



$S[O(2) \times O(2)]$ BF theory realization of non-Abelian quantum doubles

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ABSTRACT

Non-Abelian topological orders, which support anyonic excitations, are central to quantum computing but remain difficult to realize. We show that the twisted quantum double of the dihedral group—a key family of non-Abelian topological orders—can emerge via Higgsing from a parent $O(2)$ gauge theory. This correspondence is established by matching anyon statistics with Wilson loop observables in a BF theory framework. We further construct lattice models realizing these phases and, through renormalization group analysis, propose that they can undergo a direct transition to a gapless, deconfined $U(1)$ phase at a multicritical point with emergent $O(3)$ symmetry.

TOPOLOGICAL ORDER: $D^\omega(G)$

A **topological order** is completely determined by the anyon excitations $\mathcal{A} = \{a, b, c, \dots\}$ and their statistics:

- An anyon $a \in \mathcal{A}$ is a shorthand for the corresponding quantum state $|a\rangle$;
- Any symbol $O_{ab\dots}^{cd\dots}$ denotes the transition amplitude of incoming anyons a, b, \dots to outgoing anyons c, d, \dots . They are completely determined by 1) the elementary 2-fusion N_{ab}^c , 2) the change of basis for 3-fusion, F_d^{abc} , and 3) the (half-) braiding for a given 2-fusion channel, R_c^{ab} .

The **twisted quantum double**, $D^\omega(G)$, is a systematic construction of topological order starting from a finite group G and a cocycle $\omega \in H^3(G, U(1))$.

- The anyons are labeled by $a = \{(Cl, \rho)\}$ where Cl runs over the conjugacy classes of G and ρ the irreducible representations (irreps) of G .

Anyon Label	Dim.	Number	Conjugacy Class Cl	Centralizer	Irrep. ρ
1	1	1	{1}	D_n	$\mathbf{1}_+$
η	1	1	{1}	D_n	$\mathbf{1}_-$
ψ_l	2	$(n-1)/2$	{1}	D_n	$\mathbf{2}_l$
$\psi_{l,r}$	2	$n(n-1)/2$	$\{C_n^l, C_n^{n-l}\}$	\mathbb{Z}_n	$\mathbf{1}_r$
σ_\pm	n	2	$\{M_1, \dots, M_n\}$	\mathbb{Z}_2	$\mathbf{1}_\pm$

Anyon content of $D^\omega(D_n)$ for odd n . $l = 1, \dots, (n-1)/2$ and $r = -(n-1)/2, \dots, (n-1)/2$.

- The F symbols for the non-twisted quantum double $D(G)$ can be chosen to be trivial (i.e. the usual D_n gauge theory in which all the pure gauge fluxes carry bosonic exchange statistics). The F -symbols are identified with the twist ω (i.e. the 3-cocycle); and R -symbols of $D^\omega(G)$ will only depend on ω : they assume the form of $F_{g_4=g_1g_2g_3}^{g_1g_2g_3}$ and $R_{g_3=g_1g_2}^{g_1g_2}$.

$S[O(2) \times O(2)]$ BF THEORY

The $S[O(2) \times O(2)]$ BF theory is obtained from a $U(1) \times U(1)$ BF theory with the same appearance by gauging the symmetry $\mathbb{Z}_2^C : a \mapsto -a, b \mapsto -b$:

$$\mathcal{L}_{D_n} = \frac{ik}{2\pi} ada + \frac{in}{2\pi} adb, \quad k = 0, 1, \dots, n-1,$$

The **topological loop operators** in this theory can be classified into 1) the trivial identity operator, 2) the Wilson loop operators that are inherited from the $U(1) \times U(1)$ BF theory, and 3) the twist operators.

For example, the Wilson loop operators are

$$W_{p,q} = 2\mathcal{P} \cos \left(\oint_L pa + qb \right), \quad (p, q) \neq (0, 0),$$

subject to the identification $(p, q) \sim (p+n, q) \sim (p+2k, q+n) \stackrel{\mathbb{Z}_2}{\sim} (-p, -q)$.

EQUIVALENCE: $D^\omega(D_n)$ & $S[O(2) \times O(2)]$

Our central claim: the twisted quantum double $D^\omega(G)$ is realized by the $S[O(2) \times O(2)]$ BF theory.

