

217B notes

March 14, 2016

Jan 21

$$\mathcal{G}(\vec{r}_{12}, \tau_{12}) = \langle T[\hat{\psi}(\vec{r}_1, \tau_1)\hat{\psi}^\dagger(\vec{r}_2, \tau_2)] \rangle = \frac{1}{V} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}_{12}} \langle [\hat{a}_{\vec{p}}(\tau_1)\hat{a}_{\vec{p}}^\dagger(\tau_2)] \rangle$$

$$\mathcal{G}(\vec{p}, \tau_{12}) \equiv \langle [\hat{a}_{\vec{p}}(\tau_1)\hat{a}_{\vec{p}}^\dagger(\tau_2)] \rangle \equiv \mathcal{G}^{\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger}(\tau_{12})$$

$$\mathcal{G}^{\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger}(\vec{p}, \omega_l) = \int_0^\beta d\tau e^{i\omega_l\tau} \mathcal{G}^{\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}^\dagger}(\tau)$$

$$\hat{K} = \hat{H} - \mu\hat{N}, \quad \hat{K}|n\rangle = K_n|n\rangle$$

Feb 2

Adiabatic $p = p_F$. In fermi liquid all spectral function is

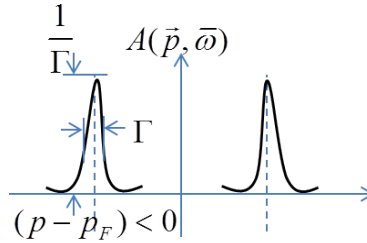
$$A(p_F, \omega) = z\delta(\omega) + A_{\text{smooth}}(\omega), \quad 0 < z < 1 \text{ (in Fermi liquid)}$$

$$\lim_{p \rightarrow p_F} |\langle \text{exact eigenstate of } \vec{p} | \hat{a}_{\vec{p}}^\dagger | \text{VAC} \rangle| = Vz$$

when $p \neq p_F$,

$$A(\vec{p}, \omega) = \frac{z}{\pi} \frac{\Gamma(\vec{p}, \omega) + \eta}{(\omega - v_F(p - p_F) + \delta\Sigma_{1r})^2 + (\Gamma(\vec{p}, \omega) + \eta)^2},$$

$$\Gamma(\vec{p}, \omega) = z(-1)\text{Im}\Sigma_r(\vec{p}, \omega).$$



check: $\Gamma(\vec{p}, \omega) \Big|_{\omega=v_F(p-p_F)} \propto (p - p_F)^2$, where p is close to p_F .

Recall: $\langle \hat{n}_{\vec{p}} \rangle = \langle \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} \rangle = \int_{-\infty}^{+\infty} d\omega f(\omega) A(\vec{p}, \omega)$, where $f(\omega)A(\vec{p}, \omega) = A_e(\vec{p}, \omega)$ is what is seen in ARPES!!

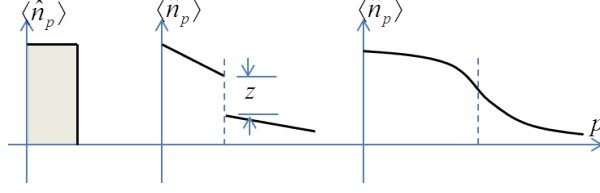


Figure 1: From left to right are number versus momentum in non-interacting fermio, fermi liquid and non-fermi-liquid type interacting systems. In an non-fermi-liquid type interacting system, at $p = p_F$ typically is some power law behavior, e.g. in 1D Luttinger liquid by duncane Haldane.

$$\text{fermi function } f(\omega) = \frac{1}{1+e^{\beta\omega}} \xrightarrow{T \rightarrow 0} \Theta(-\omega),$$

$$\lim_{p \rightarrow p_F} \langle \hat{n}_{\vec{p}} \rangle = \lim_{p \rightarrow p_F} \int_{-\infty}^{+\infty} d\omega A_e(\vec{p}, \omega) = \lim_{\varepsilon \rightarrow 0^-} \int_{-\infty}^0 d\omega A(p_F - \varepsilon, \omega) = z + \int_{-\infty}^0 d\omega A_{\text{smooth}}(p_F, \omega),$$

$$\lim_{p \downarrow p_F} \langle \hat{n}_{\vec{p}} \rangle = \lim_{p \rightarrow p_F} \int_{-\infty}^{+\infty} d\omega A_e(\vec{p}, \omega) = \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^0 d\omega A(p_F + \varepsilon, \omega) = 0 + \int_{-\infty}^0 d\omega A_{\text{smooth}}(p_F, \omega),$$

Thus at $T = 0$, the $\langle n_p \rangle$ - p graph will be different for non-interacting fermion and fermi liquid and non-fermi-liquid interacting system in fig. (1).

Note: if Γ_p were independent of ω , $\Gamma(p, \omega) = \Gamma(p)$, then $\mathcal{G}_{\text{ret}}(p, \omega) = \frac{z}{\omega - [v_F(p - p_F) + \delta\Sigma_{1r}^{-1} - i\Gamma_p - i\eta]}$.

Feb 4

Feb 9

$$S = S_0 + S_{\text{int}},$$

$$S_0 = \int_{\tau} \int_{\vec{r}} \bar{\psi}_{\sigma}(\vec{r}\tau) \left[\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu \right] \psi_{\sigma}(\vec{r}\tau),$$

$$S_1 = \frac{1}{\tau} \int_{\tau} \int_{\tau'} \int_{\vec{r}} \int_{\vec{r}'} \bar{\psi}_{\sigma}(\vec{r}\tau + 0^+) \psi_{\sigma}(\vec{r}, \tau) V(\vec{r} - \vec{r}') \delta(\tau - \tau') \bar{\psi}_{\sigma'}(\vec{r}', \tau' + \delta) \psi_{\sigma'}(\vec{r}'\tau'),$$

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_{\alpha}(\vec{r}\tau) \psi_{\alpha}(\vec{r}\tau)] e^{-S_0} e^{-S_{\text{int}}} = \text{Const.} \int \mathcal{D}[\bar{\psi}_{\alpha}(\vec{r}\tau) \psi_{\alpha}(\vec{r}\tau)] \int \mathcal{D}[\chi_{\alpha}(\vec{r}\tau)] e^{-(S_0 + S_{\text{int}})},$$

$$S = S_0 + S_{\text{int}}, \quad S_0 = \int_{\tau} \int_{\vec{r}} \bar{\psi}_{\sigma}(\vec{r}\tau) \left[\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu \right] \psi_{\sigma}(\vec{r}\tau),$$

$$S_{\text{int}} = (-i) \int_{\tau} \int_{\vec{r}} n(\vec{r}\tau) \chi(\vec{r}\tau) = (-i) \int_{\tau} \int_{\vec{r}} (\bar{\psi}_{\alpha}(\vec{r}\tau) \psi_{\alpha}(\vec{r}\tau)) \chi(\vec{r}\tau).$$

Perturbation theory of $S = S_0 + S_{\text{int}}$: diagrams corresponding to different green's functions, as shown in fig.(2). For small parameter: rescale:

$$\chi(\vec{r}, \tau) = \sqrt{V_0} \phi(\vec{r}\tau), \quad V(\vec{r} - \vec{r}') = V_0 v(\vec{r} - \vec{r}'),$$

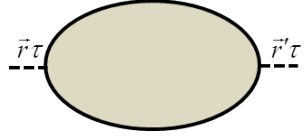
where v is the unit strength potential. Thus we have $\chi V^{-1} \chi = \phi v^{-1} \phi$, and

$$S_{\text{int}} = (-i) \int_{\tau} \int_{\vec{r}} (\bar{\psi} \psi) \chi = (-i) \sqrt{V_0} \int_{\tau} \int_{\vec{r}} (\bar{\psi}_{\alpha}(\vec{r}\tau + 0^+) \psi_{\alpha}(\vec{r}\tau)) \phi(\vec{r}\tau)$$

$$\begin{aligned}
\begin{array}{c} \psi \\ \vec{r}\tau \end{array} \xrightarrow{\alpha=1} \begin{array}{c} \bar{\psi} \\ \vec{r}'\tau' \end{array} &= \langle \psi_{\alpha=1}(\vec{r}\tau) \bar{\psi}_{\alpha=1}(\vec{r}'\tau') \rangle_0 = \mathcal{G}_0(\vec{r}-\vec{r}', \tau-\tau') \\
\begin{array}{c} \text{---} \\ \vec{r}\tau \end{array} \text{---} \begin{array}{c} \text{---} \\ \vec{r}'\tau' \end{array} &= \langle \chi(\vec{r}\tau) \chi(\vec{r}'\tau') \rangle_0 = g_0(\vec{r}-\vec{r}', \tau-\tau') = \delta(\tau-\tau') V(\vec{r}-\vec{r}') \\
\begin{array}{c} \nearrow \\ \text{---} \\ \searrow \end{array} &= (-i) \delta_{\alpha,\alpha'}
\end{aligned}$$

Figure 2: Diagram representation of green's functions in perturbation theory, note the subscript 0 means expectation value with respect to S_0 .

Large- n limit: Fully interacting density-density correlation function is related to the diagram below in the new theory:



$$\begin{aligned}
\mathcal{D}(\vec{r}-\vec{r}', \tau-\tau') &= \langle n(\vec{r}\tau) n(\vec{r}'\tau') \rangle_s - [\langle n(\vec{r}\tau) \rangle_s]^2, \\
g(\vec{r}-\vec{r}', \tau-\tau') &= \langle \chi(\vec{r}\tau) \chi(\vec{r}'\tau') \rangle_s = [\langle \chi(\vec{r}\tau) \rangle_s]^2.
\end{aligned}$$

Perturbation theory: statement:

$$\begin{aligned}
\begin{array}{c} \vec{r}\tau \\ \text{=====} \\ \vec{r}'\tau' \end{array} &= \begin{array}{c} \vec{r}\tau \\ \text{-----} \\ \vec{r}'\tau' \end{array} + \begin{array}{c} \vec{r}\tau \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \vec{r}'\tau' \\ \text{---} \end{array} + \begin{array}{c} \vec{r}\tau \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \vec{r}'\tau' \\ \text{---} \end{array} \\
+ \begin{array}{c} \vec{r}\tau \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \vec{r}'\tau' \\ \text{---} \end{array} + \begin{array}{c} \vec{r}\tau \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \vec{r}'\tau' \\ \text{---} \end{array} \\
= \begin{array}{c} \vec{r}\tau \\ \text{-----} \\ \vec{r}'\tau' \end{array} + \begin{array}{c} \vec{r}\tau \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \vec{r}'\tau' \\ \text{---} \end{array}
\end{aligned}$$

Figure 3: diagrammatic statement corresponding to eq.(1)-(2).

$$g(\vec{r}-\vec{r}', \tau-\tau') = g_0(\vec{r}-\vec{r}', \tau-\tau') - \int_{\tau_1} \int_{\tau_2} \int_{\vec{r}_1} \int_{\vec{r}_2} g_0(\vec{r}-\vec{r}_1, \tau-\tau_1) \mathcal{D}(\vec{r}_1-\vec{r}_2, \tau_1-\tau_2) g_0(\vec{r}_2-\vec{r}', \tau_2-\tau') \quad (1)$$

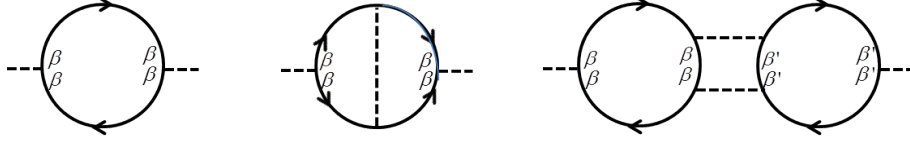


Figure 4: The first diagram is Π_0 , which only has sum of spin once. as $n \rightarrow +\infty$, we have $\frac{1}{n}\Pi_0 \rightarrow \frac{1}{n}\Pi_0$. The second diagram, the spin index sum give same number of n as the first diagram, but the interaction line gives a $\frac{1}{n}!$ thus the second diagram is of $1/n$ order of the first one and will $\rightarrow 0$ as $n \rightarrow +\infty$. The third one, still, the interaction line is 2 and the spin sum is 2, thus it is still of order $1/n$ compared to Π_0 and will vanish as $n \rightarrow +\infty$. Thus $-\frac{1}{n}\mathcal{D}(\vec{p}, \bar{\omega}) \xrightarrow{n \rightarrow +\infty} \frac{\frac{1}{n}\Pi_0}{1 - \frac{1}{n}\Pi_0 \overline{V}(\vec{p})}$.

