\chi_{2j}+i\chi_{2j+1}}{\sqrt{2}}, a_j^\dagger = \frac{\chi_{2j}-i\chi_{2j+1}}{\sqrt{2}}$
- Hibert space is $(\mathbb{C}^2)^{\otimes N/2}$ vector space, with basis of the form(the occupation basis)

$$|1010001...\rangle = |1\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 \otimes |0\rangle_4 \otimes |0\rangle_5 \otimes \cdots \tag{16}$$

- Action of a_j and $a_j^\dagger$ is to flip the occupation number:

$$\boxed{a_j|...1_j...\rangle = |...0_j...\rangle, \quad a_j^\dagger|...0_j...\rangle = |...1_j...\rangle} \tag{17}$$

  this completely determines the operators a_j and $a_j^\dagger$.

The (right) eigenvector of the annihilation operator a_j is called the *coherent state*.

- Only an annihilation operator has right-eigensolution.
- Eigenspectrum is continuous, but Grassmann number!

$$a|\eta\rangle = \eta|\eta\rangle. \tag{18}$$

  Concretely, $|\eta\rangle = e^{-\eta a^\dagger}|0\rangle = |0\rangle - \eta|1\rangle$. And we have $\langle\bar{\eta}|a^\dagger = \langle\bar{\eta}|\bar{\eta}$.

Coherent state path integral for fermions: derivation, cont'd

- Grassmann calculus:

$$\int d\eta = 0, \quad \int d\eta\eta = 1, \quad \langle\bar{\eta}|\eta\rangle = e^{\bar{\eta}\eta} \tag{19}$$

- Coherent state relation:

$$1 = \int d\bar{\eta}d\eta\, e^{-\bar{\eta}\eta}|\eta\rangle\langle\bar{\eta}|, \qquad \mathrm{Tr}[\mathcal{O}] = \int d\bar{\eta}d\eta\, e^{-\bar{\eta}\eta}\langle-\bar{\eta}|\mathcal{O}|\eta\rangle. \tag{20}$$

Example: a two-Majorana fermion Hailtonian $H[\chi_1, \chi_2]$:

$$\boxed{\mathrm{Tr}e^{-\beta H[\chi_1,\chi_2]} = \int D\psi_1 D\psi_2 e^{-\int_0^\beta d\tau\left(\frac{1}{2}\sum_{i=1}^{2}\psi_i\partial_\tau\psi_i + H[\psi_1,\psi_2]\right)}} \tag{21}$$

- RHS: functional integral, where $\psi_i = \psi_i(\tau)$ are Grassmann valued functions
- Boundary condition fixed by the trace: $\psi_i(\beta^-) = -\psi_i(0^+)$
- The term $\psi_i\partial_\tau\psi_i$ reflects quantum fluctuation (operator non-commutativity) and is intrinsic to a quantum system and universal!

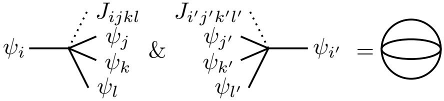
Path integral for SYK

Goal: conduct disorder average and obtain expression in terms of bosonic fields.

$$Z(\{J_{i_1\ldots i_q}\}) = \mathrm{Tr} e^{-\beta H_q} = \int [D\psi_i] e^{-\int_0^\beta d\tau \left(\frac{1}{2}\sum_i \psi_i \partial_\tau \psi_i + H_q[\{\psi_i\}]\right)} \tag{22}$$

- The disorder average $\overline{Z} = \langle Z(\{J_{i_1\ldots i_q}\})\rangle_J$:

Example at $q = 4$:



after which flavor indexes are decoupled

$$\sum_{i<j<k<l} \overline{|J_{ijkl}|^2} \psi_i(\tau)\psi_i(\tau') \cdots \psi_l(\tau)\psi_l(\tau') = \frac{J^2}{4N^3}\left(\sum_i \psi_i(\tau)\psi_i(\tau')\right)^4 \tag{23}$$

- Q: Possible to integrate out the ψ_i fields?
- A: Yes! at the cost of introducing other fields. Resolution of identity

$$1 = \int DGD\Sigma e^{-\frac{1}{2}\int d\tau d\tau' \Sigma(\tau,\tau')\left(NG - \sum_i \psi_i(\tau)\psi_i(\tau')\right)} \tag{24}$$

