220 Advanced Statistical Mechanics Notes

2016 年 6 月 9 日

重整化群

Derive from perturbative QFT, but apply beyound perturbation theory.

Key words in RG: fixed point, univesality class.

Fixed points 是 [高频积分 -scaling] 这个 procedure 不再改动的场论: 记为 $S_{\text{eff}}[\phi_<] = D^*[\phi]$. Scaling 不变, 说明这时系统有 Scaling invariance, 只能 (i) 关联尺度 $\xi \to 0$ 且能隙 $\to \infty$ (stable phase), 或 (ii) 关联尺度 $\xi \to \infty$ 且能隙 $\to 0$(quantum/thermal criticality, unstable fixed point, 对应相变发生处).

概念: 算符的尺度维数 [Scaling dimension]: 设 $b = e^{-\delta l} \to 1^- < 1$, 若 $\phi(xb^{-1}) = b^{\Delta}\phi(x)$, 则称 Δ 为 $\phi(x)$ 的尺度维数. 这个概念事实上是数学上的齐次函数 [Homogeneous function] 的概念: 如果一个函数满足 $\phi(\alpha x) = \alpha^{\triangle}\phi(x)$, 则称其为 $\triangle$ 次其次函数. 注意我们这里的定义有 $\alpha = b^{-1}$, $\Delta = -\triangle$. 注意尺度维数 (或说齐次函数) 是个动态的概念. 积分的 measure d^dx 是 x 次齐次函数, ϕ 是 $\triangle$ 次齐次函数, 但谈论 λ 的次数没有意义, 因为它并不是 x 的函数. 然而在 RG flow 的过程中可以赋予其意义: 在稳定点处, 用所有可能算符 $\phi_n(x)$ 微扰 [$\phi_n(x)$ 是任意算符]:

$$S[\phi] = S^*[\phi] + \int_{x>a} d^dx \sum_n \lambda_n \phi_n(x),$$

动量对应积分是 $|k| < \Lambda$, 注意 $b \to 1^-$. 在路径积分中积掉 $a < x < a/b$ 的部分, 则只剩下 $x > a/b$ 的积分, 令 $x' = xb$ 则 $x' > a$, 有 (这里省略求和, 用一个 ϕ 代表所有的 ϕ_n):

$$\int_{x>\frac{a}{b}} d^dx\lambda\phi(x) = \int_{x'>a} d^dx'\lambda b^{-d}\phi(x'/b) = \int_{x'>a} d^dx'\lambda b^{-d-\triangle}\phi(x') = \int_{x>a} d^dx(\lambda b^{-d-\triangle})\phi(x),$$

从而在这一过程 (积掉 UV 部分) 后, 如果我们令 $\lambda' = \lambda b^{-d-\triangle}$, 则作用量形式上没有变. 且在这一过程和形式不变性的意义下, 可以给 λ 一个” 尺度维数”[注意 λ 的尺度维数与前面提到的尺度维数意义完全不同: 前面提到的尺度维数就是数学上的齐次函数, 然而这里 λ 的尺度维数不是数学上的齐次函数的意义]: 注意每一个 b(即 l) 对应一个积掉 UV 部分的过程, 也就对应

着一个 λ 的变换 $\lambda \to \lambda'$, 从而可以把 λ 看成 b 的函数 (b 充当” 积掉 UV 部分” 这个过程的指标), 得到 (注意到 $b = e^{-\delta l} = 1 - \delta l$)

$$\lambda(b = 1 - \delta l) = \lambda(b = 1) b^{-d-\triangle} = \lambda(b = 1)(1 + \delta l(d + \triangle)),$$

或

$$\lambda(b = 1) + \delta l \frac{d\lambda}{dl} = \lambda(b = 1) + \delta l \lambda(d + \triangle),$$

得到

$$\frac{d\lambda}{dl} = \lambda(d + \triangle),$$

注意, 此处和 Fradkin 书上 (4.12) 有不同, 是括号内的不同, 因为 $\triangle = -\Delta$. 另外由于 Fradkin 认为 $(\lambda' - \lambda)/\delta l = d\lambda/dl$, 即他认为 λ' 中的 $l > 1$ 而 λ 中 $l = 1$ 所以应该这样减; 而我们则认为 λ 首先是 b 的函数, 注意 b 与 δl 的单调性方向正好是相反的, 所以作 Taylor 展开时是对 b 展开, 而且是对右边的 $b = 1$ 为展开的基点, 所以也是相加. 这样我们就得到了在积掉 UV 部分时, 为使原场论看起来不变, 耦合系数 λ 应满足的 flow 的方程.

现在, 注意到 $\phi(x)$ 的尺度维数, 也就是它的量纲 (注意: 作用量现在只有一种单位, x; $\phi(x)$ 的尺度维数往往是由作用量中别的项求出的). 这样我们可以求出 λ 的量纲: 显然就为 $-d - \triangle$. 这样, 给 λ 乘上 $a^{d+\triangle}$: 得到 $\bar{\lambda} = \lambda a^{d+\triangle}$ 为无量纲量. 注意, 我们上面的重整化过程并没有改变 a, 即一直把 a 视为常量, 我们引入的变量 (用来 label 重整化过程) 为 b 或 l. 现在, 先把无量纲参数 $\bar{\lambda}$ 代入上面方程得

$$\frac{d\bar{\lambda}}{dl} = \bar{\lambda}(d + \triangle),$$

再把截断变换写为 $a \to a' = a + da$, 从而 $a' = a/b = a e^{\delta l} = a(1 + \delta l)$, 或 $a' - a = a\delta l$, 或 $da = a dl$, 代入上面方程得到

$$a \frac{d\bar{\lambda}}{da} = \bar{\lambda}(d + \triangle),$$

注意, 也可以用 λ 与 a 无关, 故 $\partial\lambda/\partial a = 0$ 来得到上面的方程 (与上面是一致的). 这样, 我们就得到了在树图阶近似下的 RG 方程, 也称为 β 方程, 因为习惯称 $\beta(\bar{\lambda}) = a\frac{\partial\bar{\lambda}}{\partial a} = \frac{\partial\bar{\lambda}}{\partial \ln a}$. 以后我们将从用无量纲的参数 $\bar{\lambda}$ 来写 RG 方程. RG 方程的分析是显然的: 当 ϕ 的尺度维数 $\Delta = -\triangle$ 大于维数 d, 则耦合常数随积掉 UV 部分而减小, 即越低能该项越不起作用, 故 ϕ 是无关的 (irrelevant); 反之当 ϕ 的尺度维数 $\Delta = -\triangle$ 小于 d 则耦合常数随积掉 UV 部分而增大, 故 ϕ 是相关算符 (relevant operator) 并使系统 flow 到新的 fixed-point. 当 $\triangle = d$ 称为 marginal 的, 需要算更高阶修正 (loop correction).

例 (自由标量场 fixed point): $S = \int dx^d (\partial\phi)^2$, 根据 S 无量纲得到 ϕ 的量纲 (也即齐次阶数)$-(d-2)/2$, 设新加项都是 ϕ 或 $\partial\phi$ 的组合, 故由上面的方程的只有在新的项的齐次阶数与 d 直和大于等于零才 relevant: 设不含导数项: 由 $-m(d-2)/2 + d \geq 0$, 当 $d = 2$ 无约束; 当 $d > 2$ 得 $m \leq 2d/(d-2)$, $d = 3$ 得 $m \leq 6$, $d = 4$ 得 $m \leq 4$.

例 (non-linear sigma model)

First problem: Schrodinger equation of 2D finite square wall

$$\left[-\nabla^2+V_0\left(1-\frac{1}{(1+e^{(x-l)/\xi})(1+e^{-(x+l)/\xi})(1+e^{(y-l)/\xi})(1+e^{-(y+l)/\xi})}\right)\right]\psi(x,y)=E\psi(x,y)$$

$$\int dk_x dk_y dk'_x dk'_y e^{i(k_x+k'_x)x}e^{i(k_y+k'_y)y}V(k'_x,k'_y)\psi(k_x,k_y)=\int dk_x dk_y dk'_x dk'_y$$

$$\left[-E-(k_x^2+k_y^2)\right]\psi(k_x,k_y)+\int dk'_x dk'_y V(k_x+k'_x,k_y+k'_y)\psi(k_x-k'_x,k_y-k'_y)=0$$

Make it periodic: $[-L/2,L/2]\times[-L/2,L/2]$, $V(x,y)=0$ when $(x,y)\in[-a/2,a/2]\times[-a/2,a/2]$, and $V(x,y)=V_0$ for the rest. Note that V is the limit for the potential above in the Schrodinger equation when $\xi\to 0$. Let $k_x^n=k_y^n=\frac{2\pi}{L}n$,

$$\psi(x,y)=\sum_{m,n=-\infty}^{+\infty}\psi_{m,n}e^{i(k_x^m x+k_y^n y)},\qquad \psi_{m,n}=\frac{1}{L^2}\int_{-\frac{L}{2}}^{\frac{L}{2}}dx\int_{-\frac{L}{2}}^{\frac{L}{2}}dy\psi(x,y)e^{-i(k_x^m x+k_y^n y)},$$

$$V(x,y)=\sum_{m,n=-\infty}^{+\infty}V_{m,n}e^{i(k_x^m x+k_y^n y)},\qquad V_{m,n}=\frac{1}{L^2}\int_{-\frac{L}{2}}^{\frac{L}{2}}dx\int_{-\frac{L}{2}}^{\frac{L}{2}}dyV(x,y)e^{-i(k_x^m x+k_y^n y)},$$

$$\begin{aligned}V_{m,n}&=\frac{V_0}{L^2}\left(\frac{1}{-ik_x^m}e^{-ik_x^m x}\Big|_{a/2}^{L/2}+\frac{1}{-ik_x^m}e^{-ik_x^m x}\Big|_{-L/2}^{-a/2}\right)\left(\frac{1}{-ik_y^n}e^{-ik_y^n y}\Big|_{a/2}^{L/2}+\frac{1}{-ik_y^n}e^{-ik_y^n y}\Big|_{-L/2}^{-a/2}\right)\\&=\frac{V_0}{-L^2k_x^mk_y^n}\left(e^{-i\pi m}-e^{-i\frac{\pi ma}{L}}+e^{i\frac{\pi ma}{L}}-e^{i\pi m}\right)\left(e^{-i\pi n}-e^{-i\frac{\pi na}{L}}+e^{i\frac{\pi na}{L}}-e^{i\pi n}\right)\end{aligned}$$

$$V_{m,n}=-\frac{V_0}{L^2k_x^mk_y^n}\left(e^{i\frac{\pi ma}{L}}-e^{-i\frac{\pi ma}{L}}\right)\left(e^{i\frac{\pi na}{L}}-e^{-i\frac{\pi na}{L}}\right)=\frac{4V_0}{L^2k_x^mk_y^n}\sin\frac{\pi ma}{L}\sin\frac{\pi na}{L},mn\neq 0;$$

$$V_{0,n}=\frac{2V_0}{Lk_y^n}\sin\frac{\pi na}{L}\frac{L-a}{L},\quad V_{m,0}=\frac{2V_0}{Lk_x^m}\sin\frac{\pi ma}{L}\frac{L-a}{L},\quad V_{0,0}=V_0\frac{(L-a)^2}{L^2}.$$

Plug in Schrodiner equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V(x,y)\right]\psi(x,y)=E\psi(x,y)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2-E+\sum_{\mu,\nu=-\infty}^{+\infty}V_{\mu,\nu}e^{i(k_x^\mu x+k_y^\nu y)}\right]\sum_{m,n=-\infty}^{+\infty}\psi_{m,n}e^{i(k_x^m x+k_y^n y)}=0,$$

$$\sum_{m,n=-\infty}^{+\infty}\left[\frac{\hbar^2}{2m}(k_x^{m2}+k_y^{n2})-E+\sum_{\mu,\nu=-\infty}^{+\infty}V_{\mu,\nu}e^{i(k_x^{\mu}x+k_y^{\nu}y)}\right]\psi_{m,n}e^{i(k_x^m x+k_y^n y)}=0,$$

$$\sum_{m,n=-\infty}^{+\infty}\left[\left(\frac{\hbar^2}{2m}(k_x^{m2}+k_y^{n2})-E\right)\psi_{m,n}+\sum_{\mu,\nu=-\infty}^{+\infty}V_{\mu,\nu}\psi_{m-\mu,n-\nu}\right]e^{i(k_x^m x+k_y^n y)}=0,$$

thus must have

$$\left(\frac{\hbar^2}{2m}(k_x^{m2}+k_y^{n2})-E\right)\psi_{m,n}+\sum_{\mu,\nu=-\infty}^{+\infty}V_{\mu,\nu}\psi_{m-\mu,n-\nu}=0,\quad \forall m,n.$$

$$\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}(m^2+n^2)-E\right)\psi_{m,n}+\sum_{\mu,\nu=-\infty,\mu\nu\neq 0}^{+\infty}\frac{V_0}{\pi^2\mu\nu}\sin\frac{\pi\mu a}{L}\sin\frac{\pi\nu a}{L}\psi_{m-\mu,n-\nu}+$$

$$\sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\frac{V_0}{\pi\nu}\sin\frac{\pi\nu a}{L}(1-\frac{a}{L})\psi_{m,n-\nu}+\sum_{\mu=-\infty,\mu\neq 0}^{+\infty}\frac{V_0}{\pi\mu}\sin\frac{\pi\mu a}{L}(1-\frac{a}{L})\psi_{m-\mu,n}+V_0(1-\frac{a}{L})^2\psi_{m,n}=0,\quad \forall m,n.$$

$$\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}(m^2+n^2)+V_0(1-\frac{a}{L})^2-E\right)\psi_{m,n}+\sum_{\mu,\nu=-\infty,\mu\nu\neq 0}^{+\infty}\frac{V_0}{\pi^2\mu\nu}\sin\frac{\pi\mu a}{L}\sin\frac{\pi\nu a}{L}\psi_{m-\mu,n-\nu}+$$

$$\sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\frac{V_0}{\pi\nu}\sin\frac{\pi\nu a}{L}(1-\frac{a}{L})\psi_{m,n-\nu}+\sum_{\mu=-\infty,\mu\neq 0}^{+\infty}\frac{V_0}{\pi\mu}\sin\frac{\pi\mu a}{L}(1-\frac{a}{L})\psi_{m-\mu,n}=0,\quad \forall m,n.$$

Note that in order to use Fourier transformation we introduced a periodicity, L, to the system. In reality $L\to+\infty$, while a is finite, so there is really no way that the well width a times any integer, m, can exceed L, i.e. we must have $ma\ll L\to+\infty$. In order to characterize this, and also for the sake of solubleness of our equation(by this we mean currently we have infinite parameters and is not soluble; to turn it into soluble we must only have finite unknowns), we must introduce cut-off to our equation: $-N\le m,n,\mu,\nu\le N$, such that $\frac{Na}{L}\ll 1$. This way, we can approximate $\sin\frac{\pi\mu a}{L}=\frac{\pi\mu a}{L}$, and the equation becomes

$$\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}(m^2+n^2)+V_0(1-\frac{a}{L})^2-E\right)\psi_{m,n}+\sum_{\mu,\nu=-N,\mu\nu\neq 0}^{N}\frac{V_0a^2}{L^2}\psi_{m-\mu,n-\nu}+$$

$$\sum_{\nu=-N,\nu\neq 0}^{N}\frac{V_0a}{L}(1-\frac{a}{L})\psi_{m,n-\nu}+\sum_{\mu=-N,\mu\neq 0}^{N}\frac{V_0a}{L}(1-\frac{a}{L})\psi_{m-\mu,n}=0,\quad \forall m,n.$$

Also note that we can really throw away second order small quantity a/L, and the equation becomes

$$\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}(m^2+n^2)+V_0-E\right)\psi_{m,n}+\sum_{\mu,\nu=-N,\mu\nu\neq 0}^{N}\frac{V_0a^2}{L^2}\psi_{m-\mu,n-\nu}+$$

$$\sum_{\nu=-N,\nu\neq 0}^{N} \frac{V_0 a}{L}\psi_{m,n-\nu} + \sum_{\mu=-N,\mu\neq 0}^{N} \frac{V_0 a}{L}\psi_{m-\mu,n} = 0, \quad \forall m, n.$$

Other thoughts: does conformal mapping work? see `https://arxiv.org/ftp/arxiv/papers/1509/1509.06344.pdf`

Second problem: polymer

$$i_+ - i_- = i, \tag{1}$$

$$j_+ - j_- = j, \tag{2}$$

$$i_+ + i_- + j_+ + j_- = N, \tag{3}$$

we want to know how many solutions for nonnegative values of (i_+, i_-, j_+, j_-), and then do permutation for them. Note (i, j, N) is given . We have $i_+ + j_+ = \frac{N+i+j}{2}$, $i_- + j_- = \frac{N-i-j}{2}$, and $i_+ = i_- + i$, $j_+ = j_- + j$, thus $i_- + j_- = \frac{N-i-j}{2}$ is the only independent equation. And the number of solution is just $\frac{N-i-j}{2} + 1$. Then, for any such a solution, we can arrange the permutation of i_+, i_-, j_+, j_-: first bipartite N to $\frac{N+i+j}{2}$ and $\frac{N-i-j}{2}$, then bypartite the $\frac{N-i-j}{2}$ to be i_- and j_-, and similarly for $\frac{N+i+j}{2}$. Thus the total number of configuration should be

$$C_N^{\frac{N-i-j}{2}} \sum_{i_-=0}^{\frac{N-i-j}{2}} C_{\frac{N-i-j}{2}}^{i_-} C_{\frac{N+i+j}{2}}^{i_-+i} = C_N^{\frac{N-i-j}{2}} \sum_{i_-=0}^{\frac{N-i-j}{2}} C_{\frac{N-i-j}{2}}^{i_-} C_{\frac{N+i+j}{2}}^{\frac{N-i+j}{2}-i_-} = C_N^{\frac{N-i-j}{2}} C_N^{\frac{N-i+j}{2}},$$

The second equality noticed the fact that the sum is like picking out in total $\frac{N-i+j}{2}$ elements out of N things, thus we get the expression on the right. Note this expression can also be obtained by using the equation for $i_- + j_+ = \frac{N-i+j}{2}$, thus we have

$$C_N^{\frac{N-i+j}{2}} \sum_{i_-=0}^{\frac{N-i+j}{2}} C_{\frac{N-i+j}{2}}^{i_-} C_{\frac{N+i-j}{2}}^{i_-+i} = C_N^{\frac{N-i-j}{2}} \sum_{i_-=0}^{\frac{N-i+j}{2}} C_{\frac{N-i+j}{2}}^{i_-} C_{\frac{N+i-j}{2}}^{\frac{N-i-j}{2}-i_-} = C_N^{\frac{N-i+j}{2}} C_N^{\frac{N-i-j}{2}}.$$

Note according to this definition we have still have 1 configuration for $N = i = j = 0$. This will be useful for numerical calculation.