Two momentum scales: Λ, p_F ; dim'less: Λ/p_F ; consider $0 < \Lambda/p_F \ll 1$. Shell in momentum space around fermi surface: momentum \vec{p} of any fermions have to satisfy:

$$|(|\vec{p}| - 0)| < \Lambda$$

$$\sum'_{\vec{p}} = \text{prime sum around shell } |(|\vec{p}| - 0)| < \Lambda$$

Mist general translationally invariant Hamiltonian

$$\hat{K} = \sum'_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{1}{2} \frac{1}{V} \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{\vec{p}_3} \sum'_{\vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \hat{a}_{\vec{p}_4}^\dagger \hat{a}_{\vec{p}_3}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

+ other terms e.g.: $\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \hat{a} + 8$ fermion operator terms

$V = L^d$, $d = 2$, Look at momentum conservation: when $\frac{\Lambda}{p_F} \ll 1$,

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = \vec{Q}$$

i) $|\vec{Q}| > \sim 2\Lambda$: gives solution I and II: $\vec{p}_3 \leftrightarrow \vec{p}_4$

ii) $|\vec{Q}| < \sim 2\Lambda$ gives solution III.

Rewrite the interaction term using the analysis:

$$\frac{1}{V} \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{\vec{p}_3} \sum'_{\vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \hat{a}_{\vec{p}_4}^\dagger \hat{a}_{\vec{p}_3}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

\vec{k} is the vector that has the length of a pencil stroke:

Now only discuss only (I) and (II) and later will discuss (III).

$$(I) = \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{|\vec{k}| < \Lambda} v(\vec{p}_1, \vec{p}_2, \vec{p}_3 = \vec{p}_2 + \vec{k}, \vec{p}_4 = \vec{p}_1 - \vec{k}) \hat{a}_{\vec{p}_1 - \vec{k}}^\dagger \hat{a}_{\vec{p}_2 + \vec{k}}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

$$(II) = \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{|\vec{k}| < \Lambda} v(\vec{p}_1, \vec{p}_2, \vec{p}_3 = \vec{p}_1 + \vec{k}, \vec{p}_4 = \vec{p}_2 - \vec{k}) \hat{a}_{\vec{p}_2 - \vec{k}}^\dagger \hat{a}_{\vec{p}_1 + \vec{k}}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

+(III).

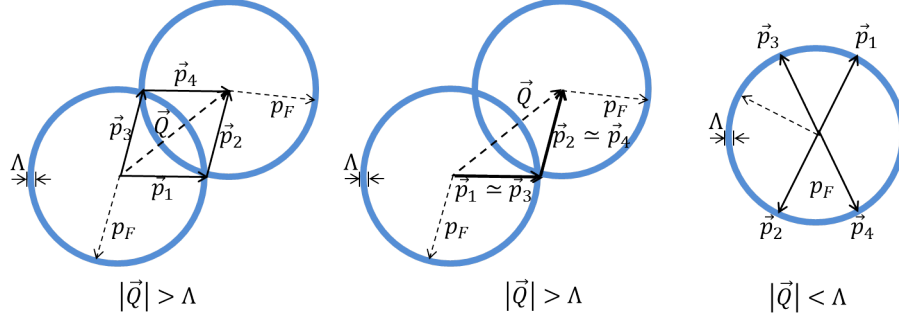


Figure 5: Three allowed momentum configuration under RG. Details see http://guava.physics.uiuc.edu/~nigel/courses/563/Essays_2013/PDF/Dwivedi.pdf

We must have

$$v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = -v(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4) = -v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$$

(I) and (II) is $\vec{p}_1 \leftrightarrow \vec{p}_2$, so $I = II$. Now relabel $2v \rightarrow v$.

$$(I) + (II) + (III) = \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{|\vec{k}| < \Lambda} v(\vec{p}_1, \vec{p}_2, \vec{p}_3 = \vec{p}_2 + \vec{k}, \vec{p}_4 = \vec{p}_1 - \vec{k}) \hat{a}_{\vec{p}_1 - \vec{k}}^\dagger \hat{a}_{\vec{p}_2 + \vec{k}}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

write $\hat{a}_{\vec{p}_1 - \vec{k}}^\dagger \hat{a}_{\vec{p}_2 + \vec{k}}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1} =: (\hat{a}_{\vec{p}_1 - \vec{k}}^\dagger \hat{a}_{\vec{p}_1}) (\hat{a}_{\vec{p}_2 + \vec{k}}^\dagger \hat{a}_{\vec{p}_2}) :=$ density operator near $\vec{p}_1 \cdot$ density operator near \vec{p}_2 . This is density density interaction near the Fermi surface.

Look at (III): $\vec{p}_1 + \vec{p}_2 = \vec{0}$, is a different term.

$$\begin{aligned} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} &= \text{resolve Kronecker } \delta \\ &= v(\vec{p}_1, \vec{p}_2 = -\vec{p}_1 - \vec{k}, \vec{p}_3, \vec{p}_4 = -\vec{p}_3 - \vec{k}) \equiv w(\vec{p}_1, \vec{p}_3, \vec{k}). \end{aligned}$$

$$(III) = \sum_{\vec{p}_1} \sum_{\vec{p}_3} \sum_{|\vec{k}| < \Lambda} w(\vec{p}_1, \vec{p}_3, \vec{k}) \hat{a}_{\vec{p}_1}^\dagger \hat{a}_{-\vec{p}_1 - \vec{k}}^\dagger \hat{a}_{\vec{p}_3} \hat{a}_{-\vec{p}_3 - \vec{k}}$$

We are done with this toy model. The first term (I)=(II) is density density operators at different points of the fermi surface, the other term, (III), is two pair operators, again, at different points of fermi surface. It will turn out: the first term (I)=(II) is responsible to fermi liquid theory. The second term (III), also an interaction, its physics will destroy the fermi surface. It will lead to superconducting instability of the fermi surface (for some value of w);

Kohn-Luttinger theorem: any interacting fermion system, will eventually undergo a superconducting instability, at some low temperature T_c . The theorem doesn't tell you what T_c it is. In reality, if we don't see SC it is because the lab temperature is not low enough.

Another thing: fermion system is ubiquitous, this analysis also applies to high energy physics.

Now we will switch off by hand w (will add them on later), and will only discuss the presence of the first type interaction. Now what we discussed has no physics.

RG: Not toy model, start with a theory which has no Λ

Since Λ is completely up to us, we can make Λ/p_F as small as we want. As long as we can do so, we can always narrow down the interaction to the term (I)=(II).

We choose $2D$ for simplicity, no pauli spin. For example we subject the fermions to a strong magnetic field so they polarize so we don't have to consider spins.

$$\begin{aligned}\langle T[\hat{a}_p(\tau)\hat{a}_p^\dagger(0)] \rangle_{\hat{K}} &= \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} e^{-i\tau\bar{\omega}} \mathcal{G}(p, \bar{\omega}) = \langle \psi_p(\tau), \bar{\psi}_p(0) \rangle_S, \\ Z &= \text{Tre}^{-\beta\hat{K}} = \int \mathcal{D}[\bar{\psi}_p(\tau), \psi_p(\tau)] e^{-S} \\ S &= \int_0^\beta d\tau \left[\sum_{\vec{p}} \bar{\psi}_{\vec{p}}(\tau) \partial_\tau \psi_{\vec{p}}(\tau) + K(\bar{\psi}_p(\tau+0^+), \psi_p(\tau)) \right]\end{aligned}$$

Some convention:

$$\begin{aligned}\hat{\psi}(\vec{r}) &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}} \hat{a}_{\vec{p}} \\ V(\vec{r}) &= \frac{1}{V} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}} \tilde{V}(\vec{p}) \\ V &= L^d \quad (d=2)\end{aligned}$$

RG is best thought as zero temperature. in $T=0$ we get rid of the scales, and the only ones we have are Λ and p_F .

Feb 16

Firt hint on the second homework: $\mathcal{D}(\tau - \tau')$ analytic continuation. (First do the analytic continuation, then take imaginary part.)

RG

Start with a theory (most general) without 'cut off Λ '.

For simplicity, start in d spatial dimensions with $d=2$, no Pauli-spin index.

System in a box of dimensionize L , momentum is discrete, $\text{Vol} = L^d$.

$$\begin{aligned}\hat{\psi}(\vec{r}) &= \frac{1}{\sqrt{\text{Vol}}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}} \hat{a}_{\vec{p}}, \\ V(\vec{r}) &= \frac{1}{\sqrt{\text{Vol}}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}} \tilde{V}(\vec{p})\end{aligned}$$

$\hat{K} = \hat{H} - \mu\hat{N}$, $\hat{K} = \sum_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}$, $\xi_{\vec{p}} = \frac{\vec{p}^2}{2m} - \mu = v_F(|\vec{p}| - p_F) + O(|\vec{p}| - p_F)^2$. We want the sys. to have translational invariance, thus

$$\hat{K} = \sum_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \hat{a}_{\vec{p}_4}^\dagger \hat{a}_{\vec{p}_3}^\dagger \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

Now we go to coherent path integral. This is the natural way to think about RG.

$$\mathcal{Z} = \text{Tre}^{-\beta\hat{K}} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}(\tau)\psi_{\vec{p}}(\tau)]e^{-S}$$

$$S = \int_{-\beta/2}^{\beta/2} d\tau \left[\sum_{\vec{p}} \bar{\psi}_{\vec{p}}(\tau)\partial_{\tau}\hat{\psi}_{\vec{p}}(\tau) + K\{\bar{\psi}_{\vec{p}}(\tau+0^+), \psi_{\vec{p}}(\tau)\} \right]$$

Next: go to Matsubara space: at $T = 0$, $\omega_n \rightarrow \bar{\omega}$, fourier transform:

$$\bar{\psi}_{\vec{p}}(\tau) = \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} e^{i\bar{\omega}\tau} \bar{\psi}_{\vec{p}}(\bar{\omega}),$$

$$\psi_{\vec{p}}(\tau) = \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} e^{-i\bar{\omega}\tau} \psi_{\vec{p}}(\bar{\omega})$$

We get

$$S = \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} \sum_{\vec{p}} \bar{\psi}_{\vec{p}}(\bar{\omega})[-i\bar{\omega} + \xi_{\vec{p}}]\psi_{\vec{p}}(\bar{\omega}) + \frac{1}{2} \frac{1}{\text{Vol}} \int_{-\infty}^{\infty} \frac{d\bar{\omega}_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\bar{\omega}_2}{2\pi} \int_{-\infty}^{\infty} \frac{d\bar{\omega}_3}{2\pi} \int_{-\infty}^{\infty} \frac{d\bar{\omega}_4}{2\pi} \sum_{\vec{p}_1\vec{p}_2\vec{p}_3\vec{p}_4} \times \\ \times \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}(2\pi)\delta(\bar{\omega}_1+\bar{\omega}_2-\bar{\omega}_3-\bar{\omega}_4)v(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\bar{\psi}_{\vec{p}_4}(\bar{\omega}_4)\bar{\psi}_{\vec{p}_3}(\bar{\omega}_3)\psi_{\vec{p}_2}(\bar{\omega}_2)\psi_{\vec{p}_1}(\bar{\omega}_1),$$

it has $U(1)$ invariance: $\psi_{\vec{p}}(\bar{\omega}) \rightarrow e^{i\alpha}\psi_{\vec{p}}(\bar{\omega})$, $\bar{\psi}_{\vec{p}}(\bar{\omega}) \rightarrow e^{-i\alpha}\bar{\psi}_{\vec{p}}(\bar{\omega})$.

Now comes the RG step: we choose some completely arbitrary ‘band width’ around p_F , band width is $\Lambda \ll p_F$. In $p_x - p_y$ plane we draw the fermi surface, we choose p_F and Λ (to give the annulus), refer to the annulus as the shell.