Large N saddle point for SYK

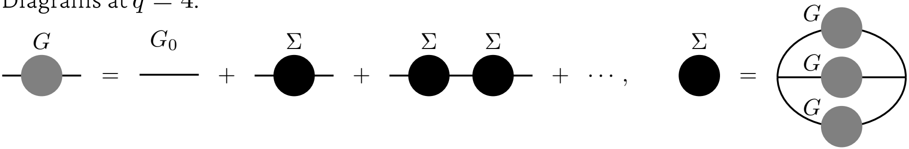
Finally: $\overline{Z} = \int DGD\Sigma e^{-NS_q}$,

$$\boxed{S_q = \frac{1}{2}\ln \mathrm{Det}(\partial_\tau\delta(\tau-\tau')-\Sigma) + \frac{1}{2}\int d\tau d\tau' \left(-\Sigma(\tau,\tau')G(\tau,\tau') + \frac{J^2}{q}G^q(\tau,\tau')\right).} \quad (25)$$

Saddle point equations $\partial S_q/\partial G = 0, \partial S_q/\partial\Sigma = 0$ give (the Schwinger-Dyson equation)

$$\boxed{\Sigma(\tau,\tau') = J^2 G^{q-1}(\tau,\tau'), \qquad G = \frac{1}{\partial_\tau\delta(\tau-\tau')-\Sigma}.} \quad (26)$$

Diagrams at $q = 4$:



SYK model is large-N exactly solvable, in the sense that the saddle point equation can be written in closed form, and solved exactly (numerically).

Results from path integral

Many (large N) properties can be conveniently studied from path integral:

- Thermodynamical properties (free energy, entropy, heat capacity)
- Correlation functions (two point, four point)
- The eigenvalue distribution function (i.e. density of states) is the inverse Laplace transform to Z:
$$\rho(E) = \frac{1}{2\pi i} \int_{\gamma + i\mathbb{R}} d\beta Z(\beta) e^{\beta E}. \tag{27}$$
- Emergent conformal invariance at low energy (near spectrum edge): this is the Schwarzian theory. One can obtain the edge spectrum from the Schwarzian theory (this is the Q-Hermite limit).

Maldacena, Stanford, 2016; **Stanford, Witten, 2017**; **Cotler et al. 2017**

Entanglement entropy of SYK from RMT

Subsystem entanglement entropy (see also Prof. Wei's talk)

Many-body Hamiltonian of the total system is H: $H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle, \alpha = 1, 2, ..., 2^M$.

- *Density matrix* of a pure state $|\psi_\alpha\rangle$:

$$\rho_{\text{pure state}} = |\psi_\alpha\rangle\langle\psi_\alpha|. \tag{28}$$

- *Density matrix* of a thermal ensemble: define $Z = \text{Tr}e^{-\beta H}$,

$$\rho = \frac{1}{Z}e^{-\beta H} = \frac{1}{Z}\sum_\alpha e^{-\beta E_\alpha}|\psi_\alpha\rangle\langle\psi_\alpha|. \tag{29}$$

  The ground state density matrix is obtained by $\rho_{\text{g.s.}} = \lim_{\beta\to\infty}\rho = |g.s.\rangle\langle g.s.|$.

A bipartite system $A \cup B$. Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.
Reduced density matrix for subsystem A:

$$\rho_A = \text{Tr}_B\rho, \tag{30}$$

- *Rényi entanglement entropy*:

$$S_A^{(n)} = \frac{1}{1-n}\ln \text{Tr}_A\rho_A^n. \tag{31}$$

- *von Neumann entanglement entropy*: (The "replica trick")

$$S_A = -\text{Tr}_A(\rho_A \ln \rho_A) = \lim_{n\to1} S_A^{(n)}. \tag{32}$$

Entanglement entropy for a non-interacting system

- Described by a quadratic Hamiltonian

$$H_{\text{free fermion}} = \sum_{i,j=1}^{N} J_{ij} a_i^\dagger a_j = i\boldsymbol{a}^\dagger \mathbf{J} \boldsymbol{a}, \tag{33}$$

  where the matrix $\mathbf{J}$ is Hermitian. Diagonalize $\mathbf{J}$ gives $\mathbf{J} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\dagger$, where $\mathbf{U}$ is unitary and $\boldsymbol{\Lambda}$ is diagonal.
- The two-point correlation is $C_{ij} = \langle a_i^\dagger a_j \rangle$. In matrix form: at finite (inverse) temperature β

$$\mathbf{C} = \mathbf{U}^* \frac{1}{1 + e^{\beta \boldsymbol{\Lambda}}} \mathbf{U}^T. \tag{34}$$