Note that the binomial distribution is $B(N, p)$, the distribution for X is $C_N^X p^X (1-p)^{N-X}$. The normal distribution $\mathcal{N}(\mu, \sigma^2)$ for X is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. We have

$$B(N, p) \longrightarrow \mathcal{N}(Np, Np(1-p)), \quad \text{when } N \to \infty$$

which means

$$C_N^X \left(\frac{1}{2}\right)^N \overset{N\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi(N/4)}} e^{-\frac{(x-N/2)^2}{N/2}},$$

Thus we see that

$$C_N^{\frac{N-i+j}{2}} C_N^{\frac{N-i-j}{2}} \left(\frac{1}{2}\right)^{2N} \overset{N\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{N-i+j}{2}-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{N-i-j}{2}-\mu)^2}{2\sigma^2}},$$

where $p = 1/2$, $\mu = Np = N/2$ and $\sigma^2 = Np(1-p) = N/4$. Thus the above distribution becomes

$$C_N^{\frac{N-i+j}{2}} C_N^{\frac{N-i-j}{2}} \left(\frac{1}{2}\right)^{2N} \overset{N\to\infty}{\longrightarrow} \frac{2}{\pi N} e^{-\frac{i^2+j^2}{N}},$$

A rough estimation of the upper limit of the ways for N segment polymer: should be less than 3^N.

March 29

Renormalization group, phase transitions, critical phenomena.

Introductory talk

Critical phenomena: characterized by large number of DOM of all different length scales, and they interacting with each other. Physics at different scales.

Now focus on classical statistical mechanics..

Example 1. Liquid-gas transition.

A phase diagram: a $p-T$diagram. Crital point is the point where the distinguish between liquidand gas vanishes.

generally: density $\rho(p,T)$, on the coexistence line of liquid and gas: $\Delta(\rho,T) = \rho_L(T, p_c(T)) - \rho_G(T, p_c(T)) = \text{const.}(T_c - T)^\beta + \cdots$ as $T_c - T \to 0^+$, and measure or compute we get $\beta = 0.32+$ error) simple theory (any theory at the: $\beta = 1/2$, discrepancy: tip of an iceberg.

at T_c, density fluctuations at wavelength of visible liquid. critical opalescence.

Critical point is very special.

technically: density-density correlation function: $G(r_1 - r_2, T) = \langle(\rho(r_1) - \rho)(\rho(r_2) - \rho)\rangle_{\text{stat. avg.}}$, $\rho =$average density. Then Fourier transform to $G(q,T)$, when $T \ll T_c$ or $T \gg T_c$, $G(q,T)$ is always ~ 1, as a function of $|q|$. $G$is the cross section we use as scattering. At $T \to T_c$, light scattering has largest amplitudes.

Once we understand the critical point we understand everything near this point.

Example 2. Uniaxial ("Ising") Ferromagnetic on a 3-d lattice.

$\hat{H} = -J \sum\limits_{\langle i,j\rangle} \hat{s}_i^z \hat{s}_j^z - H \sum\limits_i s_i^z$ $J > 0$; H applied magnetic field. $\hat{s}_i^z = \frac{1}{2}\sigma_i^z$ (material: YFeO_3, Rb_2NiF_4) experimentally, can vary T, H, and measure magnetization, specific heat, ... theoretically: need to compute canonical partition function, $Z(T,H,N) = \text{Tr}[e^{-\beta\hat{H}}] = e^{-N\beta f(T,H)}$, $\beta = 1/(k_B T)$, ($N$is lattice sites), volume of system is $a^d N$, $\frac{-1}{N\beta} \ln Z[T,H,Z] \overset{N\to\infty}{\to} f(T,H)$, magnetization per lattice site $= m(T,H) = -\frac{\partial f(T,H)}{\partial H}$, One then finds the phase diagram: the jump again appear at some temperature. One finds $m = m(H = 0^+, T)$, at $T \to T_c$, we have $m = \text{const}(T_c - T)^\beta + \cdots$, β is the same β of the first example. This needs to be understood! This also occurs for many other observables: specific heat, scatteing, etc... Look through: bubbles of spins goes into all length scales.

March 31

The magnetization as a func of temperature behaves similarly:

β =critical exponent. $\Delta\rho(T) = \text{const.}(T_c - T)^\beta + \cdots$, $m(T, H = 0^+) = \text{const}'(T - c - T)^{\beta'}$, turns out $\beta = \beta'$

most remarkable property: we are interesed in the phonomenon that so different things have those in common, i.e. why $\beta = \beta'$ even though the system looks completely different. Also, many other properties behave similarly. Will be given a catalog of this. e.g. specific heat, magnetic susceptibility, compressibility, also exhibit some critical exponent. We call these UNIVERSALITY: a certain aspects of complete different systems, are discribed by the same properties and more abstractly speaking, by the same theory. there most be some theory to be capable to calculate.

Universality of liquid-gas critical point.

answer questions

(a) What is the theory common to both systems? Answer: "Continuum field theory" devoid of all microscopic details.

$$\frac{\mathcal{H}}{k_B T} = \int d^3\vec{x} \left[\frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{T - T_c}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - H \cdot \phi \right]$$

$\phi(\vec{x})$ is real, $Z = \sum_{\text{all } \phi(\vec{x})} e^{-\frac{\mathcal{H}}{k_B T}}$, $\int d^3x \to \int d^4x$

(b) What quantities are "universal" (critical exponents", what quantities are not universal" $(T_c, \cdots)$

(c) How do we compute relevant universal quantities (critical exponents)? turns out to be complicated! We cannot use standard theoretical tool (perturbation theory, ...) So, what to do? Ingenious new calculational tool: renormalization group (RG). many contributors. 1982: Nobel Prize given to Ken Wilson.

Another example: Polymer Physics, (A) Ideal polymer, (B) Real Polymer, the ideal polymer: its a chain of N rigid massless rods, of length a each, the rods rotate reely about their jionts. Each rod is called monomer, or unit. All vectors $\vec{a}_i$ independent random vanishes, with same probability distribution: $\vec{a}_i = a\hat{\Omega}_i(O_i, \phi_i)$, $P(\hat{\Omega}_1, \cdots, \hat{\Omega}_N) = P(\hat{\Omega}_1)\cdots P(\hat{\Omega}_N)$, examples of a uniform distribution: P =const. require: $\langle \vec{a}_i \rangle = 0$. The limit $N \to \infty$, can be reformulated into a different universal class.

Easy to compute root mean square site of (ideal)chain , movable end: $\vec{r}_N = \vec{a}_1 + \cdots + \vec{a}_N$, $\langle (\vec{r}_N)^2 \rangle = \sum\limits_{i,j=1}^{N} \langle \vec{r}_i \cdot \vec{r}_j \rangle = \sum \langle \vec{a}_i \rangle \langle \vec{a}_j \rangle + \sum\limits_{i=1}^{N} \langle \vec{a}_i^2 \rangle = a^2 N$. radius of gyration: $\xi(N) = \sqrt{\langle (\vec{r}_N)^2 \rangle} = aN^{\nu}$, $\nu = 1/2$. $\xi(N) \to \infty$, as $N \to \infty$, $\xi(N)$ measured by light scattering for dilute set of polymers: a plot (pass).

Consider the probability distribution for the position $\vec{r}_N$ of the movable end of the (idea) chain: $P_N(\vec{r})$ = probability density that the movebla end of chain with N unity located at position $\vec{r} = \vec{r}_N$. (Centra limit theorem) when $N \to \infty$, centra limit theorem gives $P_N(\vec{r})$ = Gaussian for large $''N''$.

$$P_N(\vec{r}) = C_N \exp\left[-\frac{d}{2}\left(\frac{\vec{r}}{\xi(N)}\right)^2\right], \quad C_N = \left(\frac{1}{\sqrt{2\pi\xi^2(N)/d}}\right)^d.$$

Hold the movable end fixed at position $\vec{r}$. Entropy at fixed $\vec{r}$:

$$S(\vec{r}) = \ln[\text{const.} P_N(\vec{r})] = s_0 - \frac{d}{2}\frac{(\vec{r})^2}{(\xi(N))^2},$$

$(\xi(N) = aN^{1/2})$.

Free energy of ideal polymer at fixed elongation $r = |\vec{r}|$.

$F_N(\vec{r}) = E - T \cdot S_N(\vec{r}) = F_0(T) + \frac{1}{2}\left(\frac{d \cdot T}{\xi^2(N)}\right) r^2$,

feature: quadratic potential, pure entropic.

(A): ideal polymer: $\xi(N) \propto N^{1/2}$ in all dimensions.

(B) real polymer: real polymer, cannot intersect itself! (self avoidance) calculation is complicated. Will just give the results: properties when $N \gg 1$ are universal.

real polymer: $\xi(N) = \sqrt{\langle (\vec{r}_N)^2 \rangle} \propto aN^{\nu}$, ν = universal number, $\nu = 3/4$ in 2d. (a linein $\ln(\xi(N)) - \ln N$plane: It is an exact result, calculation is really complicated. In 3d: $\nu \sim 0.589 > 1/2$. In $d \geq 4$: $\ln[\xi(N)] - \ln N$: $\nu = 1/2$: exact. (note: 1/2 is always the ideal case in all dimensions)

Similar theory which describes the universal properties of real polymer chains:

$\mathcal{H} = \int d^d\vec{x} \left[\frac{1}{2}\sum\limits_{a=1}^{n}(\nabla\phi^a)\cdot(\nabla\phi^a) + \frac{\delta_\mu}{2}\left(\sum\limits_{a=1}^{N}\phi^a\phi^a\right) + \frac{\lambda}{4!}\left(\sum\limits_{a=1}^{N}\phi^a\phi^a\right)^2\right]$, $Z = e^{-\mathcal{H}}$, then at the end let $n \to 0$.

$\vec{\phi}(\vec{x})$ a n dimensional vector, $(\phi_1, \phi_2, \cdots, \phi_N)$ introduce a chemical potential μ, when $\mu_c \geq \mu$ describes the long polymer, and $\mu_c < \mu$ is excluded. $\mu \to \mu_c^+$ corresponds to the length of polymer diverge. $\delta_\mu = \mu - \mu_c$.

April 5

Homework 1.

$d = 2$, ideal: $\nu = 1/2$, real: $\nu = 3/4$. $\xi(N) = \sqrt{\langle(\vec{r}_N)^2\rangle} \sim const.N^{\nu}$

Compact object in $d$dimensions: (the linear size$(\xi(N)))^d \propto$ Mass, $\Leftrightarrow$ linear size $\propto (Mass)^{1/d}$, $\Leftrightarrow$ ideal polymer in $d = 2$ is compact, since $\xi(N) \propto N^{1/2} = N^{1/d}$. Ideal polyer is not compact in $d \neq 2$: fractal object (in d dimensions) has fractal dimension d_f: (linear size$)^{d_f}$ $\propto$Mass $\Leftrightarrow$ linear size $\propto (Mass)^{1/d_f}$. (Benoit Mandelbrot)

Done with polymers.

Phase Transitions in Magnets (description of phenomena and observables)

Specifically, consider the Ising/uniaxial magnet. Without justification, Effects of quantum mechanics not very important, for any continuous phase transitiion which happens at finite value of temperature $T = T_c$.

Square lattice, sites, $i = (i_1, j_2)$, lattice spacing is a, $r_i = ai = a(i_1, i_2)$, High dims: hypercubic lattice, $i = (i_1, \cdots, i_d)$.

Classical variable "spin" = "bit", $\sigma_i = \pm 1$, N is the number of lattice sites,

classical energy assignment:

$$\mathcal{H}\{\sigma_k\} = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - H\sum_{i=1}^{N}\sigma_i$$

where $J_{ij} = J_{ji}$, symmetric real, exchange coupling constant. Assume translational invariance $J_{ij} = J(r_i - r_j)$, when $J_{ij} \geq 0$, ferromagnetism; $J_{ij} \leq 0$, anti-ferromagnetism.

DEF: range of interactions is a length scale

$$R^2 = \frac{\sum_i (r_i)^2 |J(r_i)|}{\sum_i |J(r_i)|}$$

speciak case: $J_{ij} = J$, when i, j are nearest neighbors; $J_{ij} = 0$, otherwise. Thus

$$\frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j = J\sum_{\langle i,j\rangle}\sigma_i\sigma_j,$$

where $\langle i, j\rangle$ is sum over all nearest neighbors each n.n. counted once.

Need to compute canonical partition function: N =number of lattice sites. Often, when convenient, choose periodic boundary condition in the direction of the lattice.

$$Z[T, H, N] = \sum_{\sigma_1=\pm}\sum_{\sigma_2=\pm}\cdots\sum_{\sigma_N=\pm} e^{-\beta\mathcal{H}} = \sum_{\{\sigma_k\}} e^{-\beta\mathcal{H}} = \mathrm{Tr}_{\{\sigma_k\}}e^{-\beta\mathcal{H}} = e^{-\beta N f_N(T,H)}$$

free energy/density, free enegy per site.

Crucial to consider thermaldynamics limits ($N \to \infty$), cab be shiwb that a becessary condition for $f(T,H) = \lim_{N\to\infty} f_N(T,H) = \lim_{N\to\infty} \frac{-1}{\beta N} \ln Z[T,H,N]$ is $\sum_{j=1}^{N} |J_{ij}| < \infty$, finite size scaling, what happens to critical point when N is finite but large: $T, H, 1/N$ Descriptive discussion Ising ferromagnet:

$-\beta\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{N}(\beta J_{ij})\sigma_i\sigma_j + (\beta H)\sum_{i=1}^{N}\sigma_i,$

First, Remind Magnetization persite: $m = \langle\sigma_i\rangle = \frac{\mathrm{Tr}_{\{\sigma_k\}}\sigma_i e^{-\beta\mathcal{H}\{\sigma_k\}}}{\mathrm{Tr}_{\{\sigma_k\}} e^{-\beta\mathcal{H}\{\sigma_k\}}} = \frac{\partial}{\partial\beta H}\frac{lnZ}{N} = \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i\rangle = \frac{\partial f}{\partial H}$, $N \to +\infty$.

Magnetic susceptibility: $\chi = \frac{\partial m}{\partial H} = -\frac{\partial^2 f}{\partial H^2} = \beta\frac{\partial}{\partial\beta H}\frac{\partial}{\partial(\beta H)}\frac{\ln Z}{N} = \beta\frac{1}{N}\left[\langle s^2\rangle - \langle s\rangle^2\right] = \beta\frac{1}{N}\langle(s - \langle s\rangle)^2\rangle \geq 0$, $\chi = \beta\frac{1}{N}\times$(fluctuation of the totla spin $= s$).

Next: $\chi \leftrightarrow \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$; also specific heat C.

April 07

$m\langle\sigma_i\rangle$, $\chi = \frac{\partial m}{\partial H} = -\frac{\partial^2 f}{\partial H^2} = \beta\frac{1}{N}\langle(S - \langle S\rangle)^2\rangle \geq 0$, canonical average $\langle ...\rangle$. $P\{\sigma_k\} = \frac{1}{Z[T,H,N]}e^{-\beta\mathcal{H}\{\sigma_k\}}$

spin-spin correlation function: translational symmetry gives $G[r_i, r_j] = G[r_i - r_j] = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle = \langle(\sigma_i - \langle\sigma_i\rangle)(\sigma_i - \langle\sigma_i\rangle)\rangle$. $\chi = \beta\sum_j G[r_i - r_j]$, fluctuation dissipation theorem.

spec. heat: $c_V = k_B\beta^2\frac{\partial^2}{\partial\beta^2}\frac{\ln Z}{N} = k_B\beta^2\frac{1}{N}[\langle\mathcal{H}^2\rangle - \langle\mathcal{H}\rangle^2] = \langle(\mathcal{H} - \langle\mathcal{H}\rangle)^2\rangle$

Z_2 symmetry: $\mathcal{H}_H\{\sigma_k\} = -\mathcal{H}_{-H}\{-\sigma_k\}$, $m(-H) = -m(H)$, odd function; $\chi(H) = \chi(-H)$, even function.

plots in iphone: PHASE DIAGRAM, and the behaviour of $m(H), \chi(H), f$ in $T < T_c$, $T = T_c$, $T > T_c$, and spontaneous magnetization.

Def of critical exponents: $f(H < T) = -\frac{\ln Z}{\beta N}$ exhibit non-analytic behaviour. Def: reduced temperature, $t = (T - T_c)/T_c \to 0$.

(1) $m(H = 0^+, T) \sim C(-t)^\beta + \cdots (|t| \to 0, T < T_c)$ (note the symbol "$\sim$" means asymptotic behavior), $m(H = 0^+, T) = C(-t)^\beta + R(t)$ ($R(t)$ is remainder), $\frac{m(H=0^+,T)}{C(-t)^\beta} = \frac{R(t)}{C(-t)^\beta} \to 0$. $|t| \to 0$ Asymptotic expansion.

(2) $\chi(H = 0, T) \sim C_\pm|t|^{-\gamma}$, $+ : t > 0$, $- : t < 0$

(3) Specific heat: $C(H = 0, T) \sim A_\pm|t|^{-\alpha}$+possibly analytic terms. (note that α can be positive or negative, unlike $\gamma > 0$..

Typically not integer.

("Critical isotherm")

(1) $m(H, T = T_c) \propto |H|^{1/\delta}$,

(2) spin-spin correlation function at $H = 0$; $t \neq 0 \rightarrow G(r_i - r_j, T) \sim \frac{e^{-\frac{|r_i - r_j|}{\xi(T)}}}{|r_i - r_j|^{\frac{d-1}{2}}} = \exp\{-\frac{|r_i - r_j|}{\xi} + \frac{d-1}{2} \ln |r_i - r_j|\} = \exp\{-\frac{|r_i - r_j|}{\xi}[1 + \frac{d-1}{2} \frac{\xi}{|r_i - r_j|}(\ln \frac{|r_i - r_j|}{\xi + \ln \xi})]\}$, $\frac{|r_i - r_j|}{\xi(T)} \gg 1$, at $\frac{|r_i - r_j|}{\xi} \rightarrow \infty$, we have $G(r_i - r_j, T) \rightarrow -\frac{|r_i - r_j|}{\xi}$, define $\xi(T) \sim \kappa_{\pm} |t|^{-\nu}$.