Then, split

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}(\bar{\omega}), \psi_{\vec{p}}(\bar{\omega})]e^{-S}$$

$$\psi_{\vec{p}}(\bar{\omega}) = \left\{ \begin{array}{l} \psi_{\vec{p}}^{\leq}(\bar{\omega}), \quad |(|\vec{p}| - p_F)| < \Lambda, \\ \psi_{\vec{p}}^{\geq}(\bar{\omega}), \quad |(|\vec{p}| - p_F)| > \Lambda, \end{array} \right\} = \Theta[\Lambda - |(|\vec{p}| - p_F)|]\psi_{\vec{p}}(\bar{\omega}) + \Theta[|(|\vec{p}| - p_F)| - \Lambda]\psi_{\vec{p}}(\bar{\omega}) \quad (3)$$

Define $\Theta[\Lambda - |(|\vec{p}| - p_F)|]\psi_{\vec{p}}(\bar{\omega}) \equiv \psi_{\vec{p}}^{\leq}(\bar{\omega})$, $\Theta[|(|\vec{p}| - p_F)| - \Lambda]\psi_{\vec{p}}(\bar{\omega}) \equiv \psi_{\vec{p}}^{\geq}(\bar{\omega})$.

Imagine integrating out in the path integral all $\psi_{\vec{p}}^{\leq}(\bar{\omega})$ and $\psi_{\vec{p}}^{\geq}(\bar{\omega})$, (we will do this implicitly),

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}(\bar{\omega}), \psi_{\vec{p}}(\bar{\omega})]e^{-S\{\bar{\psi}_{\vec{p}}(\bar{\omega}), \psi_{\vec{p}}(\bar{\omega})\}}$$

$$\int \mathcal{D}[\bar{\psi}_{\vec{p}}(\bar{\omega}), \psi_{\vec{p}}(\bar{\omega})] \stackrel{\text{means}}{=} \prod_{\vec{p}} \prod_{\bar{\omega}} \int d\bar{\psi}_{\vec{p}}(\bar{\omega})d\psi_{\vec{p}}(\bar{\omega}) = \int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})] \int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\geq}(\bar{\omega}), \psi_{\vec{p}}^{\geq}(\bar{\omega})]$$

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})] \int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\geq}(\bar{\omega}), \psi_{\vec{p}}^{\geq}(\bar{\omega})]e^{-S\{\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega})+\bar{\psi}_{\vec{p}}^{\geq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})+\psi_{\vec{p}}^{\geq}(\bar{\omega})\}}$$

define

$$\int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\geq}(\bar{\omega}), \psi_{\vec{p}}^{\geq}(\bar{\omega})]e^{-S\{\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega})+\bar{\psi}_{\vec{p}}^{\geq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})+\psi_{\vec{p}}^{\geq}(\bar{\omega})\}} \equiv e^{-S_{\text{eff}}^{\Lambda}\{\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})\}}$$

thus

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})]e^{-S_{\text{eff}}^{\Lambda}\{\bar{\psi}_{\vec{p}}^{\leq}(\bar{\omega}), \psi_{\vec{p}}^{\leq}(\bar{\omega})\}}$$

the partition function does not change under the RG transformation. now we end up having the fermions whose momentum length is much away from p_F by Λ .

now let us discuss how this effective action must look like.

But before: let's now choose to look at correlation functions.

$$\langle \psi_{\vec{p}_1}(\bar{\omega}_1) \dots \psi_{\vec{p}_n}(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}(\bar{\omega}'_n) \rangle_S$$

choose the momenta inside the corr. func. are inside the thin shell: $||\vec{p}_j| - p_F| < \Lambda, ||\vec{p}'_j| - p_F| < \Lambda$, thus

$$\begin{aligned} & \langle \psi_{\vec{p}_1}(\bar{\omega}_1) \dots \psi_{\vec{p}_n}(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}(\bar{\omega}'_n) \rangle_S = \langle \psi_{\vec{p}_1}^<(\bar{\omega}_1) \dots \psi_{\vec{p}_n}^<(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}^<(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}^<(\bar{\omega}'_n) \rangle_S, \\ & = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})] \psi_{\vec{p}_1}^<(\bar{\omega}_1) \dots \psi_{\vec{p}_n}^<(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}^<(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}^<(\bar{\omega}'_n) e^{-S\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})\}} \\ & = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})] \psi_{\vec{p}_1}^<(\bar{\omega}_1) \dots \psi_{\vec{p}_n}^<(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}^<(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}^<(\bar{\omega}'_n) \int \mathcal{D}[\bar{\psi}_{\vec{p}}^>(\bar{\omega}), \psi_{\vec{p}}^>(\bar{\omega})] e^{-S\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}) + \bar{\psi}_{\vec{p}}^>(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega}) + \psi_{\vec{p}}^>(\bar{\omega})\}} \\ & = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})] \psi_{\vec{p}_1}^<(\bar{\omega}_1) \dots \psi_{\vec{p}_n}^<(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}^<(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}^<(\bar{\omega}'_n) e^{-S_{\text{eff}}^{\Lambda}\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})\}} \\ & = \langle \psi_{\vec{p}_1}^<(\bar{\omega}_1) \dots \psi_{\vec{p}_n}^<(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}^<(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}^<(\bar{\omega}'_n) \rangle_{S_{\text{eff}}}. \\ & = \langle \psi_{\vec{p}_1}(\bar{\omega}_1) \dots \psi_{\vec{p}_n}(\bar{\omega}_n) \bar{\psi}_{\vec{p}'_1}(\bar{\omega}'_1) \dots \bar{\psi}_{\vec{p}'_n}(\bar{\omega}'_n) \rangle_{S_{\text{eff}}}. \end{aligned}$$

becomes the momenta inside the corr. func. are inside the thin shell: $||\vec{p}_j| - p_F| < \Lambda, ||\vec{p}'_j| - p_F| < \Lambda$. this is only valid when $||\vec{p}_j| - p_F| < \Lambda, ||\vec{p}'_j| - p_F| < \Lambda$.

If for some reason, we are only interested in the momentum shell $\Lambda/p_F \ll 1$, we can use the derivation above. Effective action allows the fermions inside the thin shell to be simple calculating. Now: what this effective action will look like? Using the $U(1)$ symmetry,

$$\begin{aligned} S_{\text{eff}}^{\Lambda}\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})\} & = \int \frac{d\bar{\omega}}{2\pi} \sum'_{\vec{p}} V_2(\vec{p}, \bar{\omega}) \bar{\psi}_{\vec{p}}^<(\bar{\omega}) \psi_{\vec{p}}^<(\bar{\omega}) \\ & + \int \frac{d\bar{\omega}_1}{2\pi} \int \frac{d\bar{\omega}_2}{2\pi} \int \frac{d\bar{\omega}_3}{2\pi} \int \frac{d\bar{\omega}_4}{2\pi} \frac{1}{\text{Vol}} \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{\vec{p}_3} \sum'_{\vec{p}_4} 2\pi \delta(\bar{\omega}_1 + \bar{\omega}_2 - \bar{\omega}_3 - \bar{\omega}_4) \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3, \vec{p}_4} \\ & \quad \times V_4(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4, \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \bar{\psi}_{\vec{p}_4}^<(\bar{\omega}_4) \bar{\psi}_{\vec{p}_3}^<(\bar{\omega}_3) \psi_{\vec{p}_2}^<(\bar{\omega}_2) \psi_{\vec{p}_1}^<(\bar{\omega}_1) \end{aligned}$$

+ All kinds of interactions with more than four fermion operators, e.g. six of them...

we just did Taylor expansion. We will expand all functions V_2, V_4, V_6, \dots about $\bar{\omega}_j = 0$, and $|\vec{p}_j| = p_F$, e.g.

$$V_2(\vec{p}, \bar{\omega}) = \frac{1}{Z_0} [-i\bar{\omega} + (\frac{p_F}{m^*})(|\vec{p}| - p_F) + O(\bar{\omega}^2, (|\vec{p}| - p_F)^2, \omega(|\vec{p}| - p_F))]$$

the Z_0 is some coefficient. The terms in the $O(\dots)$ is irrelevant, in the sense that if we shrink Λ more and more, then we will find, as making Λ/p_F even smaller, all the irrelevant terms will just go away.

$$V_4(\bar{\omega}_j, \vec{p}_j) = V_4(\bar{\omega} = 0, \vec{p}_j) + O(\bar{\omega}_j)$$

the $O(\bar{\omega}_j)$ is again irrelevant. We call $V_4(\bar{\omega} = 0, \vec{p}_j) \equiv K_4(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$.

$$V_6(\bar{\omega}_j, \vec{p}_j) = V_6(\bar{\omega}_j, |\vec{p}_j| = p_F) + \text{higher terms}$$

those in V_6 are already all irrelevant. Thus all in V_8 are irrelevant,.... That is the logic of RG.

Feb. 18

$$\bar{\psi}_{\vec{p}}(\bar{\omega}) = \bar{\psi}_{\vec{p}}^<(\bar{\omega}) + \bar{\psi}_{\vec{p}}^>(\bar{\omega}), \quad \psi_{\vec{p}}(\bar{\omega}) = \psi_{\vec{p}}^<(\bar{\omega}) + \psi_{\vec{p}}^>(\bar{\omega})$$

$$\text{If } (|\vec{p}_i| - p_F) < \Lambda, \quad (|\vec{p}_i^{\prime}| - p_F) < \Lambda,$$

$$\langle \psi_{\vec{p}_1}(\bar{\omega}_1) \cdots \psi_{\vec{p}_N}(\bar{\omega}_N) \bar{\psi}_{\vec{p}'_1}(\bar{\omega}'_1) \cdots \bar{\psi}_{\vec{p}'_N}(\bar{\omega}'_N) \rangle_S = \langle \psi_{\vec{p}_1}(\bar{\omega}_1) \cdots \psi_{\vec{p}_N}(\bar{\omega}_N) \bar{\psi}_{\vec{p}'_1}(\bar{\omega}'_1) \cdots \bar{\psi}_{\vec{p}'_N}(\bar{\omega}'_N) \rangle_{S_{\text{eff}}}.$$

$$e^{-S_{\text{eff}}\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})\}} = \int \mathcal{D}[\bar{\psi}_{\vec{p}}^<(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega})] e^{-S\{\bar{\psi}_{\vec{p}}^<(\bar{\omega}) + \bar{\psi}_{\vec{p}}^>(\bar{\omega}), \psi_{\vec{p}}^<(\bar{\omega}) + \psi_{\vec{p}}^>(\bar{\omega})\}}.$$

The shell $\Lambda/p_F \ll 1$. In general, last time we wrote down

$$\begin{aligned} S_{\text{eff}}^{\Lambda}\{\bar{\psi}_{\vec{p}}(\bar{\omega}), \psi_{\vec{p}}(\bar{\omega})\} &= \int \frac{d\bar{\omega}}{2\pi} \sum_{\vec{p}} V_2(\vec{p}, \bar{\omega}) \bar{\psi}_{\vec{p}}^<(\bar{\omega}) \psi_{\vec{p}}^<(\bar{\omega}) \\ &+ \int \frac{d\bar{\omega}_1}{2\pi} \int \frac{d\bar{\omega}_2}{2\pi} \int \frac{d\bar{\omega}_3}{2\pi} \int \frac{d\bar{\omega}_4}{2\pi} \frac{1}{\text{Vol}} \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{\vec{p}_3} \sum'_{\vec{p}_4} 2\pi \delta(\bar{\omega}_1 + \bar{\omega}_2 - \bar{\omega}_3 - \bar{\omega}_4) \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3, \vec{p}_4} \\ &\times V_4(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4, \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \bar{\psi}_{\vec{p}_4}^<(\bar{\omega}_4) \bar{\psi}_{\vec{p}_3}^<(\bar{\omega}_3) \psi_{\vec{p}_2}^<(\bar{\omega}_2) \psi_{\vec{p}_1}^<(\bar{\omega}_1) \end{aligned}$$

+ All kinds of interactions with more than four fermion operators, e.g. six of them...

Good news is that, as $\Lambda \rightarrow \Lambda' (< \Lambda)$, is further reduced, much of this vanish! We call those vanishing terms “irrelevant”.

Thus

$$V_2(\vec{p}, \bar{\omega}) \longrightarrow \frac{1}{Z_0} \left[-i\bar{\omega} + \frac{p_F}{m^*} (|\vec{p}| - p_F) \right]$$

$$V_4(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4, \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \longrightarrow V_4(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4),$$

$$V_6 \longrightarrow 0, \quad V_8 \longrightarrow 0,$$

that's the only dependence we have to care about. Thus it is simpler to do, in $\Lambda/p_F \ll 1$. Then, we can reconstruct the Hamiltonian from the coherent path integral.