- Define subsystem A to be the first m fermions. Suppose k eigenvalues in $\boldsymbol{\Lambda}$ are negative. At zero temperature $\beta \to \infty$, the truncated two-point correlation for subystem A is

$$\mathbf{C}_A = \mathbf{V}^\dagger \mathbf{V}, \quad \mathbf{V} \text{ is the } (1:k; 1:m) \text{ truncation of } \boldsymbol{U}^T \tag{35}$$

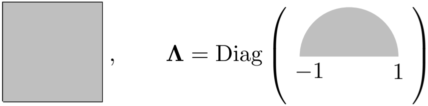
- Eigenvalues of $\mathbf{C}_A$ are $x_1, \ldots, x_m$. Then the von Neumann entropy:

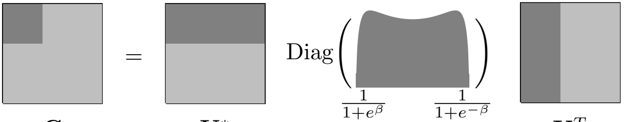
$$S_A \equiv -\sum_{\alpha=1}^{2^m} \rho_\alpha \ln \rho_\alpha = \sum_{j=1}^{m} \underbrace{\left(-x_j \ln x_j - (1-x_j)\ln(1-x_j)\right)}_{h(x_j):\ \text{binomial entropy}}. \tag{36}$$

Vidal, Latorre, Rico, Kitaev, 2003

Diagramatically, from $\mathbf{J} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\dagger$,

$\mathbf{U}^T =$, $\boldsymbol{\Lambda} = \mathrm{Diag}$ (37)



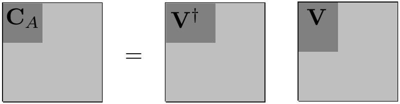


(38)

$$\mathbf{C} = \mathbf{U}^* \frac{1}{1+e^{\beta\boldsymbol{\Lambda}}} \mathbf{U}^T$$

$$\mathbf{C}_A = \mathbf{U}_A^* \frac{1}{1+e^{\beta\boldsymbol{\Lambda}}} \mathbf{U}_A^T$$

At zero temperature ($\beta = 1/T \to \infty$): suppose k out of N energies are negative, then



(39)

$\mathbf{C}_A$ belongs to the *Jacobi β-ensemble!* (The β here is unrelated to the β above)

Forrester, 2006; **CL, Chen, Balents, 2017**

Eigenstate entanglement entropy for SYK_2

$\mathbf{V}$ is a truncation of a Haar unitary matrix $\mathbf{U}$. The two-point correlation matrix $\mathbf{C}_A = \mathbf{V}^\dagger \mathbf{V}$ belongs to the *Jacobi ensemble*.

- The von Neumann entropy can be calculated:

$$S_A/N = \int \underbrace{(-x\ln x - (1-x)\ln(1-x))}_{h(x):\ \text{binomial entropy}} f(x)dx, \tag{40}$$

  $f(x)$ is the eigenvalue distribution of $\mathbf{C}_A$ (SV of $\mathbf{V}$).

- According to definition, disorder average has to be conducted at the very end:

$$S_A = -\overline{\mathrm{Tr}_A(\rho_A \ln \rho_A)} \tag{41}$$

- At leading order of N the result is the same as (disorder replica diagonal Ansatz)

$$S_A = -\mathrm{Tr}_A(\overline{\rho_A} \ln \overline{\rho_A}) \tag{42}$$

Forrester, 2006; **CL, Chen, Balents, 2017**

Limiting law of the Jacobi β-ensemble

The Wachter law **Wachter, 1980**

$$f(x,\kappa,\lambda) = \frac{1}{2\pi\lambda}\frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{x(1-x)}1_{[\lambda_-,\lambda_+]} \tag{43}$$

where

$$\lambda_\pm = (\sqrt{\kappa(1-\lambda)} \pm \sqrt{\lambda(1-\kappa)})^2 \tag{44}$$

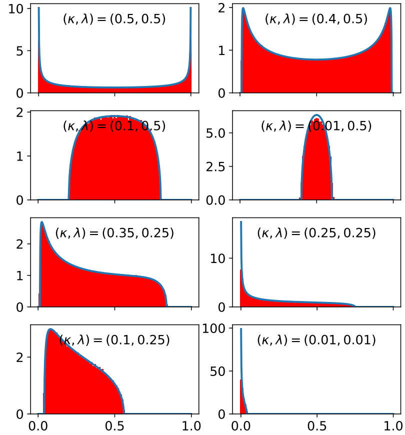
- m: size of the subsystem A;

$$\lambda = \frac{m}{N} \in (0, 1/2]; \tag{45}$$

- k: number of negative (i.e. filled) energy levels

$$\kappa = \frac{k}{N} \in (0, 1/2]. \tag{46}$$

physically, $\lambda < \kappa$.