Note:

(1)$T > T_c, t > 0$, $G(r_i - r_j), T) = \langle(\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)\rangle = \langle\sigma_i \sigma_j\rangle$. $(\langle\sigma_i\rangle = 0)$, $|r_i - r_j| \gg \xi$: σ_i and σ_j are independent, uncorrelated.

(2) $T < T_c, t < 0$, $G(r_i - r_j, T) = \langle(\sigma_i - m)(\sigma_j - m)\rangle$, $m = \langle\sigma_i\rangle$, $\delta\sigma_i = \sigma_i - m$ =deviation from average - fluctuation from average, deviation of σ_i from $(\delta\sigma_i)$ become uncorrelated.

(3) $T = T_c$: all spins are correlated with each other. more precisely: $t = 0$: $G(r_i - r_j, T = T_c) \propto \frac{1}{|r_i - r_j|^{d-2+\eta}}$, η is anomalous dimension.

Define: universal (i): $\beta, \gamma, \alpha, \delta, \nu, \eta$, critical exponents; (ii): $\frac{c_+}{c_-}$, $\frac{A_+}{A_-}$, $\frac{\kappa_+}{\kappa_-}$, amplitude ratios (but amplitude themselves are not universal).

Magnetic structure factorstatic $\widetilde{G}(\vec{q}, T) = S(\vec{q}, T) = \sum e^{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} G(\vec{r}_i - \vec{r}_j; T) \overset{(!)}{==} \frac{k_B T \chi(T)}{1 + (\vec{q} \cdot \chi)^2 + \ldots}$ another length scale: lattice dimension: a, ratio $\frac{a}{\chi} \ll 1$, scaling limit.

photo in iphone.

Next class: Landau theory derivation

April 12

magnetization $m(H = 0^+, T) \sim C(-t)^\beta$,

Susceptibility $\chi(H = 0, T) \sim C_{\pm}|t|^{-\gamma}$

specific heat $c(h = 0, T) \sim A \pm |t|^{-\alpha}$+(possible analytic in t)

correlation length $H = 0$: $\chi(T) \sim \kappa_{\pm}|t|^{-\nu}$,

correlation function $H = 0, t = 0$, $G(r_i - r_j, T = T_c) \sim \frac{1}{|r_i - r_j|^{d-2+\eta}}$ (anomolous dimension)

Numerical values for exponent, see table **??**.

$H = 0$:

$t \neq 0$: $G(r_i - r_j, T) \sim \exp\left\{-\frac{|r_i - r_j|}{\xi(T)}\right\}$;

$t = 0$, $G(r_i - r_j, T = T_c) \sim \frac{1}{|r_i - r_j|^{d-2+\eta}}$

Static structure factor $S(\vec{q}, T) = \widetilde{G}(\vec{q}, T) = \frac{k_B T}{1 + (\xi \cdot \vec{q})^2 + \cdots}$ (the ellipses goes to zero as $a/\xi \rightarrow 0$: scaling limit.)

A plot on bloackboard(see iphone). we learned $k_B T_2 \chi(T_2)$, the width (horizonta) $\sim |t|^\nu \rightarrow 0$.

Simplest theoretical discussion of PHASE TRANSITIONS: Mean Field Theory.

quantitively wrong when $d < 4$ for Ising.

Exponent	$d = 3$(Ising/aniaxial magnet)	$d = 2$ (Ising/aniaxial magnet)	$d \geq 4$
η	0.031	1/4	Mean field theories (discuss shortly)
β	0.325	1/8	as above
γ	1.241	7/4	as above
α	0.110	0 [but loy's] (ONsager,1944)	as above
ν	0.630	1 (Onsager,1944)	as above
δ	4.82	15	as above

表 1: numerical values for exponent. At $d = 1$ there is no phase transition except for $T = 0$. (In 1D there is no spontaneous symmetry breaking, and thus no phase transition.

Emphasizes importance of thermodyn limit. (reason why you need to minimize Landau's function)

Precursor for R.G. thinking (course graining)

Bragg Williams MFT.

Start from $\mathcal{H}\{\sigma_k\} = -\frac{1}{2}\sum\limits_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - H\sum\limits_{i=1}^{N}\sigma_i$,

write $\sigma_i = \frac{1}{N}\left(\sum\limits_{k=1}^{N}\sigma_k\right) + \delta\sigma_i = m\{\sigma_k\} + \delta\sigma_i = \frac{N_+ - N_-}{N_+ + N_-} + \delta\sigma_i = \frac{M}{N} + \delta\sigma_i$, N_+: number of $+1$ spins; N_-: number of -1 spins, $M = N_+ - N_-$, $N = N_+ + N_-$, $N_\pm = \frac{N \pm M}{2}$.

Note: $\sum\limits_{i=1}^{N}\delta\sigma_i = 0$, Thus

$\mathcal{H}\{\sigma_k\} = 0\frac{1}{2}\sum\limits_{i,j} J_{ij}(m + \delta\sigma_i)(m + \delta\sigma_j) - HNm = -\frac{1}{2}\sum\limits_{i,j} J_{ij}\left(m^2 + m(\delta\sigma_i\delta\sigma_j) + O(\delta\sigma)^2\right) - HNm$, then we drop $O(\delta\sigma)^2$. Thus under MFT approximation (it's an adhoc approximation):

$\mathcal{H}\{\sigma_k\} \to \mathcal{H}^{MFT}\{\sigma_k\} = (-\frac{1}{2})N \cdot \widetilde{J} \cdot m^2 - HNm$, where $\widetilde{J} = \left(\sum\limits_{j=1}^{N} J_{ij}\right)$ =no longer depends on i, $J_{ij} = J_{i-j}$, note $\sum\limits_{i,j=1}^{N} J_{ij} m\delta\sigma_j = \sum\limits_{j}\left(\sum\limits_{i,1}^{N} J_{ij}\right) m\delta\sigma_j = m\widetilde{J}\left(\sum\limits_{j=1}^{N}\delta\sigma_j\right) = 0$.

Easy to compute partition function (MFT approximation): $m = M/N = (N_+ - N_-)/N$,

$Z^{MFT}[T, H, N] = \sum\limits_{M=-N(\text{steps of 2})}^{+N} \frac{N!}{N_+! N_-!} e^{-\beta\mathcal{H}^{MFT}(m)}$

canonical Gibbs Boltzmann probability distribution (probability distribution for finding magnetism m), $P(m) = \frac{1}{Z^{MFT}} \frac{N!}{[\frac{1}{2}(N+M)]![\frac{1}{2}(N-M)]!} e^{-\beta\mathcal{H}^{MFT}(m)}$,

Know: $P(m)$ becomes sharply peaked, when $N \to \infty$, about its most probable value, m^*, determined by $\frac{\partial}{\partial m}\Big|_{m^*} \ln P(m) = 0$, Need Stirling's formula ($n! \sim e^{n \ln n - n}$):

$$\frac{N!}{[\frac{1}{2}(N+M)]![\frac{1}{2}(N-M)]!} \sim e^{N\left\{-\frac{1+m}{2}\ln\frac{1+m}{2}-\frac{1-m}{2}\ln\frac{1-m}{2}\right\}}$$

, hence: $Z^{MFT}[T,H,N] = \sum\limits_{M=-N(\text{steps }2)}^{+N} e^{-\beta N\mathcal{L}_{T,H}(m)}$, Landau function $\beta\mathcal{L}_{T,H}(m) = -\beta\frac{\widetilde{J}}{2}m^2 - \beta Hm + \frac{1+m}{2}\ln\frac{1+m}{2} + \frac{1-m}{2}\ln\frac{1-m}{2}$. The first two terms is $\mathcal{H}^{MFT}$ (which is energy), the last two terms is the combinatorics (entropy). Now:

$$P(m) = \frac{1}{Z^{MFT}[T,H,N]}e^{-\beta N\mathcal{L}_{T,H}(m)}$$

, plot in iphone(the symmetry breaking standard plot).

Note: the thermodynamic limit $N \to +\infty$ select the absolute minimum m^*:

$Z^{MFT}[T,H,N] \overset{N\to\infty}{\sim} e^{-\beta N\mathcal{L}_{T,H}(m^*)} = e^{-\beta N f^{MFT}_{T,H}}$, where $f^{MFT}_{T,H} = \mathcal{L}_{T,H}(m^*)$,

Comment: $Z^{MFT} = \sum\limits_{M=-N(\text{steps }2)}^{+N} e^{-\beta N\mathcal{L}_{T,H}(m)} = \frac{N}{2}\sum\limits_{M}(\frac{2}{M})e^{-\beta N\mathcal{L}_{T,H}(m)} \xrightarrow{N\to\infty}\sim \frac{N}{2}\int_{-1}^{+1} dm e^{-\beta N\mathcal{L}_{T,H}(m)}$, where $N/2 = e^{\ln N/2}$ is a constant contribution to $\ln Z^{MFT}$. need $\frac{\partial}{\partial m}\Big|_{m^*}\mathcal{L}_{T,H}(m) = 0$, extremum; $\frac{\partial^2}{\partial m^2}\Big|_{m^*}\mathcal{L}_{T,H}(m) > 0$: local minimum; $\mathcal{L}_{T,H}(m^*)$ =absolute minimum. simple to work out $\mathcal{L}_{T,H}(m)$ near critical point $(T,H) = (T_c, 0)$ since there $m \to 0$ (m =small) That is: we Taylor expand Landau function in m.

expand: $(1\pm x)\ln(1\pm x) = \pm x + \frac{x^2}{2} \mp \frac{x^3}{6} + \frac{x^4}{12} + \cdots$, we get $\beta\mathcal{L}_{T,H}(m) = -\ln Z + \left(1-\frac{T_c}{T}\right)\frac{m^2}{2} + \frac{m^4}{12} + \cdots - \beta Hm$, where $k_BT_c = \widetilde{J} = \sum\limits_i J_{ij}$.

April 14

Phase diagram: picture in phone.

Landau function ("derived") Taylor expansion for small m. $m(\{\sigma_k\}) = \frac{N_+-N_-}{N_++N_-} = \frac{M}{N}$.

$\beta\mathcal{L}_{T,H}(m) = -\ln Z + \left(1-\frac{T_c}{T}\right)\frac{m^2}{2} + \frac{m^4}{12} + \cdots - \beta Hm$, where $k_BT_c = \widetilde{J} = \sum\limits_i J_{ij}$, m=order parameter,

Qualitative discussion of shape of $\mathcal{L}_{T,H}(m)$: picture in iphone.

absolute minimum suddenly jumps.

Calculate $\langle\sigma_i\rangle \to \langle m\rangle$ =? using Landau MFT. expectation value of "m":

$Z^{MFT} = \sum\limits_{M=-N}^{N} e^{-\beta N\mathcal{L}_{T,H}(m)}, m = M/N$:

$\langle m\rangle = \frac{\sum\limits_{M=-N}^{N} me^{-\beta N\mathcal{L}_{T,H}(m)}}{\sum\limits_{M=-N}^{N} e^{-\beta N\mathcal{L}_{T,H}(m)}} \xrightarrow{N\to\infty} \frac{m^*e^{-\beta N\mathcal{L}_{T,H}(m^*)}}{e^{-\beta N\mathcal{L}_{T,H}(m^*)}} = m^*$,

$m = \frac{\partial}{\partial H}\frac{\ln Z^{MFT}[T,H,N]}{N}$, $P(m) = \frac{e^{-\beta N\mathcal{L}_{T,H}(m)}}{Z^{MFT}}$.

analytic calculation (for small "m"): $\frac{\partial(\beta\mathcal{L}_{T,H}(m))}{\partial m} = 0$ (Extremum), $(1-\frac{T_c}{T})m+\frac{m^3}{3}-h=0$, where $h=\beta H=\frac{H}{T}$ $(k_B=1)$. Now we want find out how m^* behaves when $h=0$. When $T>T_c$: only one sol.

$H=h=0$: when $T>T_c$: $m_*=0$; $T<T_c$: $m_*=0$ and $m_*=\pm\sqrt{3\frac{T_c-T}{T}}$. Note: local min : $\frac{\partial^2}{\partial m^2}\left(\beta\mathcal{L}_{T,H}(m)\right)\Bigg|_{m^*} > 0$.

We found: $T<T_c$: $m_*(T)\propto(T_c-T)^\beta(\beta=1/2$ in MFT).

Calculate magnetic susceptibility $H=0$: $\chi=\frac{\partial m_*}{\partial H}$, $\frac{\partial}{\partial H}$ of $(1-\frac{T_c}{T})m+\frac{m^3}{3}-h=0$: gives $\frac{T-T_c}{T}\chi+m_*^2\chi=\frac{1}{T}$ $(h=H/T)$, $T\cdot\chi=\frac{1}{\frac{T-T_c}{T}+(m_*)^2}$.

Now: $T>T_c$, $T\to T_c^+0^+$, $T\cdot\chi=\frac{T}{T-T_c}\Rightarrow\chi\sim\frac{1}{|T-T_c|^\gamma}$ with $\gamma=1$; $T<T_c$, $T\to T_c+0^-$, $T\cdot\chi=\frac{1}{\frac{T-T_c}{T}+3\frac{T_c-T}{T}}=\frac{1}{2}\frac{T}{T_c-T}$, $\Rightarrow\chi\sim\frac{1}{2}\frac{T}{|T_c-T|^\gamma}$, thus in Landau MFT: $\chi\sim c_+|\frac{T-T_c}{T_c}|^\gamma, T>T_c$, and $\chi\sim c_-|\frac{T-T_c}{T_c}|^\gamma, T<T_c$, $\gamma=1$ and $\frac{c_+}{c_-}=2$.

Magnetic critical isotherm $T=T_c$:

set $T=T_c$: $m_*(H)=\mathrm{sgm}(H)\left(3|\frac{H}{k_BT_c}|^{1/3}\right)\propto\mathrm{sgn}(H)|H|^{1/\delta}$, with $\delta=3$.

Free energy at $T=T_c$: $\chi=\frac{\partial m}{\partial H}=-\frac{\partial^2 f}{\partial H^2}$, $\frac{\partial^2 f^{MFT}}{\partial H^2}=-\frac{\partial m_*(H)}{\partial H}\propto-|H|^{-2/3}$, diverge on $f(H)-H$ plot at $(T=T_c)$.

specific heat at $H=0$: $c_V=T\frac{\partial^2}{\partial T^2}[-f(T,H=0)]$, in Landau MFT: $f^{MFT}(H,T)=\mathcal{L}_{T,H}(m^*)\leftarrow e^{-\beta Nf^{MFT}(T,H)}=Z^{MFT}[T,H,N]\overset{N\to\infty}{\sim}e^{-\beta N\mathcal{L}_{T,H}(m_*)}$,

check: a c_V-T plot in iphone.

Landau function is not unique:

$\mathcal{T},\mathcal{H}(m)$ and $m=m(\phi)$, m is order parameter, and ϕ is anoher order parameter, satisfies symmetry $m(-\phi)=-m(\phi)$, which is analytic (taylor expanson) function of ϕ, with $\partial m/\partial\phi\neq0$, then we get new Landau function $\widetilde{\mathcal{L}}_{T,H}(\phi)=\mathcal{L}_{T,H}(m(\phi))$, Clearly: minimizing $\widetilde{\mathcal{L}}$ we get the same result as minimizing $\mathcal{L}$: $\frac{\partial}{\partial\phi}\Big|_{\phi^*}\widetilde{\mathcal{L}}_{T,H}(\phi)=\left(\frac{\partial m}{\partial\phi}\right)\Bigg|_{\phi^*}\frac{\partial\mathcal{L}_{T,H}(m)}{\partial m}\Bigg|_{m(\phi_*)=m_*}=0$, and $\widetilde{\mathcal{L}}_{T,H}(\phi_*)=\mathcal{L}_{T,H}(m_*)$.

Higher order terms: do we have to worry about them? $\beta\mathcal{L}_{T,H}(m)=-\ln2+(1-\frac{T_c}{T})\frac{m^2}{2}+\frac{m^4}{12}+\cdots-\beta Hm$, say : $\mathcal{L}(m)=t\frac{m^2}{2}+\frac{m^4}{4}+\frac{\lambda}{6}m^6+\cdots-\beta Hm$, perturbation theory of λ. $t>0$ and $t<0$:

Calculate β: If $0=\frac{\mathcal{L}}{m}=tm+m^3+\lambda m^5$, $0=m[t+m^2+\lambda m^4]$, $t=-|t|$, $0=-|t|+m^2+\lambda m^4$, $m^2(1+\lambda m^2)=|t|$, thus $m^2=\frac{|t|}{1+\lambda m^2}=|t|\{1-\lambda m^2+\cdots\}$, thus $m=\sqrt{|t|}\{1-\lambda m^2\}^{1/2}=\sqrt{|t|}\{1-\frac{\lambda}{2}|t|+\cdots\}$, even when $\lambda\neq0$, still $m\sim|t|^\beta$, $\beta=1/2$, asymptotic expansion.

Landau MFT gives us unite exponents that have to do with polynomials: a $m(H=0^+,T)-T$ plot: catastrophic theory (discussed by mathematicion V.I. Arnold)

Coarse Graining:

so far (meaning Landau MFT) cannot discuss spatial fluctuations of $m = \frac{1}{N}\sum\limits_{i=1}^{N}\sigma_i$.

next step: make m spatially dependent: Landau Ginzburg theory (nobal prize 2003): (purely phenomenological): process (corase graining): microscopic: Landau Ginzburg Theory; Need to be in a regime where $a \ll \xi$.

Coarse graining procedure plot: on iphone.

This procedure we are familiar: we used this in hydrodynamics.

April 19

COARSE GRAINING handout: here $b = 3$, $a \ll l_b = ba$ is less or equal to ξ.

b^d = number in each block, Coarse grained vanish: $m_I = \frac{1}{b^d}[\sum_{i\in \text{block } I}\sigma_i]$, when b^d is large: $\sum_{m_I} = \text{Tr}_{m_I} \to \int_{-1}^{+1} dm_I$.

A coarse grain picture in iphone.

$Z_{Ising} = \text{Tr}_{\{\sigma_k\}}e^{-\beta\mathcal{H}_{Ising}\{\sigma_k\}} \overset{?}{=} \text{Tr}_{\{m_I\}}e^{-S_b\{m_I\}}$, note $\text{Tr}_{\{m_I\}} = \sum_{\{\sigma_k\}}$. This is Ginzbarg theory.