Answer(the prime in the \sum meas in shell $(|\vec{p}| - p_F) < \Lambda$):

$$\hat{K}_{\text{eff}}^{\Lambda} = \sum'_{\vec{p}} \frac{p_F}{m^*} (|\vec{p}| - p_F) \hat{a}_{\vec{p}}^{\dagger} \hat{a}_{\vec{p}} + \frac{1}{2} \frac{1}{\text{Vol}} \sum'_{\vec{p}_1} \sum'_{\vec{p}_2} \sum'_{\vec{p}_3} \sum'_{\vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \hat{a}_{\vec{p}_4}^{\dagger} \hat{a}_{\vec{p}_3}^{\dagger} \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1},$$

We dropped the Z_0 that is appearing in the effective Hamiltonian. Perhaps there is some power $v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \sqrt{Z_0^{-k}} V_4(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$ but we don't care at present.

Now our previous “toy model” has become the model that describes the real world.

Look at handout, solution (I), (II) and (III):

in the limit $\Lambda/p_F \ll 1$, $|\vec{k}| \rightarrow 0$. Thus

$$\hat{K}_{\text{eff}}^\Lambda \rightarrow \hat{K}_{\text{F.L.T.}} = \sum_{\vec{p}} \left[\frac{p_F}{m^*} (|\vec{p}| - p_F) \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{1}{2} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \frac{1}{\text{Vol}} V(\hat{p}_1 - \hat{p}_2) (\hat{a}_{\vec{p}_1}^\dagger \hat{a}_{\vec{p}_1}) (\hat{a}_{\vec{p}_2}^\dagger \hat{a}_{\vec{p}_2}) \right]$$

$\hat{K}_{\text{F.L.T.}}$ is “integrable”. i.e. exactly solvable: because $\hat{n}_{\vec{p}} = (\hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}})$ has eigenvalue 0 or 1, [finite volume, \vec{p} discrete] $[\hat{n}_{\vec{p}}, \hat{K}_{\text{F.L.T.}}] = 0$, $[\hat{n}_{\vec{p}'}, \hat{n}_{\vec{p}}] = 0$. there are infinite number of conservation laws: $\hat{n}_{\vec{p}}$. The only unknown now that describes the fermi liquid theory is $V(\vec{p}_1 - \vec{p}_2)$. \hat{p}_j is $\frac{p_F}{|\vec{p}_j|}$. Thus we see that RG plus the simple geometric consideration of momentum conservation, simplifies the theory a lot.

Note: we have not discussed solution (III). Let ignore that first.

Now: we are going to use the Large “ N ” analysis and RPA (random phase approximation). Large n parameter appears naturally: $n = p_F/\Lambda$.

Now, we need the pauli spin index of fermion to write simplest equations. Why: so far, no pauli spins (spin polarized or spinless). $v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = -v(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4) = -v(\vec{p}_1, \vec{p}_2, \vec{p}_4, \vec{p}_3)$, because of the Pauli principle. Write $v(p_1, p_2, k)$, Simplest situation: set v to be constant. But we can't, because $v(p_1, p_2, k) \rightarrow 0$ as $p_1 \rightarrow p_2$. The way out is to generalize to include Pauli spin. Rewrite $\hat{K}_{\text{eff}}^\Lambda$ with spin index:

with Pauli spin, $SU(2)$ invariant: (repeated indices $\alpha, \beta, \gamma, \delta$ are summed)

$$\hat{K}_{\text{eff}}^\Lambda = \sum_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}_1, \alpha}^\dagger \hat{a}_{\vec{p}_1, \alpha} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{\vec{p}_3} \sum_{\vec{p}_4} \times$$

$$\times \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} [v_1(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \delta_{\alpha\beta} \delta_{\gamma\delta} + v_2(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \delta_{\alpha\delta} \delta_{\beta\gamma}] \hat{a}_{\vec{p}_4, \alpha}^\dagger \hat{a}_{\vec{p}_3, \gamma}^\dagger \hat{a}_{\vec{p}_2, \delta} \hat{a}_{\vec{p}_1, \beta}$$

there are only two delta functions for the spin, because of the $SU(2)$ symmetry. (Perform a $SU(2)$ transformation, system is invariant so...) if I change $(\vec{p}_2, \delta) \leftrightarrow (\vec{p}_1, \beta)$ v picks up a minus sign. Thus v_1 and v_2 are different! $v_1(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4) = -v_2(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$, $v_1(\vec{p}_1, \vec{p}_2, \vec{p}_4, \vec{p}_3) = -v_2(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$, we can relabel. (the same logic when we argue that solution (II) is solution (I)) so we can write $v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = 2v_1(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)$, and

$$\hat{K}_{\text{eff}}^\Lambda = \sum_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}_1, \alpha}^\dagger \hat{a}_{\vec{p}_1, \alpha} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{\vec{p}_3} \sum_{\vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} v(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \delta_{\alpha\beta} \delta_{\gamma\delta} \hat{a}_{\vec{p}_4, \alpha}^\dagger \hat{a}_{\vec{p}_3, \gamma}^\dagger \hat{a}_{\vec{p}_2, \delta} \hat{a}_{\vec{p}_1, \beta}.$$

$$\delta_{\alpha\beta} \delta_{\gamma\delta} = \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\delta} \delta_{\gamma\beta}] + \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta}],$$

$$v_1(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{1}{2} [v_1(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) + v_1(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4)] + \frac{1}{2} [v_1(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) - v_1(\vec{p}_2, \vec{p}_1, \vec{p}_3, \vec{p}_4)] \equiv v_+(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) + v_-(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4),$$
 Then

$$\hat{K}_{\text{eff}} = \sum_{\vec{p}} \xi_{\vec{p}} \hat{a}_{\vec{p}_1, \alpha}^\dagger \hat{a}_{\vec{p}_1, \alpha} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{\vec{p}_3} \sum_{\vec{p}_4} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} \times$$

$$\times \left\{ \begin{array}{l} v_-(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\delta} \delta_{\gamma\beta}] + \\ v_+(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta}] \end{array} \right\} \hat{a}_{\vec{p}_4, \alpha}^\dagger \hat{a}_{\vec{p}_3, \gamma}^\dagger \hat{a}_{\vec{p}_2, \delta} \hat{a}_{\vec{p}_1, \beta}.$$

Now, generalize the idea of fermi liquid/RG to spin. To write in terms of Pauli matrices: identity:

$$\sum_{a=1,2,3} \left(\frac{1}{2} \sigma^a \right)_{\alpha\beta} \left(\frac{1}{2} \sigma^a \right)_{\gamma\delta} = -\frac{1}{4} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{1}{2} \delta_{\gamma\beta} \delta_{\alpha\delta},$$

or

$$\delta_{\gamma\beta}\delta_{\alpha\delta} = 2 \sum_{a=1,2,3} \left(\frac{1}{2}\sigma^a\right)_{\alpha\beta} \left(\frac{1}{2}\sigma^a\right)_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\beta}\delta_{\gamma\delta}.$$

Thus

$$\begin{aligned} \hat{K}_{\text{eff}} &= \sum_{\vec{p}}' \xi_{\vec{p}} \hat{a}_{\vec{p}_1, \alpha}^\dagger \hat{a}_{\vec{p}_1, \alpha} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1}' \sum_{\vec{p}_2}' \sum_{|\vec{k}| < \Lambda} V_c(\vec{p}_1, \vec{p}_2, \vec{k}) : (\hat{a}_{\vec{p}_1 - \vec{k}, \alpha_1}^\dagger \hat{a}_{\vec{p}_1, \alpha_1}) (\hat{a}_{\vec{p}_2 + \vec{k}, \alpha_2}^\dagger \hat{a}_{\vec{p}_2, \alpha_2}) : \\ &+ \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1}' \sum_{\vec{p}_2}' \sum_{|\vec{k}| < \Lambda} V_s(\vec{p}_1, \vec{p}_2, \vec{k}) \sum_{a=1,2,3} : (\hat{a}_{\vec{p}_1 - \vec{k}, \alpha_1}^\dagger \left(\frac{\sigma^a}{2}\right)_{\alpha_1 \beta_1} \hat{a}_{\vec{p}_1, \beta_1}) (\hat{a}_{\vec{p}_2 + \vec{k}, \alpha_2}^\dagger \left(\frac{\sigma^a}{2}\right)_{\alpha_2 \beta_2} \hat{a}_{\vec{p}_2, \beta_2}) : + W\text{-function} \end{aligned}$$

This is the standard fermi liquid theory.

Feb 23

Last time we studied spin liquid with Pauli degree of freedom. In the handout: V_x , $x = c, s$ denote the potential for charge or spins, \vec{p}_1 and \vec{p}_2 are both of the magnitude of p_F , $\nu(0)V_x(\vec{p}_1, \vec{p}_2) = \sum_{l=0}^{\infty} F_l^x P_l(\cos \theta)$, $V_x(\vec{p}_1, \vec{p}_2; \vec{k})$, $\sum_{l=-\infty}^{\infty} F_l^x e^{il\theta_{12}}$, $\theta_{12} = \langle \vec{p}_1, \vec{p}_2 \rangle$, $F_l^x = \text{fermi liquid parameter}$, $\hat{p}_j = \vec{p}_j / |\vec{p}_j|$. Recall $d = 3$: $P_l(\cos \theta_{12}) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{p}_1) Y_{lm}(\hat{p}_2)$, density of states: $\nu(\xi)$, $\xi \rightarrow \xi_{\vec{p}} = \frac{\vec{p}^2}{2m} - \frac{p_F^2}{2m} = (|\vec{p}| - p_F) + O(|\vec{p}| - p_F)^2$,

$$\nu(\xi) = \frac{\sum_{\vec{p}}' \delta(\xi - \xi_{\vec{p}})}{\text{Vol}} = \frac{\# \text{ of states at energy } \xi}{\text{Vol}} \rightarrow \int \frac{d^d \vec{p}}{(2\pi)^d} \delta(\xi - \xi_{\vec{p}}).$$

$$\nu(\xi = 0) = \int \frac{d^d \vec{p}}{(2\pi)^d} \delta(\xi - \xi_{\vec{p}}) \delta[v_F(|\vec{p}| - p_F)] = \int_{\text{angles}} \frac{d\hat{\Omega}_{\vec{p}}}{(2\pi)^d} \int_0^\infty p^{d-1} dp \delta[v_F(p - p_F)] = \frac{1}{v_F} \int \frac{d\Omega_{\vec{p}}}{(2\pi)^d} p_F^{d-1},$$

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \cdot \Gamma(1/2) = \sqrt{\pi}.$$

Get back to large n calculation: $n = p_F / \Lambda$

$$\begin{aligned} \hat{K}_{\text{eff}} &= \sum_{\vec{p}}' \xi_{\vec{p}} \hat{a}_{\vec{p}_1, \alpha}^\dagger \hat{a}_{\vec{p}_1, \alpha} + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1}' \sum_{\vec{p}_2}' \sum_{\vec{p}_3}' \sum_{\vec{p}_4}' \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_3 + \vec{p}_4} \times \\ &\times \left\{ \begin{aligned} &v_-(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\delta} \delta_{\gamma\beta}] + \\ &v_+(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \frac{1}{2} [\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta}] \end{aligned} \right\} \hat{a}_{\vec{p}_4, \alpha}^\dagger \hat{a}_{\vec{p}_3, \gamma}^\dagger \hat{a}_{\vec{p}_2, \delta} \hat{a}_{\vec{p}_1, \beta}. \end{aligned}$$

where v_+ is even function, can take as $v_+ = v$ as a const, while the upper term v_- is zero !

In the following we proceed ASSUMING v_- IS ZERO, and use the notation $v_+ = v$.