The von Neumann entanglement entropy of SYK_2

- The von Neumann entanglement entropy for eigenstate of SYK_2 is

$$S_A/N = \int \underbrace{(-x \ln x - (1-x)\ln(1-x))}_{h(x)} f(x,\kappa,\lambda)dx, \tag{47}$$

- Integral can evaluated in closed form and get ($s = S_A/N$)

$$s(\kappa,\lambda) = h(\kappa)\lambda - (1-\lambda)\ln(1-\lambda) - \lambda \tag{48}$$

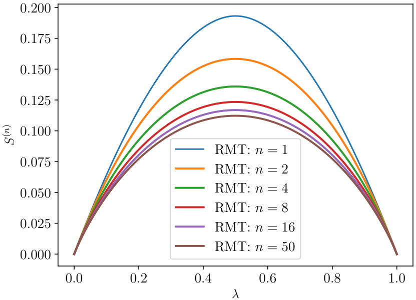
- $s(\lambda)$ has expansion

$$s = h(\kappa)\lambda - \lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n(n+1)}. \tag{49}$$

  This has also been obtained using moment method in a recent work, by deduction from evaluating the first few moments of the correlation matrix.

CL, Chen, Balents, 2017; **Łydżba et al., 2020**;

The Rényi entanglement entropy of SYK_2



- Similarly, the n-th Renyi entropy can be evaluated in closed form:

$$S^{(n)}/N = \int \ln(x^n + (1-x)^n) f(x, \kappa, \lambda) dx, \tag{50}$$

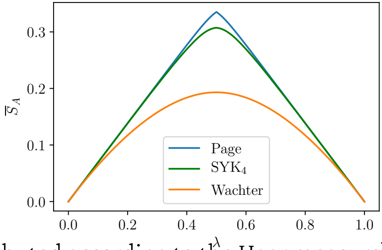
- As an example: for $\kappa = 1/2$, we have

$$S^{(2)}(\lambda)/N = 3\lambda \ln 2 + (1-2\lambda) \ln \left(\frac{\sqrt{1+4\lambda-4\lambda^2}+1-2\lambda}{1-\lambda} \right) - \ln \left(\sqrt{1+4\lambda-4\lambda^2}+1 \right). \tag{51}$$

however, no simple series exists, indicating moment method may be hard.

Zhang, CL, Chen, 2020

Maximal randomness: many-body vs single-particle



A Haar state (a state distributed according to the Haar measure):

	single particle state	many-body state
Vector dimension	N	2^N
Entropy	$\int dx h(x) f(x)$	$\int dE(-E\ln E)\varrho(E)$
Distribution type	$f(x)$: eigenvalue of $\mathbf{C}_A$	$\varrho(E)$: eigenvalue of ρ_A
Random state ensemble	Jacobi ensemble	Wishart ensemble
Limiting law	Watchter	Marchenko-Pastur
Entropy form	$h(\kappa)\lambda - \lambda\sum_{n=1}^{\infty}\frac{\lambda^n}{n(n+1)}$	$h(1/2)\lambda - \frac{2^{\lambda N}}{2^{(1-\lambda)N+1}N}$
Physical context	single particle chaos	many-body chaos

Page, 1993

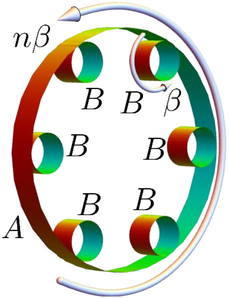
Entanglement entropy from path integral

Path integral method for entanglement entropy

$$S_A^{(n)} = \frac{1}{1-n}\left(\ln \underbrace{\mathrm{Tr}_A \left(\mathrm{Tr}_B e^{-\beta H}\right)^n}_{Z_n} - n \ln Z \right), \quad (52)$$

The essence is to expression Z_n! (The n-replica trick)

- The different tracing rules for A and B affect the boundary conditions (BCs):