Note: $\prod_I(a_Ib_I) = \prod_I a_I \prod_I b_I$. we have

$$\prod_I^{\text{all blocks}}\left[\int_{-1}^{+1} dm_I\delta\left(m_I - \frac{1}{b^d}[\sum_{i\in I}\sigma_i]\right)\right] = 1, \qquad \int_{-1}^{+1} dm_I\delta\left(m_I - \frac{1}{b^d}[\sum_{i\in I}\sigma_i]\right) = 1,$$

$$\left[\prod_I^{\text{blocks}}\int_{-1}^{+1} dm_I\right]\left[\prod_I^{\text{blocks}}\delta\left(m_I - \frac{1}{b^d}[\sum_{i\in I}\sigma_i]\right)\right] = 1.$$

Result: $S_b\{m_I\} = S_b\{m(\vec{R})\} = \beta\int d^d\vec{R}\left[\frac{\gamma_b}{2}(\nabla m(\vec{R}))^2 + \mathcal{L}_b(m(\vec{R}))\right] = \beta L_b\{m(\vec{R})\}$. note that $\gamma_b > 0$. This is Landau Ginzbarg functional.

note $\vec{R}_I$ = center of block. $m(\vec{R}_I) = m_I$, $\sum\limits_{I}^{blocks}(l_b)^d \simeq \int d^d\vec{R}$.

exact computation of $S_b\{m_I\}$:

$$Z_{Ising} = \text{Tr}_{\{\sigma_k\}}e^{-\beta\mathcal{H}\{\sigma_k\}} = \text{Tr}_{\{m_I\}}\left[\text{Tr}_{\{\sigma_k\}}\left(\prod_I^{block}\delta(m_I - \frac{1}{b^d}[\sum_{i\in I}\sigma_i])\right)e^{-\beta\mathcal{H}\{\sigma_j\}}\right]$$

Gaussian Integrals:

One dimension: A, J are real numbers, $A > 0$, $I_1[A,J] = \int_{-\infty}^{+\infty} dxd^{-\frac{1}{2}Ax^2+Jx} = \int_{-\infty}^{+\infty} dxe^{-\frac{A}{2}(x-\frac{J}{A})^2+\frac{J^2}{2A}} == e^{\frac{J^2}{2A}}\int dxe^{-\frac{A}{2}x2} = \sqrt{\frac{2\pi}{A}}e^{\frac{J^2}{2A}}$, analytic continuation: $A = |A|e^{i\alpha}$.

N dimensions:

A = real symmetric $N \times N$ matrix (invertible) A positive eigenvalues. $J = N$ dimensional column, vector/real) $\vec{x} = (x_1, \cdots, x_n)^T$, $\vec{x}^t = (x_1, \cdots, x_N)$, $\vec{J} = (J_1, \cdots, J_N)^T$, $\vec{J}^t = (J_1, \cdots, J_N)$.

matrix vector $I_N[A,\vec{J}]=\int_{-\infty}^{+\infty}d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}+\vec{J}^t\cdot\vec{x}}$, $\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}A_{ij}x_ix_j=x_iA_{ij}x_j$,

repeated indices summed, $A=\frac{A+A^t}{2}+\frac{A-A^t}{2}=A_s+A_a$.

shift $\vec{x}\to\vec{y}$: $\vec{x}=\vec{y}+A^{-1}\vec{y}$, $\vec{x}^t=\vec{y}^t+\vec{J}^t(A^{-1})$, we get $\vec{x}^tA\vec{x}=(\vec{y}^t+\vec{J}^tA^{-1})A(\vec{y}+A^{-1}\vec{J})=\vec{y}^tA\vec{y}+\vec{J}^tA^{-a}\vec{J}$, $\vec{J}^t\vec{x}=\vec{J}^tA^{-1}\vec{J}$, $-\frac{1}{2}\vec{x}^tA\vec{x}+\vec{J}^t\cdot\vec{x}=-\frac{1}{2}\vec{y}^tA\vec{y}+\frac{1}{2}\vec{J}tA^{-1}\vec{J}$,

$I_N[A,\vec{J}]=e^{\frac{1}{2}\vec{J}^tA^{-1}\vec{J}}\int d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}}=e^{\frac{1}{2}\vec{J}^tA^{-1}\vec{J}}I_N[A,\vec{J}=0]$,

Now compute: $I_N[A,\vec{0}]$ is ? diagonalize A yb orthogonal matrix O. $O^tAO=\Lambda=\text{diag}(\lambda_1,\cdots,\lambda_N)$, change integration variables: $\vec{x}\to\vec{v}\equiv O^{-1}\vec{x}$, $d^N\vec{x}=(\text{det}O)d^N\vec{v}=d^N\vec{v}$,

Hence $I[A,\vec{J}=0]=\int d^n\vec{v}e^{-\frac{1}{2}\sum\limits_{i=1}^{N}\lambda_iv_i^2}=\prod_{i=1}^N\left(\int_{-\infty}^{+\infty}dv_ie^{-\frac{1}{2}\lambda_iv_i^2}\right)=\prod\limits_{i=1}^{N}(\frac{2\pi}{\lambda_i})^{1/2}=(2\pi)^{N/2}\frac{1}{(\text{det}A)^{1/2}}$,

$=(2\pi)^{N/2}\frac{1}{(\lambda_1\cdots\lambda_N)^{1/2}}=(2\pi)^{N/2}e^{-\frac{1}{2}\sum\limits_{i=1}^{N}\ln\lambda_i}=(2\pi)^{N/2}e^{-\frac{1}{2}\text{Tr}\ln A}$. Summarize:

$$I_N[A,\vec{J}]=(2\pi)^{N/2}(\text{det}A)^{-1/2}e^{\frac{1}{2}\vec{J}^tA^{-1}\vec{J}}$$

Gaussian probability distributions: N real random variables $\vec{x}=(x_1,\cdots x_N)^T$, real symmetric matrix (positive eigenvalues), prob distribution is $P_A(\vec{x})=\frac{e^{-\frac{1}{2}\vec{x}^tA\vec{x}}}{[\int d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}}]}$. Expectation values: pick $n(n\le N)$ numbers, $i_1,\cdots,i_n$, from $1,\cdots,N$, pairwise distinct $i_a\ne i_b$ if $i\ne b$. $\langle x_{i_1}\cdots x_{i_n}\rangle_A=\frac{\int d^N\vec{x}x_{i_1}\cdots x_{i_N}e^{-\frac{1}{2}\vec{x}^tA\vec{x}}}{[\int d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}}]}=$ (note: vanishes when$n=$ odd) $=\frac{\frac{\partial^n}{\partial J_{i_1}\cdots\partial J_{i_n}}\Big|_{\vec{J}=0}[\int dx^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}+\vec{J}^t\vec{x}}]}{[\int d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}}]}=\frac{\partial^n}{\partial J_{i_1}\cdots\partial J_{i_n}}\Big|_{\vec{J}=0}\left(\frac{I_N[A,\vec{J}]}{I_N[A,\vec{0}]}\right)=\frac{\partial^n}{\partial J_{i_1}\cdots\partial J_{i_n}}\Big|_{\vec{J}=0}e^{\frac{1}{2}\vec{J}^tA^{-1}\vec{J}}$. Example: $\langle x_ix_j\rangle_A=A^{-1}_{ij}$.

April 21

$I_N[A,\vec{J}]=\int d^N\vec{x}e^{-\frac{1}{2}\vec{x}^tA\vec{x}+\vec{J}^t\vec{x}}=(2\pi)^{N/2}\frac{1}{\sqrt{\text{det}A}}e^{\frac{1}{2}\vec{J}^tA\vec{J}}$.

Note here $J_i=\sigma_i$, $(A^{-1})_{ij}=K_{ij}$.

$$e^{\frac{1}{2}\sum\limits_{i,j=1}^{N}\sigma_iK_{ij}\sigma_j}=\frac{1}{(2\pi)^{N/2}}(\text{det}K)^{-\frac{1}{2}}\int d^N\vec{x}\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N}X_i(K^{-1})_{ij}X_j+\sum_{i=1}^{N}\sigma_iX_i\right\}$$

Ising nagmet d-dimensional hypercubic lattice (N lattice sites)

Notation: $K_{ij}=-\beta J_{ij}$ ($N\times N$ matrix); $h_i=\beta H_i$ (N dimensional vector, where H_i is inhomogeneous magnetic fields)

$$Z[K_{ij},h_i,N]=\text{Tr}_{\{\sigma_k\}}\exp\left[\frac{1}{2}\sum_{i,j=1}^{N}\sigma_iK_{ij}\sigma_j+\sum_{i=1}^{N}h_i\sigma_i\right]$$

(where $\text{Tr}_{\{\sigma_k\|}=\sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1}$)

$$=\frac{1}{(2\pi)^{N/2}}\frac{1}{\sqrt{\text{det}K}}\left[\prod_{i=1}^{N}\int_{-\infty}^{\infty}dX_i\right]\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N}X_i(K^{-1})_{ij}X_j\right\}\left[\text{Tr}_{\{\sigma_k\}}\exp\left\{\sum_{i=1}^{N}(h_i+X_i)\sigma_i\right\}\right]$$

Note that

$$\left[\mathrm{Tr}_{\{\sigma_k\}} \exp\left\{\sum_{i=1}^{N}(h_i + X_i)\sigma_i\right\}\right] = \prod_{i=1}^{N}\left[\sum_{\sigma_i=\pm 1} e^{(h_i+X_i)\sigma_i}\right] = \prod_{i=1}^{N}[2\cosh(h_i + X_i)] = 2^N \prod_{i=1}^{N} e^{A(h_i+X_i)} = 2^N e^{\sum\limits_{i=1}^{N} A(h_i+X_i}$$

where $A(y) \equiv \ln\cosh(y)$.

To summarize:

$$Z[K_{ij}, h_i, N] = \left(\frac{2}{\sqrt{2\pi}}\right)^N \frac{1}{\sqrt{\det K}} \left[\prod_{i=1}^{N}\int_{-\infty}^{\infty} dX_i\right] \exp\left(-\mathcal{S}\{X_k\}\right),$$

where

$$-\mathcal{S}\{X_k\} = -\frac{1}{2}\sum_{i,j=1}^{N} X_i(K^{-1})_{ij}X_j + \sum_{i=1}^{N} A(X_i + h_i),$$

now let $y_i = X_i + h_i$. Alternative form: shift integration variables:

$$-\mathcal{S}\{X_k\} = -\frac{1}{2}\sum_{i,j=1}^{N}(y_i - h_i)(K^{-1})_{ij}(y_j - h_j) + \sum_{i=1}^{N} A(y_i) = -\frac{1}{2}\sum_{i,j=1}^{N} h_i(K_{ij}^{-1}h_j - \frac{1}{2}\sum_{i,j=1}^{N} y_i(K^{-1})_{ij}y_j$$

$$+\sum_{i=1}^{N} A(y_i) + \sum_{i=1}^{N} J_i y_i = -\frac{1}{2}\sum_{i,j=1}^{N} h_i(K^{-1})_{ij}h_j - S\{y_k\}$$

where $J_i = \sum\limits_{j=1}^{N}(K^{-1})_{ij}h_j$, $S\{y_k\} = -\frac{1}{2}\sum\limits_{i,j=1}^{N} y_i(K^{-1})_{ij}y_j + \sum\limits_{i=1}^{N} A(y_i) + \sum\limits_{i=1}^{N} J_i y_i$.

All together:

$$Z[K_{ij}.h_i, N] = e^{-\frac{1}{2}\sum\limits_{i,j=1}^{N} h_i(K^{-1})_{ij}h_j} \left(\frac{2}{\sqrt{2\pi}}\right)^N \frac{1}{\det K}\int d^N\vec{y}e^{-S\{y_k\}}$$

$$= \mathrm{Tr}_{\{\sigma_k\}}\left[e^{\frac{1}{2}\sum_{i,j=1}\sigma_i K_{ij}\sigma_j + \sum\limits_{i=1}^{N} h_i\sigma_i}\right].$$

let

$$\mathcal{Z}[K_{ij}, J_i, N] = \left(\frac{2}{\sqrt{2\pi}}\right)^N \frac{1}{\det K}\int d^N\vec{y}e^{-S\{y_k\}},$$

$\mathcal{Z}[K_{ij}, J_i, N]$ is the partition function (probability distribution) for $\{y_k\}$:

$$P(y_1, y_2, \cdots, y_N) = \frac{1}{\mathcal{Z}}e^{-S\{y_k\}},$$

note the reation between $X_i \leftrightarrow m(R_I)$.

What do the variables y_i and X_i mean physically? (All repeated indices summed) We will see $X_i = (Y_i - h_i)$ "represents" the spin "σ_i": to see this, express $\langle\sigma_i\rangle$, which is the local magnetization, and $\langle(\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)\rangle$, the spin-spin correlation function, in terms of X_i and Y_i.

Local magnetization:

$$\langle\sigma_i\rangle = \frac{\partial}{\partial h_i}\left(\ln Z[K_{ij}, h_i, N]\right) = \frac{\partial}{\partial h_i}\left(-\frac{1}{2}h_k(K^{-1})_{kl}h_l + \ln \mathcal{Z}[K_{ij}, J_i, N]\right)$$

$$= -(K^{-1})_{ik}h_k + \frac{\partial J_k}{\partial h_i}\left(\frac{\partial}{\partial J_k}\ln \mathcal{Z}[K_{ij}, J_i, N]\right) = -(K^{-1})_{ik}h_k + (K^{-1})_{ik}\langle y_k\rangle_S =$$

(where $\partial J_k/\partial h_i = K_{ki} = K_{ik}$)

$$\frac{\partial}{\partial J_k}\ln \mathcal{Z}[K_{ij}, J_i, N] = \frac{1}{\mathcal{Z}}\frac{\partial}{\partial J_k}\mathcal{Z} = \frac{1}{\mathcal{Z}}\int d^N\vec{y}y_k e^{-S\{y_k\}} = \left[\prod_{j=1}^{N}\int_{-\infty}^{\infty}dY_j\right]Y_jP(y_1, \cdots, y_N) = \langle Y_k\rangle_S = (K^{-1})_{ik}\langle X_k\rangle.$$

$Z = \sum_{\phi(\vec{x})} e^{-\beta\mathcal{H}\{\phi(\vec{x})\}}$, $\mathcal{H}(\vec{x}) = \int d^d\vec{x}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{t}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]$

Spin-spin correlation function:

G_{ij} is $N \times N$ matrix:

$$G_{ij} = G[r_i, r_j] = \langle[\sigma_i - \langle\sigma_i\rangle][\sigma_j - \langle\sigma_j\rangle]\rangle = \frac{\partial}{\partial h_i}\frac{\partial}{\partial h_j}\ln \mathcal{Z}[K_{ij}, h_i, N] = \frac{\partial}{\partial h_i}\frac{\partial}{\partial h_j}\left(-\frac{1}{2}h_k(K^{-1})_{kl}h_l + \ln \mathcal{Z}[K_{ij}, J_i, N]\right) =$$

$$= -(K^{-1})_{ij} + \left(\frac{\partial J_{i'}}{\partial h_i}\frac{\partial}{\partial J_{i'}}\right)\left(\frac{\partial J_{j'}}{\partial h_j}\frac{\partial}{\partial J_{j'}}\right)\ln \mathcal{Z}[K_{ij}, J_i, N] = -(K^{-1})_{ij} + (K^{-1})_{ii'}\left(\frac{\partial}{\partial J_{i'}}\frac{\partial}{\partial J_{j'}}\ln \mathcal{Z}[K_{ij}.J_i, N]\right)(K^{-1})_{j'j}$$

(where $\frac{\partial J_{j'}}{\partial h_j} = (K^{-1})_{jj'}$)

$$= -(K^{-1})_{ij} + (K^{-1})_{ii'}\mathcal{G}_{i'j'}(K^{-1})_{j'j}$$

where $\mathcal{G}_{ij} = \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\ln \mathcal{Z}[K_{ij}, J_i, N] = \langle[y_i - \langle y_i\rangle][y_j - \langle y_j\rangle]\rangle_S = \langle[X_i - \langle X_i\rangle][X_j - \langle X_j\rangle]\rangle_{\mathcal{S}}$, to summarize:

$$G_{ij} = -(K^{-1})_{ij} + (K^{-1})_{ii'}\mathcal{G}_{i'j'}(K^{-1})_{j'j}$$

$$G = -K^{01} + (K^{-1})\mathcal{G}(K^{-1}), KGK = -K + \mathcal{G}$$

thus

$$\mathcal{G} = KGK + K$$

$$\mathcal{G}_{ij} = \langle(K_{ii'}[\sigma_{i'} - \langle\sigma_{i'}\rangle])(K_{jj'}[\sigma_{j'} - \langle\sigma_{j'}\rangle]\rangle$$

, $X_i \to Y_i \equiv X_i + h_i$ Again shows correspondence: $X_i = (y_i - h_i) \leftrightarrow ??$

April 26

$$Z[k_{ij}, h_i, N] = \mathrm{Tr}_{\{\sigma_k\}}\exp\left[\frac{1}{2}\sum_{i,j=1}^{N}\sigma_i K_{ij}\sigma_j + \sum_{i=1}^{N}h_i\sigma_i\right] = e^{-\frac{1}{2}\sum_{i,j=1}^{N}h_i(K^{-1})_{ij}h_j}\left[\left(\frac{2}{\sqrt{2\pi}}\right)^N\int d^N\vec{y}e^{-S\{y_k\}}\right] = \mathcal{Z}[K_{ij}, J_i$$

$\sigma_i = \pm 1, -\infty < y_i < \infty$, $x_i = y_i - h_i$, $= S\{y_k\} = \frac{1}{2}\sum_{i,j=1}^{N} y_i(K^{-1})y_j + \sum_{i=1}^{N} A(y_i) + \sum_{i=1}^{N} J_i y_i$, $J_i = \sum_j (k_{ij}^{-1} h_j$, $A(y) = \ln\cosh(y)$.

New generalized def of MFT:

$$\int d^N \vec{y} e^{-S\{y_k\}} \overset{MFT approx.}{\longrightarrow} e^{-S\{\bar{y}_k\}}$$

$$0 = \left.\frac{\partial S\{y_k\}}{\partial y_i}\right|_{\bar{y}_i} = \sum_{k=1}^{N} (K^{-1})_{ik}(\bar{y}_k - h_k) - A'(\bar{y}_i)$$

$A'(\bar{y}_i) = \tanh(\bar{y}_i)$.

Last time: $x_i = (y_i - h_i) \leftrightarrow \sum_{k=1}^{N} K_{jk}\sigma_k = K_{ik}\sigma_k$ Does not affect ritical properties (crit. exponents,...) Exponent "β": $H_i = H = 0 \to h_i = h = 0$ (translational invariance) $\langle x_i\rangle_s = \langle y_i\rangle_s = \sum_{k=1}^{N} K_{ik}\langle\sigma_k\rangle \equiv (\sum K_{ik})m$, $(\frac{\sum_{k=1}^{N} J_{ik}}{k_B T}) = \frac{T^{MFT}}{T}$,

$m(T) \propto |t|^\beta$, $t = (T - T_c)/T_i$, $\langle y_i\rangle = \langle x_i\rangle \propto |t|^\beta$, extract same β from either formulation.