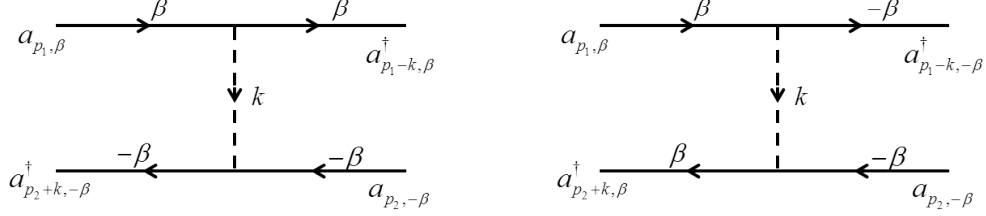
comment: $[\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta}] = \frac{1}{2} \delta_{\alpha\beta} \delta_{\gamma\delta} - 2 \sum_{a=1,2,3} \left(\frac{1}{2}\sigma^a\right)_{\alpha\beta} \left(\frac{1}{2}\sigma^a\right)_{\gamma\delta}$, the first term is v_c , the second

is v_s , $v_c = \frac{1}{2}v$, $v_s = -2v$, check:

$$\sum_{\alpha\beta\gamma\delta} [\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\delta} \delta_{\gamma\beta}] : (a_{\vec{p}_1 - \vec{k} = \vec{p}_4, \alpha}^\dagger a_{\vec{p}_1, \beta}) (a_{\vec{p}_2 + \vec{k} = \vec{p}_3, \gamma}^\dagger a_{\vec{p}_2, \delta}) :$$

$$= \sum_{\beta=\pm 1} \left(: (a_{p_1-k,\beta}^\dagger a_{p_1,\beta}) (a_{p_2,k,-\beta}^\dagger a_{p_2,-\beta}) : - : (a_{p_1-k,-\beta}^\dagger a_{p_1,\beta}) (a_{p_2+k,\beta}^\dagger a_{p_2,-\beta}) : \right)$$

diagram for the last two terms are shown below:



Now we discuss emerging large n , or zero's sound collective excitation in a fermi liquid. We will go back to density-density correlation function.

$$\mathcal{D}(r - r', \tau - \tau') = \langle (\bar{\psi}_\sigma(r, \tau) \psi_\sigma(r, \tau)) (\bar{\psi}_{\sigma'}(r', \tau') \psi_{\sigma'}(r', \tau')) \rangle_{c,s}$$

= diagrams below:

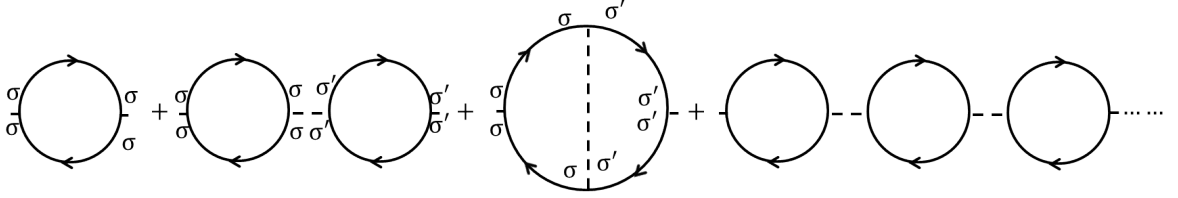


Figure 6: diagrams for $\mathcal{D}(r - r', \tau - \tau')$.

Look at second and third diagram, call A and B. diagram A is shown in fig.(7).

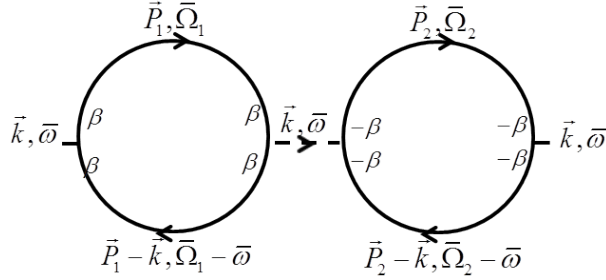


Figure 7: diagram A. Note that the interaction used is the first term: $: (a_{p_1-k,\beta}^\dagger a_{p_1,\beta}) (a_{p_2,k,-\beta}^\dagger a_{p_2,-\beta}) :$

fourier transformed:the labels on the diagram from left to right (and up and down) is $\mathcal{D}(\vec{k}, \bar{\omega})$, $\vec{k}, \bar{\omega} || \vec{p}_1, \bar{\Omega}_1, \vec{p}_1 - \vec{k}, \bar{\Omega}_1 - \bar{\omega}, || \vec{k}, \bar{\omega}, || \vec{p}_2 + \vec{k}, \bar{\Omega}_2 + \bar{\omega}, \vec{p}_2, \bar{\Omega}_2, || \vec{k}, \bar{\omega}$, The vetice from left to right: four

only a factor of 2 shows up). v : constant strength of interaction; this is valid when $n|\vec{k}| < \Lambda$ and $p_F/\Lambda \gg 1$.

Next time: discuss the retarded greens function associated with that.

Feb 25

Fermi liquid

$$\mathcal{D}(r-r', \tau-\tau') = \langle (\bar{\psi}_\sigma(r\tau)\psi_\sigma(r\tau))(\bar{\psi}_{\sigma'}(r'\tau')\psi_{\sigma'}(r'\tau'))_s \rangle - \langle (\bar{\psi}_\sigma(r\tau)\psi_\sigma(r\tau)) \rangle \langle (\bar{\psi}_{\sigma'}(r'\tau')\psi_{\sigma'}(r'\tau'))_s \rangle$$

Fourier interaction $=v$ 0 constant in \vec{p} space.

$$\mathcal{D}(\vec{k}, \bar{\omega}) = \frac{2\Pi(\vec{k}, \bar{\omega})}{1 - \Pi(\vec{k}, \bar{\omega})v}$$

$$(-1)\mathcal{D}(\vec{k}, \bar{\omega})|_{i\bar{\omega} \rightarrow \omega + i0^+} = D_{\text{ret}}(\vec{k}, \omega)$$

Why care about $\mathcal{D}(\vec{k}, \bar{\omega})$? One reason: Linear response.

Apply external potential $U(r, t)$ coupling to density of fermions, $\hat{K} \rightarrow \hat{K} + \int d^d r \hat{n}(r)U(r, t) \rightarrow$ fourier:
 $U(\vec{k}, \bar{\omega})$ and measure resulting change in expectation value of density operator $\delta\langle \hat{n} \rangle(\vec{k}, \omega) = D_r(\vec{k}, \omega)u(\vec{k}, \omega)$.

Assume $T = 0$, in general

$$D_r(\vec{k}, \omega) = D_r^{\hat{n}_{\vec{k}}\hat{n}_{\vec{k}}^\dagger}(\omega) = \sum_n \left[\frac{\langle 0|\hat{n}_{\vec{k}}|n\rangle\langle n|\hat{n}_{\vec{k}}^\dagger|0\rangle}{\omega - (k_n - k_0 - i0^+)} - \frac{\langle 0|\hat{n}_{\vec{k}}^\dagger|n\rangle\langle n|\hat{n}_{\vec{k}}|0\rangle}{\omega - (-(k_n - k_0) - i0^+)} \right]$$

Tell us: response blows up whenever ω equals one of the excitation energies $k_n - k_0$. (unless matrix element in numerator vanishes for some reason.)

infinite volume limit, $D_r(\vec{k}, \omega)$ develops in general branch cut. see explicitly in example below: analytic continuation: $\Pi_{0,r}(\vec{k}, \omega) = \Pi_0(\vec{k}, \omega)|_{i\bar{\omega} \rightarrow \omega + i0^+}$,

$$D_r(\vec{k}, \omega) = \frac{2\Pi_{0,r}(\vec{k}, \omega)}{1 - \Pi_{0,r}(\vec{k}, \omega)v}$$

Non-interacting system $v = 0$:

$$D_r^{(0)}(\vec{k}, \omega) = 2\Pi_{0,r}(\vec{k}, \omega) = \sum_n \left[\frac{\langle 0|\hat{n}_{\vec{k}}|n\rangle\langle n|\hat{n}_{\vec{k}}^\dagger|0\rangle}{\omega - (k_n - k_0 - i0^+)} - \frac{\langle 0|\hat{n}_{\vec{k}}^\dagger|n\rangle\langle n|\hat{n}_{\vec{k}}|0\rangle}{\omega - (-(k_n - k_0) - i0^+)} \right]$$

$$\hat{K}^{(0)}|n\rangle_0 = k_n^{(0)}|n\rangle_0,$$

Interactions: $D(\vec{k}, \omega) =$ poles from numerator plus poles from vanishing of denominator (these are due to interactions)

extra poles: from vanishing of denominator

$\frac{1}{v} = \Pi_{0,r}(\vec{k}, \omega) = \frac{1}{2}D_{0,r}(\vec{k}, \omega)$. Solve this equation first graphically, plot at fixed value of \vec{k} as a function of ω .

last pole will stay isolated: zero sound collective mode.

An explicit calculation:

Evaluation of $\Pi_0(\vec{q}, \bar{\omega})$ for $|\vec{q}| \ll p_F$ any $\bar{\omega}$, at $T = 0$.

$\Pi_0(\vec{q}, \bar{\omega})$: a one loop diagram (see fig.9), $\vec{q}, \bar{\omega}, \vec{P} + \frac{\vec{q}}{2}, \vec{\Omega} + \bar{\omega}, \vec{P} - \frac{\vec{q}}{2}, \vec{\Omega}, \vec{q}, \bar{\omega}, \xi_{\vec{P}} = \frac{\vec{P}^2}{2m} - \mu = \frac{\vec{P}^2}{2m} - \frac{P_F^2}{2m}$,

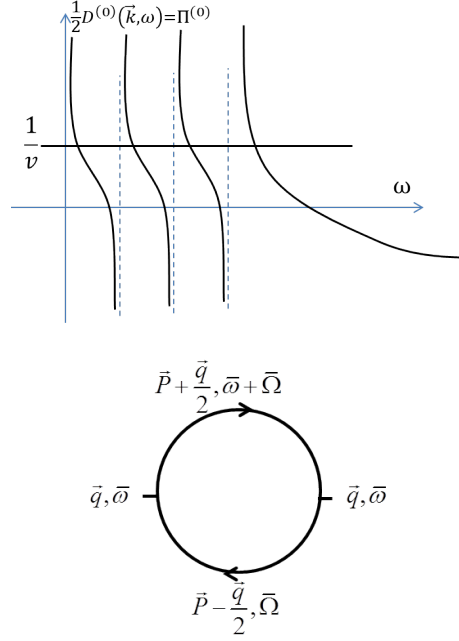


Figure 9: The one loop diagram that contributes to $\Pi_{p,r}(\vec{k}, \omega)$.

$$\begin{aligned}
& (-i)^2 (\text{two yukawa vertices}) (-1) (\text{fermi loop}) \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \int \frac{d^d \vec{P}}{(2\pi)^d} \frac{(-1)}{i\bar{\Omega} - \xi_{\vec{p}-\vec{q}/2}} \frac{(-1)}{i(\bar{\Omega} + \bar{\omega}) - \xi_{\vec{p}+\vec{q}/2}} \\
&= \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \int \frac{d^d \vec{P}}{(2\pi)^d} \frac{(-1)(-i)}{\bar{\Omega} - (-i)\xi_{\vec{p}-\vec{q}/2}} \frac{(-1)(-i)}{(\bar{\Omega} - (-\bar{\omega} - i\xi_{\vec{p}+\vec{q}/2}))} \equiv \int \frac{d^d \vec{P}}{(2\pi)^d} I \\
I &= (-i)^2 \frac{2\pi i}{2\pi} \left\{ \frac{[1 - \Theta(\xi_{\vec{p}-\vec{q}/2})]\Theta(\xi_{\vec{p}+\vec{q}/2})}{(-i)\xi_{\vec{p}-\vec{q}/2} + \bar{\omega} + i\xi_{\vec{p}+\vec{q}/2}} - \frac{\Theta(\xi_{\vec{p}-\vec{p}/2})[1 - \Theta(\xi_{\vec{p}+\vec{q}/2})]}{(-i)\xi_{\vec{p}-\vec{q}/2} + \bar{\omega} + i\xi_{\vec{p}+\vec{q}/2}} \right\} \\
I &= (-i) \frac{\Theta(\xi_{\vec{p}+\vec{q}/2}) - \Theta(\xi_{\vec{p}-\vec{q}/2})}{(-i)\xi_{\vec{p}-\vec{q}/2} + \bar{\omega} + i\xi_{\vec{p}+\vec{q}/2}} = \frac{\Theta(\xi_{\vec{p}+\vec{q}/2}) - \Theta(\xi_{\vec{p}-\vec{q}/2})}{i\bar{\omega} - (\xi_{\vec{p}+\vec{q}/2} - \xi_{\vec{p}-\vec{q}/2})}
\end{aligned}$$