- Tr_B: antiperiodic BC on every β-interval

$$\psi_{j\in B}((m+1)\beta^-) = -\psi_{j\in B}(m\beta + 0^+), \quad (53)$$

- Tr_A: antiperiodic BC on the large time interval $[0, n\beta)$:

$$\psi_{j\in A}(n\beta^-) = -\psi_{j\in A}(0^+), \quad \psi_{j\in A}(m\beta^-) = \psi_{j\in A}(m\beta^+). \quad (54)$$

Path integral method for entanglement entropy, cont'd

Next we need to average over disorders. Importantly, we make the approximation

$$\overline{S_A^{(n)}} = \frac{1}{1-n}(\overline{\ln Z_n - n \ln Z}) \approx \frac{1}{1-n}(\ln \overline{Z_n} - n \ln \overline{Z}). \tag{55}$$

This allows to conduct the disorder average within the path integral:

$$\overline{Z_n} = \int DG_A DG_B D\Sigma_A D\Sigma_B \exp\left(-NS_n[\Sigma_A, \Sigma_B, G_A, G_B]\right) \tag{56}$$

where the bilocal fields G_A and G_B inherit the boundary condition of $\psi_{j\in A}$ and $\psi_{j\in B}$, respectively.

$$\begin{aligned} S_n =& \frac{\lambda}{2} \ln \det(\partial_\tau^A - \Sigma_A) + \frac{1-\lambda}{2} \ln \det(\partial_\tau^B - \Sigma_B) \\ &+ \int_{[0,n\beta)} d\tau d\tau' \left(-\frac{\lambda}{2}\Sigma_A G_A - \frac{1-\lambda}{2}\Sigma_B G_B + \frac{J^2}{2q}(\lambda G_A + (1-\lambda) G_B)^q\right) \end{aligned} \tag{57}$$

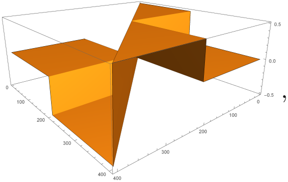
Path integral method for entanglement entropy, cont'd

Saddle point equations are

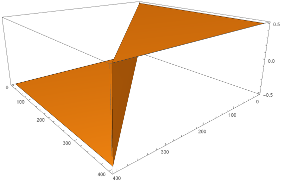
$$\boxed{G_A = (\partial_\tau^A - \Sigma_A)^{-1}, \quad G_B = (\partial_\tau^B - \Sigma_B)^{-1}, \quad \Sigma_A = \Sigma_B = J^2(\lambda G_A + (1-\lambda) G_B)^{q-1}.} \tag{58}$$

These equations can be solved numerically with high accuracy using iteration method.

$G_{0A} =$

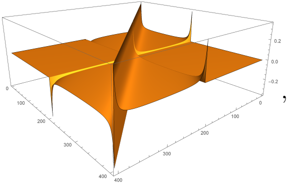


, $G_{0B} =$

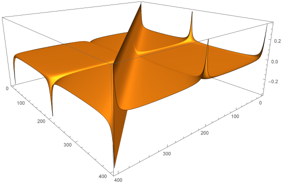


(59)

$G_A =$



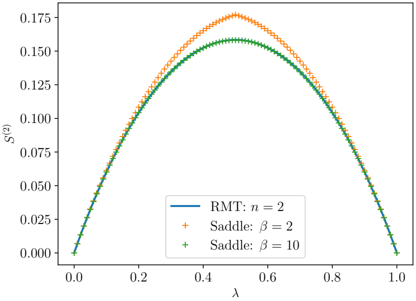
, $G_B =$



(60)

CL, Chen, Balents, 2017; **Zhang, CL, Chen, 2020**

Numerical result for SYK_2



- Numerically fit for computing thermal ensemble (finite β); ground state entropy ($\beta \to \infty$)
- Numerically can solve for integer $n \geq 2$ (i.e Rényi) and any q
- Analytic solution needed to obtain the von Neumann entropy ($n \to 1$)

More numerical results for SYK_q

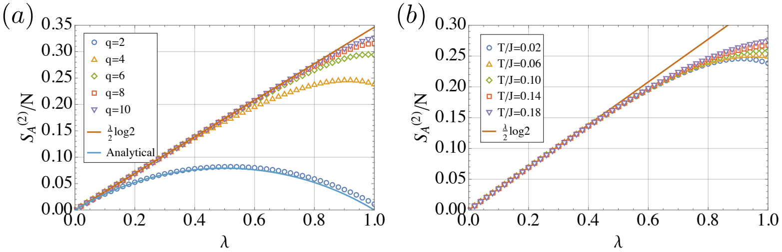


Figure 1: (a). Subsystem entropy $\mathcal{S}_A^{(2)}/N$ for different q. We fix $\beta\mathcal{J} = 50/\sqrt{2}$, which corresponds to $\beta J = 50$ for $q = 4$. We also show the analytical result for the SYK_2 model, as computed in the appendix. (b). Subsystem entropy $\mathcal{S}_A^{(2)}/N$ for different temperature T/J with $q = 4$. In both figures we have also plotted $\lambda \log 2/2$ for reference.