$H_i = h_i = 0$,$G_{ij} = G[r_i - r_j] = \langle[\sigma_i - \langle\sigma_i\rangle][\sigma_j - \langle\sigma_j\rangle]\rangle$, $\mathcal{G}_{ij} = \mathcal{G}[r_i - r_j] = \langle[\sigma_i - \langle\sigma_i\rangle][\sigma_j - \langle\sigma_j\rangle]\rangle_S$, $\mathcal{G}[r_i - r_j] = K[r_i - r_{]}G[r'_i - r'_j]K[r'_j - r_j] + K[r_i - r_j]$ (convolution:$\mathcal{G}_{ij} = K_{ii'}G_{i'j'}K_{j'j} + K_{ij}$)

Fourier transform on the lattice:

$$\hat{F}[\vec{q} = \sum_{j=1}^{N} e^{-i\vec{q}\cdot\vec{r}_j} F[\vec{r}_j]$$

$$F[\vec{r}_j] = \frac{1}{N}\sum_{\vec{q}}^{BZ} e^{i\vec{q}\vec{r}_j}\hat{F}(\vec{q})$$

lattice of $\vec{q}$ is called Brillouin zone or Dual lattice. Reminder: Duscrete Fourier transforms: pass.

Picture for $\hat{G}(\vec{q})$: exponential decay: width $\propto 1/\xi(T) \propto |t|^\nu$. $\hat{K}[\vec{q}]$ = featureless: $\hat{K}[\vec{q} = 0] = \sum_{i=1}^{N} K_{ij} = \beta\left(\sum_{i=1}^{N} J_{ij}\right) = \tilde{J}/T = T^{MFT}/T \simeq$ of order unity

will show shortly: $T_c^{MFT}/T \leq \hat{K}[\vec{q}] \leq T_c^{MFT}/T$. Wiil show: $\hat{K}[\vec{q}] = \hat{K}[0]\left\{1 - \frac{R^2}{2d}(\vec{q}\cdot\vec{q}) + O(q^4x + q^2q^2y + qy^4\right.$ Hence: $\hat{K}[\vec{q}] \simeq 1$ for those values of $\vec{q}$, where $\hat{G}[\vec{q}]$ is not small. comments: use find to sicuss the eigenvalues of the matrix $K_{ij} = K[\vec{r}_i - \vec{r}_j]$: $K_{ij} = \beta J_{ij}$

Fourier: $\hat{K}[\vec{q}] = \sum_{j=1}^{N} e^{-i\vec{q}(\vec{r}_j - \vec{r}_i)} K[\vec{r}_j - \vec{r}_i] = \sum_{j=1}^{N} \cos[\vec{q}(\vec{r}_j - \vec{r}_i)]K_{ji}$, note $\sum K_{ij} e^{-i\vec{q}\vec{R}_j} = \hat{K}[\vec{q}]e^{-i\vec{q}\vec{r}_i}$. (matrix, eigenvector, eigenvalue, eigenvector) $\{\hat{K}[q]\}$ the spectrum of eigenvalue of matrix K_{ij} Also: $-1 \le \cos \le +1$, $-\hat{K}[\vec{q}=0] \le \hat{K}[\vec{q}] \le \hat{K}[\vec{q}=0]$.

MFT: $\sqrt{2/2\pi}^N \mathcal{Z} = \int d^N \vec{y} e^{-S\{y_k\}} \to (MFT)e^{-S\{\bar{y}_k\}}$, $0 = \partial S/\partial y_i|_{\hat{y}_i} = (K^{-1}_{ik}(\bar{y}_k - h_k) - A'(\bar{y}_i) = \tanh(\bar{y}_i)$. physically most interesting case: $h_i = h = \beta H = H/k_B T$ =uniform magnitude fields: translational inv. saddle point: $\hat{y}_i \hat{y}$ = uniform. improve on MFT as : a photo in iphone. t

April 28

We will finish Ising model, and soo start renormalization group.

$d = 1$: classical Ising stat mech

$H - T$ plot (phase diagram):

$$Z[K_{ij}h_i, N] = \mathrm{Tr}_{\{\sigma_k\}} \exp\left[\frac{1}{2}\sum_{i,j=1}^{N} \sigma_i K_{ij} \sigma_j\right] = e^{-\frac{1}{2}\sum\limits_{i,j=1}^{N} h_i (K^{-1}_{ij}) h_j} \left[\left(\frac{2}{\sqrt{2\pi}}\right)^N \frac{1}{\sqrt{\det K}} \int d^N \vec{y} e^{-S\{y_k\}}\right]$$

, where $\left(\frac{2}{\sqrt{2\pi}}\right)^N \frac{1}{\sqrt{\det K}} \int d^N \vec{y} e^{-S\{y_k\}} = \mathcal{Z}[K_{ij}, J_i, N]$, where $x_j = y_j - h_j$, $J_i = \sum\limits_{k=1}^{N} (K^{-1})_{ik} h_k$,

New generalization DEF of MFT:

$\int d^N \vec{y} e^{-S\{y_k\}} \overset{MFT aprox.}{\longrightarrow} e^{-S\{\bar{y}_k\}}$, saddle point approx: $- = \frac{\partial S\{y_k\}}{\partial y_i}\Big|_{\bar{y}_i} = \sum\limits_{k=1}^{N} (K^{-1})_{ik}(\bar{y}_k - h_k) - A'(\bar{y}_i)$, where $A'(\bar{y}_i) = \tanh(\bar{y}_i)$,

imporve on this : $h_i = h = \beta H = \frac{H}{k_B T}$, uniform applied field), tanslational invariant. $\bar{y}_i = \bar{y}$ =unifor, $y_i = \bar{y} + \delta y_i$, $\int d^N \vec{y} = \int d^N (\delta \vec{y})$,

$$-S\{y_k\} = -S\{\bar{y} + \delta y_k\} = \underbrace{-S\{y_k = \bar{y}\}}_{\text{uniform part}} \cancel{-\sum_{i=1}^{N} \left[\frac{\partial S}{\partial y_i}\right]\Big|_{y_k = \bar{y}} \delta y_i} \underbrace{-\frac{1}{2}\sum_{i,j=1}^{N} \left[\frac{\partial^2 S}{\partial y_i \partial y_j}\right]\Big|_{y_k = \bar{y}} \delta y_i \delta y_j}_{\text{quadratic functuation of } y_k} + O(\delta \vec{y})^3$$

Thus $-S\{y_k\} = -\frac{1}{2}\sum\limits_{i,j=1}^{N} y_i (K^{-1})_{ij} y_j + \sum\limits_{i=1}^{N} A(y_i) + \sum\limits_{i=1}^{N} J_i y_i$,

uniform part: $-S\{y_k = y\} = -\frac{1}{2}\sum\limits_{ij} y (K^{-1})_{ij} y + NA(y) + \sum\limits_{ij,} h (K^{-1})_{ij} h$, we will write: $\sum\limits_{j=1}^{N} K_{ij} = \hat{K}[\vec{q}=0]$, an eigenvalues ofit is K_{ij}, $\sum\limits_{j=1}^{N} (K^{-1})_{ij} = 1/\hat{K}[\vec{q}]$, an eigenvalue of matrix: $(K^{-1})_{ij}$. $\hat{K}[\vec{q}=0] = \frac{T^{MFT}}{T} = \frac{T_c^0}{T}$,

$\Rightarrow -S\{y_k = y\} = -N\left\{\frac{1}{2}\frac{1}{\hat{k}[\vec{0}]} y^2 - A(y) - h\frac{1}{\hat{K}[\vec{0}]} y\right\}$ defines $\equiv = -N\beta \mathcal{L}(y)$, Landau function. remember that Landau func is not unique.

$$N \to \infty(\text{ thermal dynamic limit}) \to \text{ minimize } \mathcal{L}_{T,H}(y)$$

quadratic fluctuation: matrix $M_{ij} = \left[\frac{\partial^2 S}{\partial y_i \partial y_i}\right]_{y_k=\bar{y}=const.} = (K^{-1})_{ij} - \delta_{ij} A''(\bar{y})$ $A'(y) = \tanh(y) \to A''(y) = \tanh'(y) = \frac{1}{\cosh^2(y)}$, uniform field: $\bar{m} = \bar{m}_i = \langle\sigma_i\rangle^{MFT} \overset{(!)}{=} \tanh(\bar{y})$.

(!): $\langle\sigma_i\rangle \overset{ingeneral}{=} \sum_{j=1}^{N} (K^{-1})_{ij}[\langle\bar{y}_j\rangle - h_j] \overset{\text{at saddle point; under MFT}}{=} A'(\bar{y}_i) = \tanh(\bar{y}_i)$, translationan invariance case: $\langle\sigma_i\rangle^{MFT} = \tanh(\bar{y})$.

we get $\bar{y} = \operatorname{arctanh}(\bar{m})$, $A''(\bar{y}) = \frac{1}{\cosh^2[\operatorname{arctanh}(\bar{m})]} = 1 - (\bar{m})^2$, $(\cosh[\operatorname{arctanh}(x)] = (1-x^2)^{-1/2})$

$-S\{y_k\} = -S\{\bar{y} + \delta y_k\} = -N\beta\mathcal{L}_{T,H}(\bar{y}) - \frac{1}{2}\sum_{i,j}^{N} \delta y_i A_{ij} \delta y_j + O(\delta\vec{y})^2$

have a landau function to be minimized:

$\beta\mathcal{L}_{T,H}(y) = \frac{1}{2}\frac{1}{\hat{K}(0)} y^2 - A(y) - h\frac{1}{\hat{K}[0]} \cdot y$, $\bar{y} = minimum.$

$\hat{K}[0] = \sum_j K_{ij} = \beta \sum_j J_{ij} \beta\tilde{J} = \frac{T_c^{MFT}}{T} = \frac{T_c^0}{T}$. $A(y) = \ln\cos(y) = \frac{1}{2}y^2 - \frac{1}{12}y^4 + O(y^6)$,

$*M_{ij} = (K^{-1})_{ij} - \delta_{ij}[1 - \bar{m}^2]$ where $\bar{m} = \tanh(\bar{y})$ =magnetization per site in MFT.

*have recovered formulation at MFT using a Landau function.

*But noe can also discuss fluctuations δy_i in MFT as follows:

For simplicity: $H = h = 0$, $T > T_c^0$: since $\bar{m} = 0$ at $T > T_c$, we have $\bar{m} = \tanh(\bar{y})$, $\bar{y} = 0$. But $y_i = \bar{y}_i + \delta y_i = \delta y_i$,

$$-S\{y_k\} \to -S_0\{y_k\} = -\frac{1}{2}\sum_{i,j=1}^{N} y_i h_{ij} y_j + O(y^4)$$

Now can compute spin-spin correlation function with this DEF of MFT:

$\mathcal{G}^0[r_i - r_j] = \mathcal{G}^0_{ij} = \langle y_i y_j\rangle_{S_0} = \frac{\int d^N\vec{y} y_i y_j e^{-\frac{1}{2}y_k M_{kl} y_l}}{\int d^N\vec{y} e^{-\frac{1}{2}y_k M_{kl} y_l}} = (h^{-1})_{ij} = [(K^{-1} - 1)^{-1}]_{ij} = [(K^{-1}(1 - K))^{-1}]_{ij} = [(1-K)^{-1}K]_{ij}$, (++)

$M_{ij} = (K^{-1})_{ij} - \delta_{ij} = [K^{-1} - 1]_{ij}$, $H = 0, T > T_c^0 \to \bar{m} = \tanh(\bar{y}) = 0$,

$\mathcal{G}^0[r_i - r_j] = \langle\sigma_i\sigma_j\rangle_{MFT} = G^0_{ij}$, $\mathcal{G}^0[r_i - r_j] = K[r_i - r_{i'}]G^0[r_{i'} - r_{j'}]K[r_{j'} - r_j] + K[r_i - r_j]$, best discussed in Fourier space

eq. (++) becomes $\hat{\mathcal{G}}[\vec{q}] = (1 - \hat{K}[\vec{q}])^{-1}\hat{K}[\vec{q}] = \frac{\hat{K}[\vec{q}]}{1 - \hat{K}[\vec{q}]}$, $\hat{F}[\vec{q}] = \sum e^{-i\vec{q}\vec{r}_j} F[\vec{r}_j]$, $F[\vec{r}_j] = \frac{1}{N}\sum_{\vec{q}}^{BZ} e^{i\vec{q}\cdot\vec{r}_j}\hat{F}[\vec{q}]$, we get $\hat{\mathcal{G}}^0[\vec{q}] = \hat{K}[\vec{q}]\mathcal{G}^0(\vec{q})\hat{K}[\vec{q}] + \hat{K}[\vec{q}] = (\hat{K}[\vec{q}])^2\hat{G}^0[\vec{q}] + \hat{K}[\vec{q}]$.

We get

$$\hat{G}^0[\vec{q}] = \frac{1}{1 - \hat{K}[\vec{q}]}.$$

General DEF of correlation length

$\xi^2 = \frac{\frac{1}{2d}\sum\limits_{i=1}^{N}(\vec{r}_i-\vec{r}_j)^2\mathcal{G}[\vec{r}_i-\vec{r}_j]}{\sum\limits_{i=1}^{N}\mathcal{G}[\vec{r}_i-\vec{r}_j]}$

Express in terms of Fourier:

$$\hat{\mathcal{G}}[\vec{q}] = \sum_{j=1}^{N} e^{-i\vec{q}\cdot\vec{r}_j}\mathcal{G}[\vec{r}_j] = \underbrace{\sum_{j=1}^{N}\mathcal{G}[\vec{r}_j]}_{\hat{\mathcal{G}}[\vec{q}=0]} - \frac{1}{2}\left[(qx)^2\sum_{j=1}^{N}(x_j)^2\mathcal{G}_j[\vec{r}_j] + (qy)^2\sum(y_i^2)\mathcal{G}[\vec{r}_j] + \cdots\right] + O(\vec{q}^4),$$

where the x and y terms are same by rotation symmetries about 90°, thus we get

$$\hat{\mathcal{G}}[\vec{q}] = \hat{\mathcal{G}}[\vec{q}=0] - \frac{1}{2}(\vec{q}\cdot\vec{q})\left(\sum_{j=1}^{N}(x_j)^2\mathcal{G}[\vec{r}_j]\right) + O(\vec{q}^4) = \hat{\mathcal{G}}[\vec{q}=0]\left\{1-(\vec{q}\cdot\vec{q})\xi^2 + O(\vec{q}^4)\right\} = \hat{\mathcal{G}}[\vec{q}]$$

May 3

$$\mathcal{G}_{ij} = \mathcal{G}_{[}\vec{r}_i - \vec{r}_j] = \langle[Y_i - \langle Y_i\rangle][Y_j - \langle Y_j\rangle]\rangle, \quad \hat{\mathcal{G}}[\vec{q}] = \sum_{j=1}^{N} e^{-i\vec{q}\cdot\vec{r}_j}\mathcal{G}[\vec{r}_j]$$

$$\frac{\hat{\mathcal{G}}[\vec{q}]}{\hat{\mathcal{G}}[\vec{q}=0]} = 1 - \xi^2(\vec{q}\cdot\vec{q}) + O(\vec{q}^4), \quad \xi^2 = \frac{\frac{1}{2}\frac{1}{d}\sum\limits_{j=1}^{N}(\vec{r}_j)^2\mathcal{G}[\vec{r}_j]}{\sum_j \mathcal{G}[\vec{r}_j]}$$

$$\frac{\hat{K}[\vec{Q}]}{\hat{K}[\vec{q}=0]} = 1 - (\frac{1}{2}\frac{1}{d}R^2)(\vec{q}\vec{q}) + O(\vec{q}^4), \quad \frac{1}{2}\frac{1}{d}R^2 = \frac{\frac{1}{2d}\sum\limits_{j=1}^{N}(\vec{r}_j)^2 K[\vec{r}_j]}{\sum_j K[\vec{r}_j]}$$

Now work out $\xi \to \xi_0$ (MFT) in MFT: use $\hat{\mathcal{G}}^0[\vec{q}] = \frac{\hat{K}[\vec{Q}]}{1-\hat{K}[\vec{q}]}$,

$$\frac{\hat{\mathcal{G}}^0[\vec{q}=0]}{\hat{\mathcal{G}}^0[\vec{q}]} = \frac{\hat{K}[0]}{1-\hat{K}[0]}\frac{1-\hat{K}[\vec{q}]}{\hat{K}[\vec{q}]} = \frac{\hat{K}[0]}{\hat{K}[\vec{q}]}\frac{1-\hat{K}[\vec{q}]}{1-\hat{K}[0]} = \frac{\hat{K}[0]}{\hat{K}[\vec{q}]}\frac{(1-\hat{K}[0]) + (\hat{K}[0]-\hat{K}[\vec{q}])}{1-\hat{K}[0]} = \frac{\hat{K}[0]}{\hat{K}[\vec{q}]}(1+\frac{K[0]-K[\vec{q}]}{1-\hat{K}[0]})$$

$$= \left(1 + \frac{1}{2d}R^2\vec{q}^2 + O(\vec{q}^4)\right)\left(1 + \frac{\hat{K}[0]-\hat{K}[\vec{q}]}{1-\hat{K}[0]}\right) = 1 + (1 + \frac{\hat{K}[0]}{1-\hat{K}[0]})\frac{1}{2d}R^2(\vec{q}^2) + O(\vec{q})^4) = 1 + \frac{1}{1-\hat{K}[0]}\frac{1}{2d}R^2\vec{q}^2 + O(q^4$$

thus $\xi_0^2 = \frac{1}{1-\hat{K}[0]}\frac{1}{2}\frac{1}{d}R^2$.

$\hat{K}[0] - \hat{K}[\vec{q}] = \hat{K}[0](\frac{1}{2d}R^2\vec{q}^2 + O(\vec{q}^4))$, recall $\hat{K}[0] = \sum\limits_{j=1}^{N} K_{ij} + \beta\sum\limits_{j=1}^{N} J_{ij} = \beta\tilde{J} = \frac{k_B T_c^0}{k_B T} = \frac{T_c^0}{T}$, where $t = (T - T_c^0)/T_c^0$, thus $\frac{1}{1-\hat{K}[0]} = \frac{1}{1-\frac{T_c^0}{T}} = 1 + \frac{1}{t}$, thus $\xi_0^2 = (1+\frac{1}{t})\frac{1}{2d}R^2$ → in MFT: $\xi_0 \propto t^{-\gamma}$, where $\gamma = \frac{1}{2}$ in MFT.