Thus

$$\Pi_0(\vec{q}, \bar{\omega}) = \int \frac{d^d \vec{P}}{(2\pi)^d} \frac{\Theta(\xi_{\vec{p}+\vec{q}/2}) - \Theta(\xi_{\vec{p}-\vec{q}/2})}{i\bar{\omega} - (\xi_{\vec{p}+\vec{q}/2} - \xi_{\vec{p}-\vec{q}/2})}$$

Now we have condition $|\vec{q}| \ll p_F$, $\xi_{\vec{p} \pm \vec{q}/2} - \xi_{\vec{p}} = \frac{1}{2} \left((\vec{P} \pm \vec{q}/2)^2 - \vec{P}^2 \right)^2 = \pm \frac{\vec{P} \cdot \vec{q}}{2m} - \frac{(\vec{q})^2}{8m}$, the second term is much smaller than the first. Thus

$$\Theta(\xi_{\vec{p}+\vec{q}/2}) - \Theta(\xi_{\vec{p}-\vec{q}/2}) = [\xi_{\vec{p}} + (\xi_{\vec{p}+\vec{q}/2} - \xi_{\vec{p}})] \Theta'(\xi_{\vec{p}}) + \frac{1}{2} (\xi_{\vec{p}+\vec{q}/2} - \xi_{\vec{p}})^2 \Theta''(\xi_{\vec{p}}) + \dots - [\xi_{\vec{p}} + (\xi_{\vec{p}-\vec{q}/2} - \xi_{\vec{p}})] \Theta'(\xi_{\vec{p}}) + \frac{1}{2} (\xi_{\vec{p}-\vec{q}/2} - \xi_{\vec{p}})^2 \Theta''(\xi_{\vec{p}}) + \dots$$

$$= (\xi_{\vec{P}+\vec{q}/2} - \xi_{\vec{P}-\vec{q}/2})\Theta'(\xi_{\vec{P}}) + \dots = \frac{\vec{P} \cdot \vec{q}}{m} \delta(\xi_{\vec{P}})$$

Collect:

$$\Pi_0(\vec{q}, \bar{\omega}) = \int \frac{d^d \vec{P}}{(2\pi)^d} \frac{\vec{P} \cdot \vec{q} \delta(\xi_{\vec{P}})}{i\bar{\omega} - \frac{\vec{P} \cdot \vec{q}}{m}} + \dots$$

$$\delta \text{ function: } \delta(\xi_{\vec{P}}) - \delta[v_F(P - p_F) + O(P - P_F)^2] = \frac{1}{v_F} \delta((p - p_F) + \dots) = \frac{1}{v_F} \delta(p - p_F) = \frac{m}{p_F} \delta(p - p_F)$$

$$\Pi(\vec{q}, \bar{\omega}) = \int \frac{d^d \vec{P}}{(2\pi)^d} \frac{\vec{P} \cdot \vec{q} \delta(p - p_F)}{i\bar{\omega} - \frac{\vec{P} \cdot \vec{q}}{m}}$$

Now specialize to $d = 3$: ($d = 2$ analogous)

$$\begin{aligned} \Pi_0(\vec{q}, \bar{\omega}) &= \int \frac{p^2 dp}{(2\pi)^3} \int d \cos \theta d\phi \frac{Pq \cos \theta \frac{1}{p_F} \delta(p - p_F)}{i\bar{\omega} - \frac{Pq}{m} \cos \theta} \\ &= \frac{p_F^2}{(2\pi)^2} \int_{-1}^1 dx \frac{qx}{i\bar{\omega} - \frac{p_F}{m} qx} = \frac{p_F^2}{4\pi^2} \frac{1}{v_f} \left\{ -2 - \frac{i\bar{\omega}}{v_F q} \ln \left[\frac{1 - \frac{i\bar{\omega}}{v_F q}}{-1 - \frac{i\bar{\omega}}{v_F q}} \right] \right\} \\ &= \frac{1}{2} \frac{p_F^2}{\pi^2} \frac{1}{v_F} = \nu_{3D}(\xi = 0) \text{ is the fermi states density at 3D.} \end{aligned}$$

March 1

$$D_r(\vec{k}, \omega) = \frac{2\Pi_{0,r}(\vec{k}, \omega)}{1 - \Pi_{0,r}(\vec{k}, \omega)v}$$

is the retarded version of fermion density: $\mathcal{D}(x, x') = \langle \bar{\psi}_\sigma(x) \psi_\sigma(x) \bar{\psi}_{\sigma'}(x') \psi_{\sigma'}(x') \rangle$, $x = (\vec{r}, \tau)$.

We calculated Π_0 .

fixed \vec{k} : $D_r(\vec{k}, \omega)$ is of the $\cot(x)$ function type, but the rightmost pole is different. the poles. At $\Pi = 1/v$ has some extra holes. The continuum poles become branch cut.

$$\Pi_0(\vec{q}, \bar{\omega}) = \nu_{3D}(\xi = 0) \left\{ \frac{1}{2} \frac{i\bar{\omega}}{v_F q} \ln \left[\frac{1 - \frac{i\bar{\omega}}{v_F q}}{-1 - \frac{i\bar{\omega}}{v_F q}} \right] - 1 \right\}$$

the prefactor is the fermi states density at 3D.

$$\text{at } T = 0: G_r^{A\bar{A}}(\omega) = \sum_n \left[\frac{\langle 0|\hat{A}|n\rangle \langle n|\hat{A}^\dagger|0\rangle}{\omega - (k_n - k_0 - i0^+)} - \frac{\langle 0|\hat{A}|n\rangle \langle \hat{A}^\dagger|0\rangle}{\omega - (k_n - k_0) - i0^+} \right]. \text{ Lenman representation.}$$

Now, to analytically continue it:

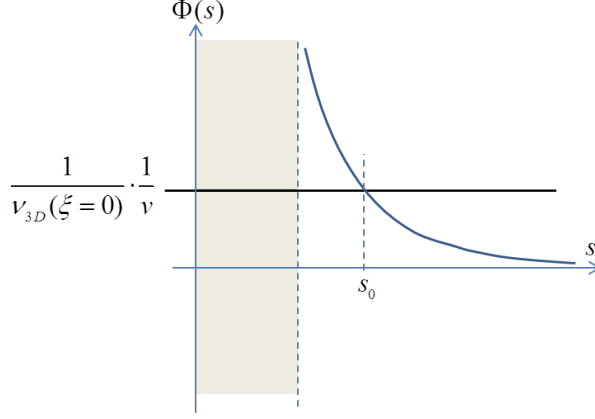
$$\Pi_{0,r}(\vec{q}, \omega) = \nu_{3D}(\xi = 0) \Phi \left(\frac{\omega + i0^+}{v_F q} \right)$$

$\Phi(z) = \frac{z}{2} \ln \frac{z+1}{z-1} - 1$. analytic properties of this function: branch cut at $Re z \in [-1, 1]$. around $+1$: $z' = (z - 1) = re^{i\phi}$, $z = 1 + re^{i\phi}$, $iIm\Phi(x + i0^+) = iIm\Phi(1 + re^{i\pi}) = iIm \left(\frac{1-r}{2} \ln \left[\frac{2-r}{e^{i\pi}} \right] \right) = \frac{1-r}{2} (-i\pi) = \frac{\pi}{2} (-i\pi)$, only when $|x| < 1$. Thus $i\Im\Phi(x + i0^+) = \frac{\pi}{2} (-i\pi)$,

$$\Phi(x + i0^+) = \left(\frac{|x|}{2} \ln \frac{|x| + 1}{|x| - 1} - 1 \right) - i \frac{\pi}{2} x \Theta(1 - |x|).$$

$$\Pi_{0,r}(\vec{q}, \omega) = \nu_{3D}(\xi = 0) \left[\frac{|s|}{2} \ln \frac{|s|+1}{|s|-1} - 1 - \frac{i\pi}{2} s \Theta(1-|s|) \right], \quad s = \frac{\omega}{v_F q}.$$

Now go back to vanishing the denominator, $\frac{i}{v} = \Pi_{0,r}(\vec{k}, \omega)$,



The plot of $\Phi(s)$ - s : continuum branch cut on the left of $s = 1$ and a single pole on the right (the zero sound mode). this is the zero sound mode of the interacting fermions system: $\omega = s_0 v_F k$, energy propto wavevector, and $s_0 v_F$ is like the sound velocity.

$\omega - k = |\vec{k}|$ plot: zero sound mode at $\omega = s_0 v_F k$, $s_0 > 1$. at $\omega \leq v_F k$, particle-hole continuum, can excite arbitrarily close to fermi surface excitations. They are called p-h continuum because any of these excitations create pair of them. The zero sound mode will not decay, it will live on forever. what is zero sound mode physically? It appears already at $T = 0$, thus it has nothing to do with actually sound. It is a pure quantum phenomenon. second, to visualize it:

If we take a fermi sea $p_x - p_y$, collective mode involving all fermions moving in the same directions. A non-interacting fermion system doesn't have this collective mode, so it is due to interactions.

$$D_r(\vec{k}, \omega) = \frac{2\nu_{3D}(0)\Phi(s)}{1 - \nu_{3D}(0)\Phi(s)v}.$$

Expand the denominator about $s = s_0$: $[1 - \nu_{3D}v\Phi(s)] = [1 - \nu_{3D}v\Phi(s_0)] + (-1)\nu_{3D}v((s - s_0)\Phi'(s_0) + \frac{(s-s_0)^2}{2}\Phi''(s_0) + \dots)$. we see there is pole pf D_r in $s - s_0$!

use fact $\Phi(s) = \Phi(-s)$ is symmetric $\Phi'(s) = -\Phi'(-s)$, therefore

$$\begin{aligned} D_r(\vec{k}, \omega) &= \frac{2\nu_{3D}\Phi(|s_0|)}{(-1)\nu_{3D}v(s - |s_0|)\Phi'(|s_0|)} + \frac{2\nu_{3D}\Phi(-|s_0|)}{(-1)\nu_{3D}v(s + |s_0|)\Phi'(-|s_0|)} + \text{other isolated poles} \\ &= \frac{2\Phi(|s_0|)}{(-1)v\Phi'(|s_0|)} \left[\frac{1}{s - |s_0|} - \frac{1}{s + |s_0|} \right] = \frac{4\Phi(|s_0|)|s_0|}{(-1)v\Phi'(|s_0|)} \frac{1}{s^2 - s_0^2} + \text{other}. \end{aligned}$$

Recall HW #3, represent interactions in terms of additional/auxiliary field, $\chi(x)$, $x = (r, \tau)$, $\langle \chi(x)\chi(x') \rangle_s = g(x-x') = g(r-r', \tau-\tau')$, fourier transform it to get $g(\vec{p}, \bar{\omega}) = g_0(\vec{p}, \bar{\omega})$ [the original potential, which is a constant.] $-g_0(\vec{p}, \bar{\omega})\tilde{\mathcal{D}}(\vec{p}, \bar{\omega})g_0(\vec{p}, \bar{\omega})$, $\tilde{\mathcal{D}} = \frac{1}{2}\mathcal{D}$ is the density density correlation function. $g(p, \bar{\omega}) = \frac{1}{(g_0)^{-1} - \Pi}$, in the RPA proximation, become $\frac{1}{(g_0)^{-1} - \Pi_0} = \frac{1}{\frac{1}{v} - \Pi_0}$.

Here in $D_r(\vec{k}, \omega) = \frac{2\nu_{3D}(0)\Phi(s)}{1-\nu_{3D}(0)\Phi(s)v}$, $\Pi_{0,r} = \nu_{3D}\Phi(s)$, $g_r(p, \omega) = A\frac{1}{s^2-s_0^2} + \text{other}$, $= A\frac{1}{(\frac{\omega}{v_F q})^2-s_0^2} = A\frac{v_F^2 q^2}{\omega^2-(v_F s_0 q)^2}$, $\omega = \pm v_F s_0 q$ (pole). No different except for q^2 numerator, from the $\langle \psi \psi \rangle$ correlators in Phy 217A for magnetism: $\mathcal{L} = \frac{1}{2}[(\partial_\tau \phi)^2 - (\nabla \phi)^2] + \phi^4 + \dots$.

We are done with Fermi liquid. Next topic: deal with term solution III (W term) that appear in the shell, that leads to SC: How to handle the W term .