This is achieved by checking several topological properties:

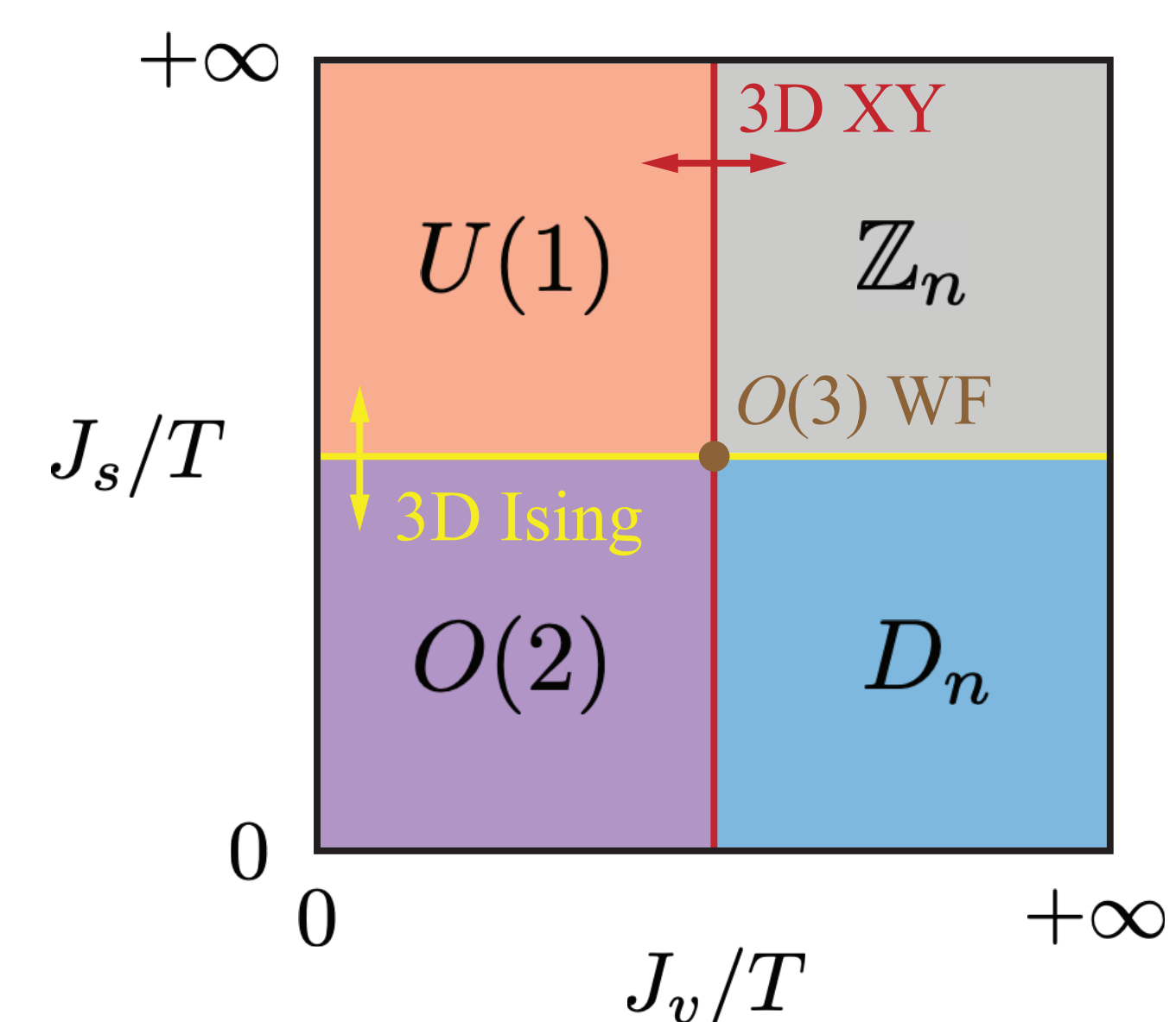
- From the ground state perspective, we show that the two theories have the **same ground state degeneracy** on Riemann surfaces with arbitrary genus g (this also determines the quantum dimension of each anyon).
- From the excitation perspective, we show that the fusion rules and the spins of the anyons in $D^\omega(D_n)$ are identical to those of the Wilson loops in the $S[O(2) \times O(2)]$ BF theory; more concretely, we show the two theories have **identical modular S and T matrices** on the torus ($g = 1$).

TWO LATTICE REALIZATIONS

In the $O(2, \mathbb{R})$ case, Consider the following classical model on a 3D lattice

$$H = H_{\text{Maxwell}} + \sum_{\langle ij \rangle} -J_s s_i |\mathbf{u}_{ij}| s_j - J_v \mathbf{v}_i \cdot \mathbf{u}_{ij} \cdot \mathbf{v}_j.$$

\mathbf{v}_i : two-component unit-norm real vector field; $s_i = \pm 1$: Ising field; \mathbf{u}_{ij} : the $O(2, \mathbb{R})$ gauge field with determinant $|\mathbf{u}_{ij}| = \pm 1$.