- Close to the Page law: SYK_q ($q \geq 4$) is very close to maximal random many-body state
- Assuming eigenstate thermalization hypothesis (ρ_A and ρ have the same spectrum up to rescaling), the curve can be estimated and agrees well with numerics.

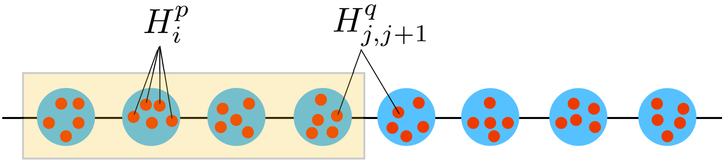
Zhang, CL, Chen, 2020; **Huang, Gu, 2019**

Path integral method: summary

- The path integral method allows to compute the large N properties of related SYK models in the same way.
- As an example, it allows to study the evolution of a quantum state under SYK like Hamiltonian

$$|\psi(t)\rangle = e^{-iHt}|\psi(t=0)\rangle. \tag{61}$$

Example: non-unitary evolution under an SYK chain



$$H \equiv J H^p_{\text{inter-cluster}} - iV H^q_{\text{intra-cluster}} \equiv J \sum_x H^p_{x,x+1} - iV \sum_x H^q_x, \tag{62}$$

$$S^{(n)} = -\frac{\log \overline{\mathrm{Tr}_A \rho_A^n}}{n-1} \simeq -\frac{\log \overline{\mathrm{Tr}_A \rho_A^n}}{n-1} = -\frac{1}{n-1}\left(\log \overline{Z_n(t, L_A)} - \log \overline{Z_n(t,0)}\right), \tag{63}$$

$$\overline{Z_n(t)} = \int_{\mathcal{C}} DG_x D\Sigma_x e^{-NS_n[G_x,\Sigma_x]}, \tag{64}$$

with

$$S_n[G_x,\Sigma_x] = -\frac{N}{4}\sum_x \Big\{ \log\det\left[\partial_{s,x} - \Sigma_x\right] - \int_0^{4nt} ds \int_0^{4nt} ds' \left[\Sigma_x G_x - f(s)f(s')\frac{J^2}{p}\left(G_x G_{x+1}\right)^{\frac{p}{2}} P - \frac{V^2}{q} G_x^q P\right]\Big\}. \tag{65}$$

The data specific to the system lies in the boundary conditions. **CL, Zhang, Chen, 2020**

Conclusion

- The SYK_q model $q > 2$ is a sparse random matrix model with correlated randomness in the entries. At $q = 2$, it is GOE/GUE/GSE.
- $q > 2$ are many-body chaotic (energy level repulsion); $q = 2$ single-particle chaos.
- Many quantities can be conveniently computed in the path integral formalism: infrared theory (edge spectrum), correlation function, entanglement entropy.
- The subsystem two-point correlation matrix of SYK_2 at zero temperature belongs to the Jacobi β-ensemble; this allows to compute entanglement entropy in closed form.
- The entanglement entropy of SYK_4 follows argument from ETH.
- Path integral is convenient in physics: one can design higher dimensional SYK model whose physics is governed by the saddle point.

Future directions

- The SYK model opens up new interests in RMT: a disordered many-body system is almost defined by a sparse matrix with correlated randomness. It is interesting to develop general statements/approach for these random matrices.
  - Ensemble for reduced density matrix (many-body)?
  - At finite temperature?
- The connection between the RMT approach and the path integral method
  - Especially in the context of entanglement entropy: how to derive from one to the other?
  - A path integral treatment of von Neumann?
  - Moment method for the entanglement entropy of SYK_4?
- Non-interacting fermion systems: explore matrix ensembles beyond standard random theory ensembles studied so far
- The SYK model is an all-to-all model with limited physical realizability. Other more realistic models?