March 3

Term below SU(2) invariant interactions generalization to include pauli spin. ($\alpha = +1 = \uparrow$, $\alpha = -1 = \downarrow$), $\xi_p = \frac{p^2}{2m} - \mu$.

$$S = \int d\tau \left\{ \sum_p \bar{\psi}_{p,\alpha}(\tau) \left[\frac{\partial}{\partial \tau} + \xi_p \right] \psi_{p,\alpha}(\tau) + [SU(2) \text{ invariant interactions}] + \right. \\ \left. + \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{|\vec{k}| < \Lambda} V_c(\vec{p}_1, \vec{p}_2, \vec{k}) : (\hat{a}_{\vec{p}_1 - \vec{k}, \alpha_1}^\dagger \hat{a}_{\vec{p}_1, \alpha_1}) (\hat{a}_{\vec{p}_2 + \vec{k}, \alpha_2}^\dagger \hat{a}_{\vec{p}_2, \alpha_2}) \right. \\ \left. \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{|\vec{k}| < \Lambda} V_s(\vec{p}_1, \vec{p}_2, \vec{k}) \sum_{a=1,2,3} : (\hat{a}_{\vec{p}_1 - \vec{k}, \alpha_1}^\dagger \left(\frac{\sigma^a}{2}\right)_{\alpha_1 \beta_1} \hat{a}_{\vec{p}_1, \beta_1}) (\hat{a}_{\vec{p}_2 + \vec{k}, \alpha_2}^\dagger \left(\frac{\sigma^a}{2}\right)_{\alpha_2 \beta_2} \hat{a}_{\vec{p}_2, \beta_2}) : + W\text{-function} \right.$$

W -function is

$$+ \frac{1}{2} \frac{1}{\text{Vol}} \sum_{\vec{p}_1} \sum_{\vec{p}_2} \sum_{\vec{k}} [W_0(p_1, p_2; k) (\bar{\psi}_{p_3, \alpha_1} (s\sigma^y)_{\alpha_1 \beta_1} \bar{\psi}_{-p_3+k, \beta_1})_\tau (\psi_{-p_1, \alpha_2} (-i\sigma^y)_{\alpha_2, \beta_2} \psi_{p_1+k, \beta_2})_\tau \\ + W_1(p_1, p_3; k) (\bar{\psi}_{p_3, \alpha_1} (\vec{\sigma} \cdot i\sigma^y)_{\alpha_1 \beta_1} \bar{\psi}_{-p_3+k, \beta_1})_\tau \cdot (\psi_{-p_1, \alpha_2} (-i\sigma^y \cdot \vec{\sigma})_{\alpha_2 \beta_2} \psi_{p_1+k, \beta_2})_\tau].$$

$$\text{note } \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \int_\tau = \int_{-\beta/2}^{\beta/2} d\tau, i\sigma_{\alpha\beta}^y = \epsilon_{\alpha\beta}, \vec{\sigma} i\sigma^y = \begin{pmatrix} i\sigma^x \sigma^y \\ i\sigma^y \sigma^y \\ i\sigma^z \sigma^y \end{pmatrix}. Z = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S}.$$

Simplest case $V_c = V_s = W_1 = 0$, only consider $W_0(\vec{p}_1, \vec{p}_3, \vec{k}) = -\frac{\lambda}{2} = \text{constant}$, and $\lambda \geq 0$, thus W_0 is attractive.

Decouple: 4-fermi interaction W_0 with auxiliary (hubbard stratonovich) field Δ, Δ^* .

For each value of k .

$$\prod_k \left[\exp \left\{ \lambda \frac{1}{\text{Vol}} \int_\tau \left(\sum_{p_3} \bar{\psi}_{p_3, \uparrow} \bar{\psi}_{-p_3+k, \downarrow} \right)_\tau \left(\sum_{p_1} \psi_{p_1+k, \downarrow} \psi_{-p_1, \uparrow} \right)_\tau \right\} \right]$$

Let us define

$$\left(\sum_{p_3} \bar{\psi}_{p_3, \uparrow} \bar{\psi}_{-p_3+k, \downarrow} \right)_\tau = \Phi_k^*(\tau), \quad \left(\sum_{p_1} \psi_{p_1+k, \downarrow} \psi_{-p_1, \uparrow} \right)_\tau = \Phi_k(\tau),$$

$$\text{use } \frac{1}{\chi} (\Delta^* - \lambda \Phi^*) (\Delta - \lambda \Phi) = \frac{1}{\chi} \Delta \Delta^* - (\Delta^* \Phi + \Delta \Phi^*) + \lambda \Phi^* \Phi,$$

$$e^{-\frac{1}{\chi} (\Delta^* - \lambda \Phi^*) (\Delta - \lambda \Phi)} e^{-\lambda \Phi^* \Phi} = \int d\Delta^* d\Delta e^{-\frac{1}{\chi} \Delta^* \Delta + (\Delta \Phi^* + \Delta^* \Phi)},$$

the previous is equal to

$$\prod_k \left[(\text{const.}) \int \mathcal{D}[\Delta_k(\tau), \Delta_k^*(\tau)] \exp \left\{ -\frac{1}{\text{vol}} \frac{1}{\lambda} \int_\tau \Delta_k^*(\tau) \Delta_k(\tau) + \frac{1}{\text{vol}} \int_\tau \Delta_k(\tau) \Phi_k^*(\tau) + \Delta_k^*(\tau) \Phi_k(\tau) \right\} \right]$$

To summarize: $V_c = V_s = W_i = 0$, W_0 is constant:

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}[\psi_{p,\alpha}(\tau), \bar{\psi}_{p,\alpha}(\tau)] e^{-S} = \int \mathcal{D}[\psi_{p,\alpha}(\tau), \bar{\psi}_{p,\alpha}(\tau)] \int \mathcal{D}[\Delta_k(\tau), \Delta_k^*(\tau)] e^{-\tilde{S}} \\ \tilde{S} &= \int_\tau \sum_p \bar{\psi}_{p,\alpha}(\tau) \left[\frac{\partial}{\partial \tau} + \xi_p \right] \psi_{p,\alpha}(\tau) + \frac{1}{\lambda} \frac{1}{\text{vol}} \sum_k \Delta_k^*(\tau) \Delta_k(\tau) - \frac{1}{\text{vol}} \sum_k [\Delta_k(\tau) \Phi_k^*(\tau) + \Delta_k^*(\tau) \Phi_k(\tau)] \end{aligned}$$

define

$$\tilde{S}_0 = \int_\tau \sum_p \bar{\psi}_{p,\alpha}(\tau) \left[\frac{\partial}{\partial \tau} + \xi_p \right] \psi_{p,\alpha}(\tau),$$

we have

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}[\Delta_k(\tau), \Delta_k^*(\tau)] e^{-\frac{1}{\lambda} \int_\tau (\frac{1}{\text{vol}} \sum_k \Delta_k^*(\tau) \Delta_k(\tau))} Z_0 \cdot \frac{1}{Z_0} \int \mathcal{D}[\bar{\psi}, \psi] e^{\int_\tau (\frac{1}{\text{vol}} \sum_k [\Delta_k(\tau) \Phi_k^*(\tau) + \Delta_k^*(\tau) \Phi_k(\tau)])} e^{-\tilde{S}_0[\bar{\psi}, \psi]} \\ &= \int \mathcal{D}[\Delta_k(\tau), \Delta_k^*(\tau)] e^{-\frac{1}{\lambda} \int_\tau (\frac{1}{\text{vol}} \sum_k \Delta_k^*(\tau) \Delta_k(\tau))} Z_0 \langle e^{\int_\tau (\frac{1}{\text{vol}} \sum_k [\Delta_k(\tau) \Phi_k^*(\tau) + \Delta_k^*(\tau) \Phi_k(\tau)])} \rangle_{\tilde{S}_0} \end{aligned}$$

Note the Z_0 above is just some constant, $Z_0 = e^{\ln Z_0} = \dots$, and will be overlooked.

$$= \int \mathcal{D}[\Delta_k(\tau), \Delta_k^*(\tau)] e^{-W[\Delta_k(\tau), \Delta_k^*(\tau)]} = Z_0 \langle \dots \rangle_{\tilde{S}_0}.$$

Now use cumulant expansions:

$$\begin{aligned} &\langle e^{\int_\tau \sum_k \left[(\frac{\Delta_k(\tau)}{\text{vol}}) \Phi_k^*(\tau) + \frac{\Delta_k^*(\tau)}{\text{vol}} \Phi_k(\tau) \right]} \rangle_{\tilde{S}_0} \\ &\int_\tau \sum_k \rightarrow \sum_a, \quad \frac{\Delta_k(\tau)}{\text{vol}} = J_z, \quad \Phi_k^*(\tau) = Z_a^*, \quad \frac{\Delta_k^*(\tau)}{\text{vol}} = J_a^*, \quad \Phi_k(\tau) = Z_a. \end{aligned}$$

Now X is real random variable, $\langle e^{JX} \rangle = \exp\{J \langle X \rangle_c \frac{J^2}{2!} \langle X^2 \rangle_c + \dots\} = \exp\left\{ \sum_{n=1}^{\infty} \frac{J^n}{n!} \langle X^n \rangle_c \right\} \langle x \rangle = \langle x \rangle_c$, $\langle x^2 \rangle = \langle x^2 \rangle_c + (\langle x \rangle_c)^2$, $\langle x^3 \rangle = \langle x^3 \rangle_c + 3 \langle x^2 \rangle_c \langle x \rangle_c + (\langle x \rangle_c)^3$

when \vec{J}, \vec{X} are vectorial numbers,

$\langle e^{\vec{J} \cdot \vec{X}} \rangle = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n!} \langle (\vec{J} \cdot \vec{X})^n \rangle_c \right\} \langle (\vec{J} \cdot \vec{X})^2 \rangle_c = J_i J_j \langle x_i x_j \rangle_c$, etc. $\langle x_i \rangle = \langle x_i \rangle_c$, $\langle x_i x_j \rangle = \langle x_i x_j \rangle_c + \langle x_i \rangle_c \langle x_j \rangle_c$, $\langle x_i x_j x_k \rangle = \langle x_i x_j x_k \rangle_c + \langle x_i \rangle_c \langle x_j x_k \rangle_c + \dots + \langle x_i \rangle_c \langle x_j \rangle_c \langle x_k \rangle_c$

$$\begin{aligned} \langle e^{J^* Z + J Z^*} \rangle &= \exp\{ \langle J^* Z + J Z^* \rangle_c + \frac{1}{2} \langle (J^* Z + J Z^*)^2 \rangle_c + \dots \} \\ &= \exp\{ J^* \langle Z \rangle_c + J \langle Z^* \rangle_c + \frac{1}{2!} (J^*)^2 \langle Z^2 \rangle_c + J^2 \langle (Z^*)^2 \rangle_c + J J^* \langle Z^* Z \rangle_c + \dots \} \end{aligned}$$

$$\begin{aligned}
\langle e^{(\vec{J} \cdot \vec{Z} + \vec{J} \cdot \vec{Z}^*)} \rangle &= \exp\{ \langle (\vec{J} \cdot \vec{Z}^*) \rangle_c + \frac{1}{2!} \langle (\vec{J}^* \cdot \vec{Z} + \vec{J} \cdot \vec{Z}^*)^2 \rangle_c + \dots \} \\
&= \exp\{ \vec{J}^* \cdot \langle \vec{Z} \rangle_c + \vec{J} \cdot \langle \vec{Z}^* \rangle_c + \frac{1}{2} \langle (\vec{J}^* \cdot \vec{Z})^2 \rangle_c + \frac{1}{2} \langle (\vec{J} \cdot \vec{Z}^*)^2 \rangle_c + \langle (\vec{J}^* \cdot \vec{Z})(\vec{J} \cdot \vec{Z}^*) \rangle_c + \dots \}
\end{aligned}$$

March 8

Define $\sum_a = \int_\tau \sum_k$, $J_a = \frac{\Delta_k(\tau)}{\text{vol}}$, $Z_a = \Phi_k(\tau)$, $\left(\sum_{p_3}' \bar{\psi}_{p_3, \uparrow} \bar{\psi}_{-p_3+k, \downarrow} \right)_\tau = \Phi_k^*(\tau)$, $\left(\sum_{p_1}' \psi_{p_1+k, \downarrow} \psi_{-p_1, \uparrow} \right)_\tau = \Phi_k(\tau)$,

$$\langle e^{\int_\tau \sum_k \left[\frac{\Delta_k(\tau)}{\text{vol}} \Phi_k^*(\tau) + \frac{\Delta_k^*(\tau)}{\text{vol}} \Phi_k(\tau) \right]} \rangle_{\tilde{S}_0} = \exp \left\{ \int_\tau \sum_k \left[\frac{\Delta_k(\tau)}{\text{vol}} \langle \Phi_k^*(\tau) \rangle_0 + \frac{\Delta_k^*(\tau)}{\text{vol}} \langle \Phi_k(\tau) \rangle_0 \right] \right\}$$

$\langle \Psi \rangle_0 = \langle \psi \psi \rangle_0 = 0$ because of $U(1)$ invariance of \tilde{S}_0 : $\psi \rightarrow \psi e^{i\alpha}$, $\bar{\psi} \rightarrow e^{-i\alpha} \bar{\psi}$.