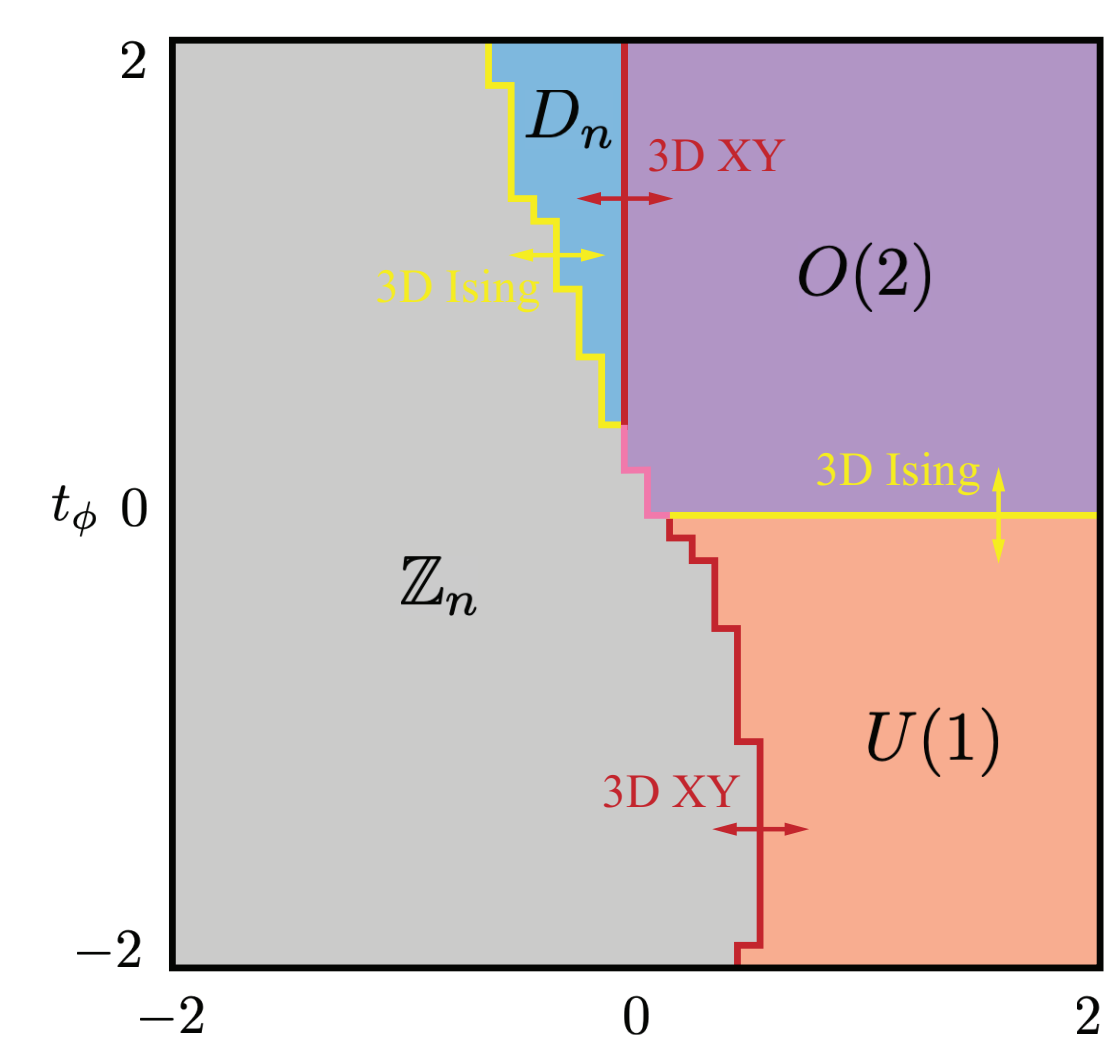


Phase diagram for the $O(2, \mathbb{R})$ model.

In the $O(2, \mathbb{C})$ case, consider the classical theory for complex bosons

$$H = - \sum_{\langle ij \rangle} J_\Phi \Phi_i^* \cdot \mathbf{u}_{ij} \cdot \Phi_j + J_\phi \phi_i |\mathbf{u}_{ij}| \phi_j + \sum_i \lambda \phi_i (|\Phi_{1,i}|^2 - |\Phi_{2,i}|^2) + \Lambda |\Phi_{1,i} - \Phi_{2,i}^*|^2,$$

where $\Phi_i = (\Phi_{1,i}, \Phi_{2,i})$ is a unit-norm two-component complex boson field, $\phi_i = \pm 1$ an Ising field, and \mathbf{u}_{ij} is the $O(2, \mathbb{C})$ gauge field.



Mean-field phase diagram for the energy of complex bosons, $(J_\Phi, J_\phi) \rightarrow (t_\Phi, t_\phi)$.

REFERENCES

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