Now: remember (use $\hat{\mathcal{G}}^0[\vec{q}=0] = \frac{\hat{K}[0]}{\frac{1}{2d}R^2}\frac{\frac{1}{2d}R^2}{1-\hat{[}0]} = \frac{\hat{K}[0]}{\frac{1}{2d}R^2}\xi_0^2$)

$$\frac{1}{\hat{\mathcal{G}}^0(\vec{q})} = \frac{1}{\hat{\mathcal{G}}^0[q \vec{=} 0]}(1+\xi_0^2\vec{q}^2+O(\vec{Q}^4)) = \frac{\frac{1}{2d}R^2}{\hat{K}[0]}(\frac{1}{\xi_0^2}+\vec{q}^2+O(\vec{q}^4))$$

Extracting the field theory at $T \simeq T_c$: simplicity $H=0$ (zero field) $T > T_c$,

$$\mathcal{Z}[K_{ij}, h_i = 0, N] = \frac{1}{\sqrt{\det K}}\left(\frac{2}{\sqrt{2\pi}}\right)^N \int d^N\vec{y}e^{-S\{Y_k\}}$$

$$S\{Y_k\} = \frac{1}{2}\sum_{i,j=1}^N Y_i(K_{ij}^{-1}Y_j - \sum_{i=1}^N A(Y_i)$$

second term $A(y) = \ln\cosh(Y_{=}\frac{1}{2}y^2 - \frac{1}{!2}y^4 + O(y^6)$, first term: in fourier space : $\leftrightarrow \hat{Y}[\vec{q}] = \sum_{i=1}^N e^{-i\vec{q}\cdot\vec{r}_i}y_i$, thus first term is $= \frac{1}{2}\frac{1}{N}\sum_{\vec{q}}^{BZ}\hat{Y}[\vec{q}]\frac{1}{\hat{K}[\vec{q}]}\hat{y}[-\vec{q}]$,

$S_0\{Y_k\}$ =quadratic part pf $S\{Y_k\}$: $= -\frac{1}{2}\frac{1}{N}\sum_{\vec{q}}^{BZ}\hat{Y}[\vec{q}]\left(\frac{1}{\hat{K}[\vec{q}]}-1\right)\hat{Y}[-\vec{q}] = \frac{1}{\hat{\mathcal{G}}[\vec{q}]} = \frac{\frac{1}{2d}R^2}{[\hat{K}[0]]}(\frac{1}{\xi_0^2} + \vec{q}^2 + O(\vec{q})^4)$

Cosmetic redefinitions

$\phi(\vec{r}_i) = \frac{1}{a^{d/2}}a[\frac{1}{\hat{K}[0]}\frac{(R/a)^2}{2d}]^{1/2}Y_i$, $\tilde{\phi}(\vec{q}) = a^{d/2}a\left[\frac{1}{\hat{K}[0]}\frac{(R/a)^2}{2d}\right]^{1/2}\hat{Y}[\vec{q}]$

second fourier convition: $\sum_{j=1}^N a^d\phi(\vec{r}_i) \leftrightarrow \int d^d\vec{r}\phi(\vec{r})$, $\frac{1}{vol}\sum_{\vec{q}}^{BZ}\tilde{\phi}(\vec{q}) \leftrightarrow \int \frac{d^d\vec{q}}{(2\pi)^d}\tilde{\phi}(\vec{q})$.

$$S\{\phi(\vec{r})\} = \int d^d\vec{r}\left[\frac{1}{2}\frac{1}{\xi_0^2}\phi^2(\vec{r}) + \frac{1}{2}(\nabla\phi)(\nabla\phi)(\vec{r}) + \text{ higher gradients } + \frac{\lambda}{4!}\phi^4(\vec{r}) + \text{ higher even powers of } \phi(\vec{r})\right]$$

, where $\lambda = 2\left[\frac{2d\hat{K}[0]}{(R/a)^2}\right]^2 a^{d-4}$, (Coeff. $\int d^d r\phi^{2n}(\vec{r})$, the Coeff. is $\left[\frac{2d\hat{K}[0]}{(R/a)^2}\right]^n a^{(n-1)d-2n}$.

This is Landau - Ginzburg Theory.

$\int d^d\vec{r}(\nabla\phi(\vec{r})(\nabla\phi(\vec{r})$, $\int d^d\vec{q}i\vec{q}\tilde{\phi}(\vec{q})i(-\vec{q})\tilde{\phi}(\vec{q})$.

General renormalization group idea. near crit. point: $a \ll r \ll \xi$, look at system at scale "r". For $d=2$: coarse grained block spin (Leo Kadanoff) .

May 5

$\sigma_i = \pm 1 = S_i$,

$-\mathcal{H}\{S_i\} = K_0 + K_1\sum_i S_i + K_2\sum_{\langle i,j\rangle}^{n.n.} S_iS_j + K_3\sum_{\langle i,j,k\rangle}^{3n.n} S_iS_jS_k + K_4\sum_{u,j}^{n.n.n.} S_iS_j + \cdots$

label all local terms (clusters) of spins.

$\mathcal{H}\{S_i\}$ (and $Ze^{-\beta\mathcal{H}}$) parameterized characterized) by a large (∞ dimensional) vector: $\vec{K} = (K_1, K_2, K_3, K_4, \cdots)$

	before	after
interac.params.=coupling const.	$K_0, \vec{K}$	$K_0' = K_0'(\vec{K}), \vec{K}' = \vec{K}'(\vec{K})$
lattice spacing	a	$a' = ba$, $b > 1$, here $b = 3$
number of sites	N	$N' = N/b^d$

Let us do ourselves a favor: Z_2-symmetry: $S_i \to -S_i$, $K_m \to (-1)K_m$,

RG flow: $\mathcal{H}^1\{S_I'\}$ $\vec{K}' = (K_1', K_2', K_3', \cdots)$

DEF of RG transformation: "Projectoin operator":

$T(S_I', s_1, s_2, \cdots, s_q) = \delta_{S_I', \mathrm{sgn}(\sum_{i\in I} s_i)}$ (Major row'l)

T has properties: 1. $T(S_I^1, \{s_i\}_{i\in I}) \geq 0$, 2. T respects the symmetries; 3. $\sum\limits_{S_I'=\pm 1} T(S_I^1, \{s_i\}_{i\in I}) =$ 1.

Starting from T we can compute $\mathcal{H}'$ from $\mathcal{H}$:

$$\mathcal{Z} = \mathrm{Tr}_{\{S_i\}} e^{-\mathcal{H}\{S_i\}} = \mathrm{Tr}_{\{s_i\}} \left[\prod_I^{blocks} 1\right] e^{-\mathcal{H}\{s_i\}} = \mathrm{Tr}_{\{s_i\}} \left[\prod_I^{blocks} \left\{(\sum_{S'=\pm 1})T(S_I', \{s_i\}_{i\in I})\right\}\right] e^{-\mathcal{H}\{s_i\}}$$

$$= \mathrm{Tr}_{S_I'=\pm 1} \left(\mathrm{Tr}_{\{s_i\}} \left[\prod_I^{blocks} T(S_I', \{s_i\}_{i\in I})\right] e^{-\mathcal{H}\{s_i\}}\right) = \mathrm{Tr}_{\{S_I'\}} e^{-\mathcal{H}'\{S_I'\}}$$

Summary: before R.G.transformation: $\mathcal{H}\{s_i\}$, after R.G. transformation: $\mathcal{H}'\{s_I'\}$,

parameterized L $\vec{K}(K_1, K_2, \cdots)$ and K_0,; after: $\vec{K}' = (K_1', K_2', \cdots)$ and K_0'. we get $K_0, \vec{K} \to K_0' = K_0'(\vec{K}), \vec{K}' = \vec{K}'(\vec{K})$, [In practice: this is a tedious calculation, which is in Goldenfeld, 962, 9.64)

$\mathcal{Z} = \mathrm{Tr}_{\{s_i=\pm 1\}} e^{-\mathcal{H}\{s_i\}} = \mathrm{Tr}_{\{S_I'=\pm 1\}} e^{-\mathcal{H}'\{S_I'\}}$

Proliforation of couplings.

If switch off nagmetic field: all $K_{2n+1} = 0$ (R.G. preserves symmetry) $\vec{K} = (0, K_2, 0, K_4, \cdots)$ thus after RG: $K_{2n+1}' = 0$, $\vec{K}' = (0, K_2', 0, K_4', \cdots)$ [$s_i \to (-s_i)$ for all i, $K_m \to (-1)^m K_m$]

Now: the RG transformation: $\vec{K} \to \vec{K}'[\vec{K}] = \vec{\mathcal{R}}_b[\vec{K}]$,

general RG Theory: Many important cnclusion can be drawn from the fact that $\vec{\mathcal{R}}_b$ exists and is analytic (Taylor). Assume that RG transformation $\vec{\mathcal{R}}_b$ has a fixed point at $\vec{K} = \ vecK^*$, $\vec{K}^* = \vec{\mathcal{R}}_b[\vec{K}^*]$

Analyticity of $\vec{\mathcal{R}}_b[\vec{K}]$ allows us to Taylor expand about fixed point $K_\alpha' = K_\alpha'[\vec{K}] = \left(\mathcal{R}_b[\vec{K}]\right)_\alpha = \left(\mathcal{R}_b[\vec{K}^*]\right)_\alpha + \sum_\beta T_{\alpha\beta}(K_\beta - K_\beta^*) + O(\vec{K} - \vec{K}^*)^2$, where $\left(\mathcal{R}_b[\vec{K}^*]\right)_\alpha = K_\alpha^*$, where $T_{\alpha\beta} = \frac{\partial K_*'}{\partial K_\beta}\Big|_{\vec{K}=\vec{K}^*} = \frac{\partial}{\partial K_\beta}(\mathcal{R}_b[\vec{K}])|_{\vec{K}=\vec{K}^*}$

or: $(K'_\alpha - K^*_\alpha) = \sum_\beta T_{\alpha\beta}(K_\beta - K^*_\beta) + O()^2$, or $(\vec{K}' - \vec{K}^*) - T(\vec{K} - \vec{K}^*) + O()^2$, in matrix form.

The Boltzmann wave is totally benign: it can be taylor expanded.

May 10

$\mathcal{Z} = \mathrm{Tr}^{sites}_{s_i=\pm 1} e^{-\mathcal{H}\{S_i\}} = \mathrm{Tr}^{blocks}_{s'_I=\pm 1} e^{-\mathcal{H}\{S'_I\}}$

$\vec{K} \to \vec{K}' = \vec{\mathcal{R}}[\vec{K}]$: existence + analyticity/ fixed point: $\vec{\mathcal{R}}(\vec{K}^*) = \vec{K}^*$.

$\vec{K} = (K_1, K_2, \cdots)$, $(K'_\alpha - K^*_\alpha) = \sum_\beta T_{\alpha\beta}(K_\beta - K^* + \beta)$.

Assume $T_{\alpha\beta}$ is diagonalizable with real eigenvalues (usually the case)

$$T_{\alpha\beta} = \frac{\partial K'_\alpha}{\partial K_\beta}\Big|_{\vec{K}=\vec{K}^*} = \frac{\partial}{\partial k_\beta}\Big|_{\vec{K}=\vec{K}^*}(\mathcal{R}_b[\vec{K}])_\alpha$$

nonisingular matrix Φ, $\Phi T \Phi^{-1} = \Lambda$, where Λ is diagonal matrix,

Hence the row-vectors of Φ are the left-eigenvectors:

$$\Phi = \begin{pmatrix} \Phi^{(1)}_1 & \Phi^{(1)}_2 & \Phi^{(1)}_3 & \Phi^{(1)}_4 & \cdots \\ \Phi^{(2)}_1 & \Phi^{(2)}_2 & \Phi^{(2)}_3 & \Phi^{(2)}_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix} = \begin{pmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \vdots \end{pmatrix}$$

$\Phi T = \Lambda \Phi$, $\sum_\alpha \Phi^{(0)}_\alpha T_{\alpha\beta} = \lambda^i \Phi^i_\beta$

Boltzmann weight in vicinity of F.P. Make a transformation from $(\vec{K} - \vec{K}^*)$ to a set of variables denoted by $\vec{u}$. hh corr. scaling variables (field)

$u_i = \sum\limits_\alpha \Phi^{(i)}_\alpha (K_\alpha - K^*_\alpha)$, u_i's transform simply under RG transformation;

$$R.G. \vec{K} \to \vec{K}', \vec{u} \to \vec{u}'$$

$u'_i = \sum_\alpha \Phi^{(i)}_\alpha (K'_\alpha - K^*_\alpha) + O(\vec{K} - \vec{K}^*)^2$

$$= \sum_\alpha \Phi^{(i)}_\alpha \sum_\beta T_{\alpha\beta}(K_\beta - K^*_\beta) = \sum_\beta \lambda^{(i)} \Phi^{(i)}_\beta (k_\beta - k^*_\beta) + \ldots = \lambda^{(i)} \sum_{beta} \Phi^{(i)}_\beta (K_\beta - K^*_\beta) + \cdots = \lambda^{(i)} u_i + O(\vec{u}^2)$$

Def: the direction u_i near the fixed points $\vec{K}^*$ is called

relevant (growing under RG) if $\lambda^{(i)} > 1$, notation : $\lambda^{(i)} = b^{y_i}, y_i > 0$;

irrelevant (decreasing) if $\lambda^{(i)} < 1$, $\lambda^{(i)} = b^{y_i}, y_i < 0$;

marginal if $\lambda^{(i)} = 1$, $b =$ rescaling factor $y_i = 0$.

$e^{\ln \lambda^{(i)}} = (e^{\ln b})^{y_i} = e^{y_i \ln b}$, y_i is called the RG eigenvalue of the variable u_i at the fixed point $\vec{K}^*$.

a picture of RG flow: in iphone.

Global properties of RG flows and universality.

Simpicity, first zero magnetic field: $H = 0 = K_j$, $K_{2m+1} - 0$ for all m.

$\vec{K} = (0, K_2, 0, K_4, 0, K_6, ...)$ $\vec{K}^* = (0, K_2^*, 0, K_4^*, \cdots)$ large space of couplings

Illustrate sgnificance of fixed point: toy example

$\vec{K} = (0, K_2, 0, K_4)$, $K_2 = \frac{J_2}{T}$, $=$ n.n. interaction; $K_4 = \frac{J_4}{T}$: $=$ n.n.n interaction.

$\mathcal{H} = K_2 \sum_{\langle i,j \rangle}^{n.n.} s_i s_j + K_4 \sum_{\langle\langle i,j \rangle\rangle}^{n.n.n} s_i s_j$

$\vec{K} = (k_2, k_4 = 0) \rightarrow \vec{K}^{(1)} = (K^{(1)}, K_4^{(1)}) \rightarrow \vec{K}^{(2)} \rightarrow ... \rightarrow \vec{K}^{(n)} = (K_2^{(n)}, K_4^{(n)})$: critical surface (or manifold) = set of points that are eventually attracted into the fixed point

Consider case: u_2 relevant, u_4 irrelevant.

Simple but important statement: when $\vec{K}(k_2, k_4)$ is on the critical manifold (surface), then $\xi[\vec{K}] = \infty$.

Behavior of correlation length under RG: $\xi =$ cm or $A°$. dimensionaless correlation length $\overline{\xi}[\vec{K}] = \xi/a$.

under RG, ξ remains unchanged

$\overline{\xi}[\vec{K}'] = \xi/a' = \xi/ba = \frac{1}{b}\xi/a = \frac{1}{b}\overline{\xi}[\vec{K}]$

At an instable fixed point $\vec{K}^*$ the dimensionless correlation length $\overline{\xi}[\vec{K}^*] = \infty$. (This will be derived next time)

The key argument: as long as at fie point: as long as there are very few relevent. Even the space we are dealing with is prohibitively big, but everything that determines the correlation exponents comes out from the fix point.

May 12

Statement: When $\vec{K} = (k_2, k_4)^T$, is on the critical surface, then $\overline{\xi}[\vec{K}] = \infty$.

i. correlation length $\overline{x}[K^{\#}] = \infty$, for $K^{\#} =$ unstable fixed point, $\overline{\xi}[\vec{K}] = b\overline{\xi}[\vec{K}'] = b\overline{\xi}[\vec{K}^{(1)}]$, $\overline{\xi}[\vec{K}^{(1)}] = b\overline{\xi}[\vec{K}^{(2)}]$, and so on.

$\overline{\xi}[\vec{K}] = b^n\overline{\xi}[\vec{K}^{(n)}]$, along n_2- axis, $\overline{\xi}[u_2, u_4 = 0] = b^n\overline{\xi}[(\lambda_2)^n u_2, u_4 = 0]$, $(\lambda_2)^n u_2 = u_2^{(n)}$, $(\lambda_2)^n u_2 = u_2^{(n)} = K_2 =$fixed, $\lambda_2 = b^{y_2}$, $(b^{y_2})^n u_2 = K_2$, $(b^n)^{y_2} = K_2/u_2$, $b^n = (K_2/u_2)^{1/y_2}$

$\overline{\xi}[u_2, u_4 = 0] = (K_2/u_2)^{1/y_2}\overline{\xi}[K_2, u_2, u_4 = 0]$ $\nu = 1/y_2$ as $u_2 \rightarrow 0$, i.e. as we approach unstable fixed point, $\overline{\xi}[u_2, u_4] \rightarrow \infty$

ii $\overline{\xi}[\vec{K}] = b^n\overline{\xi}[\vec{K}^{(n)}]$, now if $\vec{K}$ is on critical surface, then $\vec{K}^{(n)}$ is also on it. $n \rightarrow \infty$: $\overline{\xi}[\vec{K}] \rightarrow \infty, \overline{\xi}[\vec{K}^*] = \infty \cdot \infty = \infty$.

$H = 0$: have to tune only /oaramemters to set the original system to have $\overline{\xi} = \infty$, to put the Boltzmann weight onto the critical surface.

E,g, choose Boltzman weight whic has fixed J_2, J_4:

$(k_2, k_4) = (J_2/T, J_4/T)$

We don't vary u's in lab; we vary the temeperature.

Relationship between linear coordinates u_i and physical parameters such as T, H.

Toy example (2 couplings) :$(k_2, k_4) \to (u_2, u_4)$

Imagine starting with only nn couplings ($K_4 = 0$), thus on the K_2 axis

iterate many many times: $\vec{K}^{(n)} = \vec{\mathcal{R}}\vec{\mathcal{R}}...\vec{\mathcal{R}}\vec{\mathcal{R}}(k_2, k_4 = 0)$, where $(k_2, k_4 = 0)$ is in region A, and $\vec{K}^{(n)}$ is in region B, decribed by $(u_2(t), u_4(t))$, $K_2 = J_2/T$,

$t = (T - T_C)/T_C$, $u_2(t)$ and $u_4(t)$ are analytic in t (taylor), $\vec{\mathcal{R}}$ is analytic, and m =finite, thus after the interation is also analytic.

$u_2(t) = u_2^{(0)} + u_2^{(1)}t + O(t^2)$, $u_4(t) = u_4^{(0)} + u_4^{(1)} + O(t^2)$, claim: $u_2^{(0)} \equiv = 0$. (coordinates of critical surface are $u_2 = 0, u_4$; $0 = u_2(t = 0) = u_2^{(0)} + 0$.)

Generalization to more than two couplings still $H = 0$.

Assume we have M even couplings.

$\vec{K} = (0, k_2, 0, k_4, \cdots)$

What is the dimensionality of the critical surface (manifold) in this M-dimensional space?

Answer: we jniw we have to adjust one parameter (namely T) to put the system at ∞ corr. length.So, critical surface is a $(M - 1)$ dimensional surface in the M dimensional space of all M couplings. In vacinity of fixed point, we have coordinates $\vec{u} = (0, u_2, 0, u_4, \cdots, u_{2M})$, M coordinates.

Example: $\vec{K} = (0, k_2, 0, k_4, 0, k_6)$, $M = 3$,

only one of the u_i's is relevant (by convention u_2 relevant) all others are irrelavent $u_4, u_6, \cdots$

Including a non-zero nagnetic field ($H \neq 0$)

initial a Hamiltonian , containing only $K_1 = H/T, K_2 = J_2/T, K_m \equiv 0, m \geq 3$ after repeated RG transformations: $\vec{K} = (K_1, K_2, K_3, K_4, \cdots, K_{2M})$ (2M couplings)

Need to adjust two parameters to bring systems to critical point, i.e. to make $\overline{x\xi} \equiv \infty$, i.e. to put it on its critical surface. Thus critical surface is a $(2M - 2)$ dimensional surface in the $2M - dim'l$ space of all Boltzman weights.

In vicinities of fixed point, we have coordinates ($\vec{u} = (u_1, u_2, \cdots, u_{2M})$, Two of these u_i's must be relevant (convention, u_1, u_2) u_1 switches sign if we flip sign of spins by convention; u_2 doesn't. (Note: we make the assumption that RG transformation preserves all symmetries, $\vec{K}' = \vec{\mathcal{R}}[\vec{K}]$, $\vec{K}'$ should have same symmetry as $\vec{K}$)

May 17

lattice	a	$a' = ba > a$
	$\vec{K}$	$\vec{K}' = \vec{K}'[\vec{K}] = \vec{\mathcal{R}}[\vec{K}]$(analytic)
number of lattice sites	N	$N' = \#of blocks = N/b^d$

Goal for today: Close the global aspects of RG and then go to Landau-Ginzburg formulation.

RG trnsformation of free energy (density per site)

Notation: separate ont constants K_0, K_0'.

$\mathcal{Z} = e^{-NK_0}\mathrm{Tr}_{\{S_i=\pm1\}}e^{-\mathcal{H}_{\vec{K}}\{S_i\}} = e^{-N'K_0'}\mathrm{Tr}_{\{S_I'=\pm1\}}e^{-\mathcal{H}_{\vec{K}'}\{S_I'\}}$, partition function unchanged: $e^{-N'K_0'} = e^{-NK_0 - Ng[\vec{K}]}$, where energy (density) f the degrees of freedom thathave been tranced out (1D example: Notes) coarse granied.

See table above.

Now: $\mathcal{Z} = \cancel{e^{-NK_0}}e^{-Nf[\vec{K}]} = \cancel{e^{-NK_0}}e^{-Ng[\vec{K}]}e^{-N'f[\vec{K}']}$ where f is the same function in the middle and on the right (because we have the same type of lattice model and thus the same type Boltzmann weight). (this is important!) we get

$$f[\vec{K}] = g[\vec{K}] + b^{-d}f[\vec{K}']. \tag{4}$$

(important!!)

we have argued before that $\vec{\mathcal{R}}$ is analytic.

We expect at critical point ($\vec{K}$ approaches critical manifold) free energy f per site is non-analytic. We write

$$f[\vec{K}] = f_s[\vec{K}] + f_{reg}[\vec{K}]$$

where f_s is non-analytic and f_{reg} is analytic, for example

$$f(t) = \sqrt{t} + [A_1 t + A_2 t^2 + A_3 t^3 + \cdots]$$

where $f_s = \sqrt{t}$. Thus eq.(**??**) becomes

$$f_S[\vec{K}] = b^{-d}f_s[\vec{K}']$$

and $f_s[\vec{K}] + f_{reg}[\vec{K}] = g[\vec{K}] + (f_s[\vec{K}'] + f_{reg}[\vec{K}'])$, we get $g[\vec{K}] + b^{-d}f_{reg}[\vec{K}'] = g[\vec{K}] + b^{-d}f_{reg}[\vec{\mathcal{R}}_b[\vec{K}]]$ is anlytic of $\vec{K}$. and $b^{-d}f_s[\vec{\mathcal{R}}_b[\vec{K}]]$ is singular of $\vec{K}$. Write this equation in vicinity of unstable fixed point (from last lectures): $\vec{K}^*$:

$$f_s[u_1, u_2, u_3, u_4] = b^{-d}f_s[b^{y_1}u_1, b^{y_2}u_2, \cdots]$$

iterate to get LHS $= b^{-ud} f_s[b^{uy_1}u_1, b^{uy_2}u_2, b^{uy_3}u_3, \cdots]$

Recall: $u_1 (y_1 > 0)$ and u_2 $(y_2 > 0)$ are relevant, while u_3 $(u_3 < 0)$ and so on $(i > 3)$ are irrelevant. Singular behavior of f_s arises from iteration of RG.

Let $b^{uy_2}u_2 = K_2$. Iterate u_2: we get

$$b^n = \left(\frac{K_2}{u_2}\right)^{\frac{1}{y_2}}$$

$b^{ny_2}u_2 = K_2 = fixed$, we get

$$f_s[u_1, u_2, u_3, \cdots] = \left(\frac{u_2}{K_2}\right)^{\frac{d}{y_2}} f_s\left[\left(\frac{u_2}{K_2}\right)^{-\frac{y_1}{y_2}} u_1, K_2, \left(\frac{u_2}{K_2}\right)^{-\frac{y_3}{y_2}} u_3, \left(\frac{u_2}{K_2}\right)^{-\frac{y_4}{y_2}}, \cdots\right]$$

Crucial points: i. Boltzmann weight has infinite correlation length (the infinity is related to $b^n = (K_2/u_2)^{1/y_2}$. We can in principle also fix u_2 and let u_1 run. But we use u_2 running becomes of some physical reasoning), i.e. $u_1, u_2 \to 0$ (critical surface) ii. $y_1/y_2 > 0$, but $-y_3/y_2 > 0$, $-y_4/y_2 > 0$, etc. (all positive)

as we approach critical surface $(u_1, u_2 \to 0)$: $f_s[u_1, u_2, \cdots] = \left(\frac{u_2}{K_2}\right)^{\frac{d}{y_2}} f_s\left[\left(\frac{u_2}{K_2}\right)^{-\frac{y_1}{y_2}} u_1, K_2, 0, 0, 0, \cdots\right]$, f_s becomes insensitive to all the irrelevant u_i's. This is called the scaling form of free energy density. This shows RG has explained universality.

need to make connection with physical parameters: $t = (T - T_c)/T_c$ and $h = H/T$ $(k_B = 1)$

have h is odd, t is even, under $\mathbb{Z}_2$ symmetry $(S_i \leftrightarrow (-S_i))$

$u_1[t,h] = a_{100} + a_{110}h + a_{111}ht + \cdots$, $u_2[t,h] = a_{200} + a_{201}t + a_{220}h^2 + \cdots$, $u_3[t,h] = h[u_{310} + u_{311}t + \cdots]$, $u_4[t,h] = u_{400} + u_{401}t + u_{420}h^2 + \cdots$ small h and t: $u_1[t,h] = \frac{h}{h_0} + O(th), 1/h_0 = a_{110}$, $u_2[t,h] = t/t_0 + O(t^2, h^2), 1/t_0 = a_{201}$, $u_3[t,h] = ha_{310} + O(ht)$, $u_4[t,h] = u_4^0 + a_{401}t + a_{420}h^2 + \cdots, u_4^0 = a_{400}$

From above we have

$$f_s[u_1, u_2, u_3, u_4, \cdots] = (u_2)^{d/y_2}\Phi_{\pm}\left[\frac{u_1}{(u_2)^{y_1/y_2}}, (u_2)^{-\frac{y_3}{y_2}}u_3, (u_2)^{-\frac{y_4}{y_2}}u_4, \cdots\right]$$

where the $\pm$ sign on Φ indicates for $T > T_0$ and $T < T_0$. $\Phi_{\pm}$ is universal (because we started from most general Boltzmann weight)

$f_s[t,h] = f_s[u_1[t,h], u_2[t,h], u_3[t,h], \cdots] = (u_2[t,h])^{d/y_2}\Phi_{\pm}[\frac{u_1[t,h]}{(u_2[t,h])^{y_1/y_2 > 0}}, (u_2[t,h])^{-y_3/y_1 > 0}u_3[t,h], (u_2[t,h])^{-y_4/}$ $(t/t_0)^{d/y_2}\Phi[\frac{(h/h_0)}{(t/t_0)^{y_1/y_2}}, 0, 0, \cdots]$ This is the scaling form of free energy per site. Note the t_0 is non universal number.

May 19

$$u_1[t,h] = \frac{h}{h_0} + O(t,h)$$
$$u_2[t,h] = \frac{t}{t_0} + O(t^2,h^2)$$
$$u_3[t,h] = ha_{310} + O(h,t)$$
$$u_4[t,h] = u_4^0 + a_{401}t + a_{420}h^2 + \cdots$$
$$f_s[u_1,u_2,\cdots] = b^{-nd} f_s[b^{ny_1}u_1, b^{ny_2}u_2,\cdots] = \left(\frac{u_2}{K_2}\right)^{d/y_2} f_s[\left(\frac{u_2}{K_2}\right)^{-\frac{y_1}{y_2}} u_1, K_2, \left(\frac{u_2}{K_2}\right)^{\frac{y_3}{y_2}} u_3,\cdots]$$
$$= u_2^{d/y_2}\Phi_\pm[\frac{u_1}{(u_2)^{y_1/y_2}}, (u_2)^{-\frac{y_3}{y_2}}u_3, (u_2)^{-\frac{y_4}{y_2}}u_4,\cdots]$$

(note: universal: $y_1, y_2, \cdots, \Phi_\pm$; non-universal: $A, B, C, D, \cdots$)

$$= A|t|^{d/y_2}\Phi_\pm\left[B|t|^{-y_1/y_2}h, C|t|^{-y_3/y_2}ha_{310}, D|t|^{-y_4/y_2}u_4^0,\cdots\right]$$

where $-y_1/y_2 < 0, -y_3/y_2 \geq 0, -y_4/y_2 \geq 0$, $A = (\frac{1}{t_0})^{d/y_2}$, $C|t|^{-y_3/y_2}ha_{310}, D|t|^{-y_4/y_2}u_4^0 \to 0$. This is the scaling form.

Addendum: RG transformation of dimensionless correlation length

$$\bar{\xi}[u_1,u_2,u_3,\cdots] = b^n\bar{\xi}[b^{ny_1}u_1, b^{ny_2}u_2, b^{ny_3}u_3,\cdots] = \left(\frac{K_2}{u_2}\right)^{\frac{1}{y_2}}\bar{\xi}[\left(\frac{K_2}{u_2}\right)^{y_1/y_2} u_1, K_2, \left(\frac{u_2}{K_2}\right)^{-y_3/y_2} u_3,\cdots]$$

When no magnetic field ($H = 0$), $u_{2m+1} = 0$

$$\bar{\xi}[u_1 = 0, u_2,\cdots] = (\frac{1}{u_2})^{1/y_2}\bar{\xi}[0, K_1, ..., ..., ..]$$

Correlation length exponent: $\nu = 1/y_2$

$b^n = (\frac{K_2}{u_2})^{1/y_2}$

Corrections to scaling: Taylor expand $\Phi_\pm$ in “irrelevant entries”: G Ahler’s experiments on He^4:

specific heat: $C = A_\pm^{(c)}|t|^{-\alpha}[1 + D^{(c)}|t|^{-y_4/y_2} + \cdots]$,

order parameter $\rho_s = A^{(S)}|t|^{\beta}[1 + D^{(\rho)}|t|^{-y_4/y_2} + \cdots]$.

Experiment: $-y_4/y_2 = 0.5$, $\alpha = -0.026$, $\beta = 0.67$.

$Z = \mathrm{Tr}_{\{S_i=\pm1\}}e^{-\mathcal{H}\{S_i\}} = e^{-\frac{1}{2}\sum h_i(K^{-1})_{ij}h_j}\left[(\frac{2}{\sqrt{2\pi}})^N\frac{1}{\sqrt{\det K}}\int d^N\vec{y}e^{-S\{\vec{y}_n\}}\right]$ Let the thing in the square bracket to be $\mathcal{Z}[K_{ij}, J_i, N]$.

Rewrite $S\{y_k\}$

$$S\{\phi(\vec{r})\} = \int d^d\vec{r}[\frac{1}{2}\frac{1}{\xi_i^2}\phi^2(\vec{r})+\frac{1}{2}(\nabla\phi)(\nabla\phi)+(\text{higher gradients})+\frac{\lambda}{4!}\phi^4(\vec{r})+(\text{higher powers of}\phi)]$$

First: quadratic Theory:

$S_0 = \frac{1}{2}\int d^d\vec{r}[\frac{1}{\xi_0^2}\phi^2 + (\nabla\phi)^2 + (a^2(\frac{\partial^2}{\partial x^2}\phi)^2)] = \frac{1}{2}\int_{BZ}\frac{d^d\vec{q}}{(2\pi)^d}\tilde{\phi}(\vec{q})[\frac{1}{(\xi_0^2)^2} + (\vec{q})^2 + O(a^2qx^4 + a^2qy^4, a^2qx2qy^2, ...)]\tilde{\phi}(-\vec{q})$.

WilsonianRG quadratic theory:

$S_0\{\tilde{\phi}(\vec{q})\} = \frac{1}{2}\frac{1}{Na^d}\sum_{\vec{q}}^{BZ} r(\vec{q})|\tilde{\phi}(\vec{q})|^2$, $r(\vec{q}) = \frac{1}{\xi_0^2} + \vec{q}^2 + O(a^2qx^4, ...)$

May 24

RG Gaussian (Wilsonian)

$e^{-S_0\{\tilde{\phi}(\vec{q})\}}$, $S_0\{\tilde{\phi}(\vec{q})\} = \frac{1}{2}\frac{1}{Na^d=vol}\sum_{\vec{q}}^{BZ} r(\vec{q})|\tilde{\phi}(\vec{q})|^2 = \frac{1}{Na^d}\sum_{\vec{q}}^{BZ}|\tilde{\phi}(\vec{q})|^2$

$r(\vec{q}) = [\frac{1}{\xi_0^2} + (\vec{q})^2]$ has even powers of $\vec{q}$.

Integrate: $\int\mathcal{D}[\phi(\vec{q})] = const.\prod_{\vec{q}\in BZ_+}\int\frac{dRe\phi(q)}{\sqrt{\pi vol}}\int\frac{dIm\phi(q)}{\sqrt{\pi vol}}$

Other parts see Iphone

May 26

Note $\int_{|q|<\Lambda}\mathcal{D}[\phi(q)] = \prod_{q\in BZ_+,|q|<\Lambda}\int_{-\infty}^{\infty}\frac{dRe\phi(q)}{\sqrt{\pi vol}}\int_{-\infty}^{\infty}\frac{dIm\phi(q)}{\sqrt{\pi vol}}$.

Summary Gaussian RGL $S_0\{\phi(q)\} = \frac{1}{2}\frac{1}{vol}\sum_{q}^{BZ} r(q)|\phi(q)|^2$ Step 1: $\phi(q) = \phi_<(q) + \phi_>(q)$, partial trance over $\phi_>(q)$

Step 2 rescale: $\phi'(bq) = z^{-1}\phi_<(q)$, set $bq = p$, $q = p/L$

$Z_0 = \int\mathcal{D}[\phi(q)]e^{-S_0\{\phi(q)\}} = Z_0^>(z)^{N/b^d}\int\mathcal{D}[\phi'(p)]e^{-S_0'\{\phi'(q)\}}$ where the last $\int\mathcal{D}$ is over $|\vec{p}| < \Lambda$: parameterized by $r'(p)$

We set $\int\frac{d^dq}{(2\pi)^d} = \int_q$.

Wilsonian RG beyound Gaussian:

G-L:

$\to S = S\{\phi(r)\} = \int\{\phi(q)\} = S_0\{\phi(q)\} + \delta S_4\{\phi(q)\} + \delta S_6\{\phi(q)\}$: last term can do later

$\to \delta S_4\{\phi(q)\} = \frac{\lambda}{4!}\int d^dr\phi^4(r) = \frac{\lambda}{4!}\int^{BZ}\frac{d^dq_1}{(2\pi)^d}\int\frac{d^dq_2}{(2\pi)^d}\int\frac{d^dq_3}{(2\pi)^d}\int\frac{d^dq_4}{(2\pi)^d}(2\pi)^d\delta^{(d)}(q_1 + q_2 + q_3 + q_4)\phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4) = \frac{\lambda}{4!}\int_{q_1}\int_{q_2}\int_{q_3}\int_{q_4}(2\pi)^d\delta^{(d)}(q_1 + q_2 + q_3 + q_4)(\phi_<(q_1) + \phi_>(q_1))\cdots$

partition function

(Def $Z_0^> = \int\mathcal{D}[\phi_>(q)]e^{-S_0\{\phi_>(q)\}}$, for any function $X\{\phi_>(q)\}$: $\langle X\{\phi_>(q)\}\rangle_0 = \frac{1}{Z_0^>}\int\mathcal{D}[\phi_>(q)]\int X\{\phi_>(q)\}e^{-S_0\{}$

$\to Z = \int_{|q|<\Lambda}\mathcal{D}[\phi(q)]e^{-S_0\{\phi(q)\}-\delta S_4\{\phi(q)\}} = \int\mathcal{D}[\phi_<(q)]e^{-S_0\{\phi_<(q)\}}\Big[\int_{\Lambda/b<|q|<\Lambda}\mathcal{D}[\phi_>(q)]e^{-S_0\{\phi_>(q)\}}e^{-\delta S_4\{\phi_<(q),\phi_>}$

$= Z_0^>\int\mathcal{D}[\phi_<(q)]e^{-S_0\{\phi_<(q)\}}\langle e^{-\delta S_4\{\phi_<(q),\phi_>(q)\}}\rangle_0$

expectation value (?)

HW 1, Pr1: random variable $x \in \mathcal{R}$ $\langle e^{ikx} \rangle_p = \exp\left[\sum\limits_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle_c\right]$

$\Omega =$ random variable wrt any probability distribution

$\langle e^{-\Omega} \rangle = \exp\{-\langle \Omega \rangle + \frac{1}{2}[\langle \Omega^2 \rangle - \langle \Omega \rangle^2] - \cdots\}$

$\langle \Omega \rangle = \langle \Omega \rangle_c$, $\langle \Omega^2 \rangle = \langle \Omega^2 \rangle_c + (\langle \Omega \rangle_c)^2$, etc

Now: $\langle e^{-\delta S_4\{\phi_<(q), \phi_>(q)\}} \rangle_0 = \exp\left\{-\langle \delta S_4 \rangle_0 + \frac{1}{2}\left[\langle (\delta S_4)^2 \rangle - (\langle \delta S_4 \rangle_0)^2\right] + \cdots\right\} = \exp\{-vol \delta g_4 - \delta S'\{\phi_<(q)\}\}$, where δg_4 do not depende on $\phi_<(q)$.

Then: → average $\langle \cdots \rangle_0$ wrt probability distribution: get finite number M of $\vec{q}$ values in BZ_+, $\Lambda/b < |q| < \Lambda$

Then: → enumerate them, $q_1, q_2, q_3, \cdots, q_M$

$P_0\{\phi_>(q_1), \cdots, \phi_>(q_M)\} = \frac{1}{Z_0^>} e^{-\sum\limits_{i=1}^{M} \frac{r(q_i)}{vol} |\phi>(q_i)|}$

then → special properties because Gaussian:

i "2-point" function: (gaussian integrals) $\langle \phi_>(q_1) \phi_>(q_2) \rangle = \frac{vol}{r(q)} \delta_{q_1+q_2,0} G_0(q_1)$, where $G_0(q) = 1/r(q)$ $(\Lambda/b \leq |q_1|, |q_2| < \Lambda)$

ii Wick's theorem: $\langle \phi_>(q_1), \cdots, \phi_>(q_4) \rangle_0 = \langle \phi_>(q_1) \phi_>(q_2) \rangle_0 \langle \phi_>(q_3) \phi_>(q_4) \rangle_0 + \langle \phi_>(q_1) \phi_>(q_3) \rangle_0 \langle \phi_>(q_2) \phi_>(q_4) \rangle_0$ $\langle \phi_>(q_1) \phi_>(q_4) \rangle_0 \langle \phi_>(q_2) \phi_>(q_3) \rangle_0$, more general: $\langle \phi_>(q_1) \cdots \phi_>(q_{2m+1}) \rangle = 0$, $\langle \phi_>(q_1) \cdots \phi_>(q_{2m}) \rangle =$ ...

More general: $S\{\phi(q)\} = S_0\{\phi(q)\} + \delta S\{\phi(q)\}$, $S_0\{\phi(q)\} = \int_{q_1} \int_{q\,2} \frac{1}{2} r) q_0 (2\pi)^d \delta^{(d)}(q_1 + q_2) \phi(q_1) \phi(a_2)$ $\delta S\{\phi(q)\} = \int_{q_1} \int_{q_2} \int_{q_3} \int_{q_4} \frac{1}{4!} \lambda_4(q_1, q_2, q_3, q_4) (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3 + q_4) \phi(q_1) \phi(q_2) \phi(q_3) \phi(q_4) +$ higher terms

Step 1: $\phi = \phi_< + \phi_>$,

$Z = Z_0^> \int \mathcal{D}[\phi_<(q)] e^{-S_0\{\phi_<(q)\}} \langle e_0^{-\delta S\{\phi_<(q), \phi_>(q)\}} = Z_0^> e^{-vol \delta g} \int \mathcal{D}[\phi_<(q)] e^{-S_0\{\phi_<(q)\} - \delta S'\{\phi_<(q)\}}$

Let $-S_0\{\phi_<(q)\} - \delta S'\{\phi_<(q)\} \equiv -S'\{\phi'(p)\}$

Step 2: $\phi_<(q) = z\phi'(bq) = z\phi'(q)$; $p = bq$

$Z = \int_{0 > |q| < \Lambda} \mathcal{D}[\phi(q)] e^{-S\{\phi(q)\}} = Z_0^> e^{-vol \delta g} (z)^{N/b^d} \int_{0 < |p| < \Lambda} \mathcal{D}[\phi'(p)] e^{-S'\{\phi'(p)\}}$,

Before RG: $S\{\phi(q)\}$ after RG: $S'\{\phi'(p)\}$

$r(q) \to r'(p)$, $\lambda_4(q_1, \cdots, q_4) \to \lambda_4'(p_1, \cdots, p_4)$, $\lambda_6(q_1, \cdots, q_6) \to \lambda_6'(p_1, \cdots, p_6)$, etc

parametrise taylor coeff's:

$r(q) = r + (\vec{q})^2 + ...$ $\lambda_4(q_1, \cdots, q_4) = \lambda + (\vec{q}_1 \cdot \vec{q}_2 + allparameters) \lambda_4^{(2)} + ...$ etc

RG: $r(q) \to r'(q)$, $\lambda_4(q_1, \cdots, q_4) \to q_4'(q_1, \cdots, q_4)$,etc. block spin RG $\vec{K} \Leftrightarrow$ here collection of all Talor coeeff's

$r \to r'$, $\lambda_4 \to \lambda_4'$, $\lambda_4^{(2)} \to \lambda_4^{(2)}$, etc.

May 31

Setting:

$$S\{\phi(q)\} = S_0\{\phi(q)\} + \delta S\{\phi(q)\}$$

Step one:

$$Z = \int \mathcal{D}[\phi(q)] e^{-S_0\{\phi(q)\} - \delta S\{\phi(q)\}} = Z_0^{>} \int \mathcal{D}[\phi_<(q)] e^{-S_0\{\phi_<(q)\}} \langle e^{-\delta S\{\phi_<(q), \phi_>(q)\}} \rangle_0$$

$$= \exp\{-vol\delta g - \delta S'\{\phi_<(\phi)\}\} = \exp\{-\langle \delta S\rangle_0 + \frac{1}{2}[\langle(\delta S)^2\rangle_0 - \langle \delta S\rangle_0^2] + \cdots\}$$

Step two;

$\phi_<(q) = z\phi'(bq) = z\phi'(p)$,

$$= Z_0^{>} e^{-vol\delta g} (z)^{N/b^d} \int_{0 \le |p| < \Lambda} \mathcal{D}[\phi'(p)] e^{-S'\{\phi'(p)\}}$$

$$S\{\phi(p)\} \longrightarrow S'\{\phi'(p)\}$$

$$\lambda_4(q_1, q_2, q_3, q_4) \longrightarrow \lambda_4'(p_1, p_2, p_3, p_4)$$

$$\lambda_6(q_1, ..., q_6) \longrightarrow \lambda_6'(p_1, ..., p_6)$$

Now:

$$\delta S\{\phi\} = \frac{\lambda}{4!} \int d^d r \phi^4(r), (\lambda_4)_{constanttaylor} = \lambda$$

$r(q) = r + 1(\vec{q}^2) + \cdots$.

Choose $b = e^{dl}$, $0 < dl \ll 1$, S_d = area of unit sphere in d-dimensional space $= \frac{2\pi^{d/2}}{\gamma(\frac{d}{2})}$

Now result from $\langle \delta S_4\rangle_0$:

$$r \to r' + \left[2r + \frac{\lambda}{2} \frac{S_d}{(2\pi)^d} \Lambda^{d-2}\{1 - \frac{r}{\Lambda^2} + O(r^2)\} + O(\lambda^2)\right] dl + O(dl)^2$$

$$\lambda \to \lambda' + (4-d)\lambda dl + o(dl)^2$$

More convenient notation: dimensionless couplings: $\tilde{r} = \frac{r}{\Lambda^2}$, $\tilde{\lambda} = \frac{\lambda}{\Lambda^{4-d}}$, $\tilde{r}' = \frac{r'}{\Lambda^2}$, $\tilde{\lambda}' = \frac{\lambda'}{\Lambda^{4-d}}$,

Diff Eq.:

$$\frac{d\tilde{r}}{dl} = \frac{\tilde{r}' - \tilde{r}}{dl} = 2\tilde{r} + \frac{1}{2} \frac{S_d}{(2\pi)^d} \tilde{\lambda}\{1 - \tilde{r} + ...\} + O(\tilde{\lambda}^2)$$

$$\frac{d\tilde{\lambda}}{dl} = \frac{\tilde{\lambda}' - \tilde{\lambda}}{dl} = (4-d)\tilde{\lambda} + O(\tilde{\lambda})^2$$

These are RE equations. After solving Diff. eq.'s get $[\tilde{r}(l), \tilde{\lambda}(l)]$ = dimensional couplings at rescaling factor $b = e^l$.

After evaluating also $\frac{1}{2}[\langle(\delta S_4)^2\rangle_0 - \langle\delta S_4\rangle_0^2]$ we get new RG equations.

$$\frac{d\tilde{\lambda}}{dl} = \tilde{\lambda}[(4-d) - \tilde{\lambda}A] + O(\tilde{\lambda}^2\tilde{r}, \tilde{\lambda}^3)$$
$$\frac{d\tilde{r}}{dl} = 2\tilde{r} + \lambda B\{1 - \tilde{r} + O(\tilde{r})^2\}$$

$d = 4 - \epsilon$, we take $0 \leq 0 \ll 1$, $A = \frac{3}{2}\frac{\delta S}{(2\pi)^d}$, $B = \frac{1}{3}A$, Borel /Padé resummation.

Fixed points (of RG): First, Gaussian fixed point: $\tilde{\lambda}_* = \tilde{r}_* = 0$, Linearize RG equation about Gaussian fixed point.

$$\delta\tilde{r}(l) = \tilde{r}(l) - \tilde{r}_* = \tilde{r}(l)$$
$$\delta\tilde{\lambda}(l) = \tilde{\lambda}(l) - \tilde{\lambda}_* = \tilde{\lambda}(l)$$
$$\frac{d}{dl}\begin{bmatrix} \delta\tilde{r}(l) \\ \delta\tilde{\lambda}(l) \end{bmatrix} = \begin{bmatrix} 2 & B \\ 0 & \epsilon \end{bmatrix}\begin{bmatrix} \delta\tilde{r}(l) \\ \delta\tilde{\lambda}(l) \end{bmatrix}$$

Eigenvalues: for 2: eigenvectors is $\hat{e}_2 = (1, 0)$; for ϵ: it is $\hat{e}_4 = (-B/(2-\epsilon), 1)$.

Write $(\delta\tilde{r}(l), \delta\tilde{\lambda}(l)) = u_2(l)\hat{e}_2 + u_4(l)\hat{e}_4$, where

$$\frac{du_2(l)}{dl} = 2u_2(l)$$
$$\frac{du_4(l)}{dl} = \epsilon u_4(l)$$

$u_2(l) = (e^l)^2 u_2(0)$: (note $(e^l)^2 = b^{y_2^G}$) $y_2^G = 2$, $u_4(l) = (e^l)^\epsilon u_4(0)$ (note $(e^l)^\epsilon = b^{y_4^G}$) : $y_4^G = \epsilon$.

New fixed point: Wilson fisher fixed point (see iphone.)

Existence of the new fixed point to order ϵ:

condition for fixed point:

(I)$0 = \tilde{\lambda}_4[\epsilon - \tilde{\lambda}_* A] + O(\lambda_*^2\tilde{r}_*, \lambda_*^2)$

(II) $0 = 2\tilde{r}_* + \tilde{\lambda}_* B\{1 - \tilde{r}_* + o(\tilde{r}_*)^2\} + O(\lambda_*^2)$

From (I) we get $\tilde{\lambda}_* = \frac{\epsilon}{A}[1 + O(\epsilon)] = \epsilon\frac{2}{3}\frac{(4\pi)^{4-\epsilon}}{S_{4-\epsilon}}\{1 + O(\epsilon)\} = \epsilon\frac{2}{3}\frac{(2\pi)^4}{S_4}\{1 + O(\epsilon)\} = \frac{16\pi^2}{3}\epsilon\{1 + O(\epsilon)\} \simeq 50\epsilon\{1 + O(\epsilon)\}$.

Plog into (II): $\tilde{r}_* = -\frac{B}{2}\lambda_* + \cdots = -\frac{1}{6}(A\lambda_*) = -\frac{\epsilon}{6} + O(\epsilon^2)$.

Now: linearize RG equations about new fixed point:

$$\frac{d}{dl}\begin{bmatrix} \tilde{r} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} \beta_{\tilde{r}}(\tilde{r}, \tilde{\lambda}) \\ \beta_{\tilde{\lambda}}(\tilde{r}, \tilde{\lambda}) \end{bmatrix},$$

(this is the beta function) write $\delta\tilde{r}(l) = \tilde{r}(l) - \tilde{r}_*, \delta\tilde{\lambda}(l) = \tilde{\lambda}(l) - \tilde{\lambda}_*$,

$$\frac{d}{dl}\left[\begin{array}{c} \delta\tilde{r}(l) \\ \delta\tilde{\lambda}(l) \end{array}\right] = \left[\begin{array}{cc} \frac{\partial\beta_{\tilde{r}}}{\partial\tilde{r}} & \frac{\partial\beta_{\tilde{r}}}{\partial\tilde{\lambda}} \\ \frac{\partial\beta_{\tilde{\lambda}}}{\partial\tilde{r}} & \frac{\partial\beta_{\tilde{\lambda}}}{\partial\tilde{\lambda}} \end{array}\right]\left[\begin{array}{c} \delta\tilde{r}(l) \\ \delta\tilde{\lambda}(l) \end{array}\right]$$

Call the matrix in the middle M_ϵ, we get:

$$M_\epsilon = \left[\begin{array}{cc} 2 - \tilde{\lambda}_* B + O(\epsilon^2) & B\{1 - \tilde{r}_* + ...\} + O(\lambda_*) \\ 0 + O(\epsilon^2) & \epsilon - 2\tilde{\lambda}_* A + O(\epsilon^2) \end{array}\right] = (\text{taylor in } \epsilon)\left[\begin{array}{cc} 2 - \frac{\epsilon}{3} + O(\epsilon^2) & \frac{1}{16\pi^2}(1 + O(\epsilon)) \\ 0 + O(\epsilon^2) & -\epsilon + O(\epsilon^2) \end{array}\right]$$

M_ϵ has eigenvalues $2 - \frac{\epsilon}{3} + O(\epsilon^2)$, and $-\epsilon + O(\epsilon^2)$, the corresponding eigenvectors are $\hat{e}_2 + (1, 0) + O(\epsilon)$, and $\hat{e}_4 = (-\frac{1}{32\pi^2}, 1) + O(\epsilon)$.

Now as before: we write $(\delta\tilde{r}(l), \delta\tilde{\lambda}(l)) = u_2(l)\hat{e}_2 + u_4(l)\hat{e}_4$,

$$\frac{du_2(l)}{dl} = (2 - \frac{\epsilon}{2} + O(\epsilon^2))u_2(l)$$
$$\frac{du_4(l)}{dl} = (-\epsilon + O(\epsilon^2))u_4(l)$$

we get $u_2(l) = (e^l)^{2-\frac{\epsilon}{2}+O(\epsilon^2)}u_2(0)$, $u_4(l) = (e^l)^{-\epsilon+O(\epsilon^2)}u_4(0)$, this is for the WF fixed point. $y_2^{WF} = 2 - \frac{\epsilon}{3} + O(\epsilon^2)$, $y_4^{WF} = -\epsilon + O(\epsilon^2)$.

June 2

$d = 4 - \epsilon, \epsilon > 0$, $\epsilon \to 1$, not an expansion ϵ of free energy, or any physical observables!

Recall Landau Ginzburg notes 220-S16-lecture-notes-May-3...:

$r = \frac{1}{(\xi_0^2)} = \frac{1}{a^2}\frac{1-\hat{K}[0]}{\frac{1}{2R}(R/a)^2} = \frac{1}{a^2}\frac{2d}{(R/a)^2}[1 - \frac{T_c^{MFT}}{T}]$, $\lambda - 2\frac{2d\hat{K}[0]}{(R/a)^2}a^{d-4}$, where $\hat{K}[0] = T_c^{MFT}/T$. we get $\tilde{r} = \frac{r}{\Lambda^2} = \frac{2d}{(R/a)^2}[1 - \frac{T^{MFT}}{T}]\frac{1}{\pi^2}$, $\tilde{\lambda} = \frac{\lambda}{\Lambda^{4-d}} = 2(\frac{2d}{(R/a)^2}\frac{T_c^{MFT}}{T})^2\frac{1}{\pi^\epsilon}$, we write

$\pi^\epsilon\tilde{\lambda} = 2x^2$, $\pi^2\tilde{r} = \frac{2d}{(R/a)^2} - x$, eliminate x from them, we get $\pi^\epsilon\tilde{\lambda} = 2\left[\frac{2d}{(R/a)^2} - \pi^2\tilde{r}\right]^2$: parabola.

Fixed point Boltzmann weight:

$r = \Lambda^2\tilde{r}$, $\lambda = \Lambda^\epsilon\tilde{\lambda}$, $\tilde{r}_* = -\frac{\epsilon}{6} + O(\epsilon^2)$, $\tilde{\lambda}_* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2)$, $P_*\{\phi(r)\} = \frac{1}{Z_*}e^{-S_*\{\phi(r)\}}$,

$$S_*\{\phi(r)\} = \int_\Lambda d^dr\left[\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\Lambda^2[\frac{\epsilon}{6} + O(\epsilon^2)]\phi^2 + \frac{\Lambda^\epsilon}{4!}\frac{16\pi^2}{3}\epsilon[1 + O(\epsilon)]\phi^4\right]$$

Non-linear-sigma model and Ginzburg Landau theory.

n-point correlation function's behavior under RG

Continuum limit $a \to 0$

Callan-symanzik equation.