Next term: $J^2 \langle (Z^*)^2 \rangle + J^{*2} \langle Z^2 \rangle_0$ also vanishes. so only $J J^* \langle Z^* Z \rangle_0$ nonvanish.

$$+ \int_\tau \sum_k \int_{\tau'} \sum_{k'} \frac{\Delta_k^*(\tau)}{\text{vol}} \frac{\Delta_{k'}(\tau')}{\text{vol}} \langle \Phi_k(\tau) \Phi_{k'}^*(\tau') \rangle_0 + O[(\Delta^* \Delta)^2]$$

$$\Delta_k(\tau) = \frac{1}{\beta} \sum_{\bar{\Omega}} e^{-i\bar{\Omega}\tau} \Delta_k(\bar{\Omega}), \quad \Phi_k(\tau) = \frac{1}{\beta} \sum_{\bar{\Omega}} e^{-i\bar{\Omega}\tau} \Phi_k(\bar{\Omega}).$$

Thus

$$= \exp \left\{ \frac{1}{\beta} \sum_{\bar{\Omega}} \frac{1}{\text{vol}} \sum_k \frac{1}{\beta} \sum_{\bar{\Omega}'} \frac{1}{\text{vol}} \sum_{k'} \Delta_k^*(\bar{\Omega}) \Delta_{k'}(\bar{\Omega}') \langle \Phi_k(\bar{\Omega}) \Phi_{k'}^*(\bar{\Omega}') \rangle_0 + \dots \right\}$$

Since

$$\Phi_k(\bar{\Omega}) = \frac{1}{\beta} \sum_{\bar{\Omega}} \sum_p' \psi_{p+k, \downarrow}(\bar{\Omega} - \bar{\omega}) \psi_{-p, \uparrow}(\bar{\omega})$$

thus

$$\langle \Phi_k(\bar{\Omega}) \Phi_{k'}^*(\bar{\Omega}') \rangle_0 = \frac{1}{\beta} \sum_{\bar{\omega}_1} \sum_{p_1} \frac{1}{\beta} \sum_{\bar{\omega}_3} \sum_{p_3}' \langle \psi_{-p_1+k, \downarrow}(-\bar{\omega}_1 + \bar{\Omega}) \psi_{p_1, \uparrow}(\bar{\omega}_1) \bar{\psi}_{p_3, \uparrow}(\bar{\omega}_3) \bar{\psi}_{-p_3+k'}(-\bar{\omega}_3 + \bar{\Omega}') \rangle_0$$

$$\langle \psi_{p_1, \sigma}(\bar{\Omega}_1) \bar{\psi}_{p_3, \sigma'}(\bar{\Omega}_2) \rangle = \delta_{\sigma\sigma'} \delta_{p_1 p_2} \beta \delta_{\bar{\Omega}_1 \bar{\Omega}_2} \frac{-1}{i\bar{\Omega}_1 - \xi_{p_1}}$$

Thus

$$\langle \psi_{-p_1+k, \downarrow}(-\bar{\omega}_1 + \bar{\Omega}) \psi_{p_1, \uparrow}(\bar{\omega}_1) \bar{\psi}_{p_3, \uparrow}(\bar{\omega}_3) \bar{\psi}_{-p_3+k'}(-\bar{\omega}_3 + \bar{\Omega}') \rangle_0 = \delta_{p_1 p_3} \beta \delta_{\bar{\omega}_1 \bar{\omega}_3} \langle \psi_{p_1, \uparrow}(\bar{\omega}_1) \psi_{p_1, \uparrow}(\bar{\omega}_1) \rangle_0 \langle \psi(-\bar{\omega}_1 + \bar{\Omega}) \bar{\psi}_{-p_1+k, \downarrow}(-\bar{\omega}_1) \rangle_0$$

$$= \delta_{k, k'} \delta_{\bar{\Omega} \bar{\Omega}'} \sum_{\bar{\omega}} \sum_{p_1}' \frac{-1}{i\bar{\omega}_1 - \xi_{p_1}} \frac{-1}{-i\bar{\omega}_1 + i\bar{\Omega} - \xi_{k-p_1}} = \text{vol} \beta \delta_{k, k'} \delta_{\bar{\Omega} \bar{\Omega}'} \chi(k, \bar{\Omega})$$

$Z = \int \mathcal{D}[\Delta_k(\bar{\Omega}), \Delta_k^*(\bar{\Omega})] e^{-W[\Delta, \Delta^*]}$, when

$$W[\Delta, \Delta^*] = \frac{1}{\beta} \sum_{\bar{\Omega}} \frac{1}{\text{vol}} \sum_k \left[\frac{1}{\lambda} - X_k, \bar{\Omega} \right] \Delta^*(k(\bar{\Omega}) \Delta_k(\bar{\Omega}) + O(|\Delta_k^*(\bar{\Omega}) \Delta_k(\bar{\Omega})|^2)$$

For which Δ the W is minimized? First look at the static limit. Suppose $\Delta_\tau(k)$ is time independent:

$\Delta_k(\bar{\Omega}) = \int d\tau e^{i\bar{\Omega}\tau} \Delta_k(\tau)$, if $\Delta_k(\tau) = \Delta_k$ is time-independent, becomes $\Delta_k(\bar{\Omega}) = [\int d\tau e^{i\bar{\Omega}\tau}] \Delta_k = \beta \delta_{\bar{\Omega}, 0} \Delta_k$. Thus static case:

$$\begin{aligned} W[\Delta_k(\bar{\Omega} = 0), \Delta_k^*(\bar{\Omega} = 0)] &= \frac{1}{\beta} \sum_{\bar{\Omega}} \frac{1}{\text{vol}} \sum_k \left[\frac{1}{\lambda} - \chi(k, \bar{\Omega} = 0) \right] \left(\beta \delta_{\bar{\Omega}, 0} \Delta_k^* \right) \left(\beta \delta_{\bar{\Omega}, 0} \Delta_k \right) + \dots \\ &= \beta \left(\frac{1}{\text{vol}} \sum_k \right) \left[\frac{1}{\lambda} - \chi(k, \bar{\Omega}) \right] \Delta_k^* \Delta_k + O[(\Delta^* \Delta)^2] \end{aligned}$$

General:

$$W[\Delta_k, \Delta_k^*] = \left(\frac{1}{\text{vol}} \sum_k \right) \left\{ r \left(\frac{T - T_c}{T_c} \right) + \gamma(T) \vec{k} \cdot \vec{k} + O(k^4) \right\} \Delta_k^* \Delta_k$$

(Now we are all working at finite temperature T .)

$$= \beta \int d^d r \left[\gamma(r) (\nabla \Delta^*) \cdot (\nabla \Delta) + r(T) \Delta^* \Delta + B(T) (\Delta^* \Delta)^2 + \dots \right].$$

$$\beta r(T) = \left[\frac{1}{\lambda} - \chi(k = 0, \bar{\Omega} = 0) \right].$$

$$\chi(k, \bar{\Omega}) = \frac{1}{\beta} \sum_{\bar{\omega}_1} \frac{1}{\text{vol}} \sum_{p_1}' \frac{-1}{i\bar{\omega}_1 - \xi_{p_1 + \frac{k}{2}}} \frac{-1}{-i\bar{\omega}_1 + i\bar{\Omega} - \chi_{\frac{k}{2} - p_1}}$$

let $\text{vol} = \infty$, contour integration can also treat with finite temperature. Through calculation we get

$$\chi(k, \bar{\Omega}) = \int \frac{d^d p}{(2\pi)^d} \frac{1 - n_F(\xi_{p + \frac{k}{2}}) - n_F(\xi_{p - \frac{k}{2}})}{i\bar{\Omega} + \xi_{p + \frac{k}{2}} + \xi_{p - \frac{k}{2}}}$$

thus

$$\chi(k = 0, \bar{\Omega} = 0) = \int \frac{d^d p}{(2\pi)^d} \frac{1 - 2n_F(\xi_p)}{2\xi_p}$$

note $\xi_p = p^2/2m - \mu = p^2/2m - p_F^2/2m \sim v_F(p - p_F) + O(p - p_F)^2$, insert $1 = \int d\xi \delta(\xi - \xi_p)$, $\nu(\xi) = \int \frac{d^d p}{(2\pi)^d} \delta(\xi - \xi_p)$, $|p - p_F| < \Lambda$,

$$\chi(k = 0, \bar{\Omega} = 0) = \int_{-v_F \Lambda}^{v_F \Lambda} d\xi r(\xi) \frac{\frac{1}{2} - n_F(\xi)}{\xi} \sim \nu(0) \int_{-v_F \Lambda}^{v_F \Lambda} \frac{d\xi}{\xi} \left[\frac{1}{2} - n_F(\xi) \right]$$

$$n_F(\xi) = \frac{1}{1+e^x}, \quad x = \xi/T, \quad \text{thus } \frac{1}{2} - n_F(\xi) = \frac{1}{2} - \frac{1}{1+e^x} = \frac{1}{2} \tanh(x),$$

$$\chi(k=0, \bar{\Omega}=0) = \nu(0) \frac{1}{2} \int_{-\frac{v_F \Lambda}{k_B T}}^{\frac{v_F \Lambda}{k_B T}} \frac{dx}{x} \tanh\left(\frac{x}{2}\right) = \nu(0) \int_0^{\frac{v_F \Lambda}{k_B T}} \frac{dx}{x} \tanh\left(\frac{x}{2}\right)$$

split $\int_0^{\frac{v_F \Lambda}{k_B T}} = \int_0^{x_0} + \int_{x_0}^{\frac{v_F \Lambda}{k_B T}}$ choose $x_0 \gg 1$ fixed, so that $\tanh(x/2) = 1$,

$$\chi(k=0, \bar{\Omega}=0) \sim \nu(0) \ln\left(\frac{v_F \Lambda}{k_B T}\right), \quad \frac{v_F \Lambda}{k_B T}.$$

$$\frac{1}{\lambda} - \chi(k=0, \bar{\Omega}=0) = 0 \quad \text{at } T = T_c, \quad r(T_c) \text{ changes sign, } \frac{1}{\lambda} = \nu(0) \ln\left[\frac{v_F \Lambda}{k_B T_c}\right],$$

$$k_B T_c = v_F \Lambda e^{-\frac{1}{\nu(0)\lambda}}$$

then have to show this is consistent with RG: $\Lambda' < \Lambda$, we have

$$k_B T_c = v_F \Lambda' e^{-\frac{1}{\nu(0)\lambda'}}$$

March 10

SC at $T = 0$.

$$\tilde{S}[\Delta, \Delta^*, \psi, \bar{\psi}] = \frac{1}{\beta} \sum \bar{\Omega} \frac{1}{\text{vol}} \sum_k \Delta_k^*(\bar{\Omega}) \Delta_k(\bar{\Omega}) + \frac{1}{\beta} \sum_{\bar{\omega}} \frac{1}{\beta} \sum_{\bar{\omega}'} \sum_p \sum_{p'}' \left(\bar{\chi}_{p,\uparrow}(\bar{\omega}) \quad \bar{\chi}_{p,\downarrow} \right) \widetilde{M}_{\bar{\omega}p, \bar{\omega}'p'} \begin{pmatrix} \chi_{p',\uparrow}(\bar{\omega}') \\ \chi_{p',\downarrow}(\bar{\omega}') \end{pmatrix},$$

$\widetilde{M}_{\bar{\omega}p, \bar{\omega}'p'}$ is the Fourier transform of M ,

$$\psi_{p,\uparrow}(\tau) = \chi_{p,\uparrow}(\tau), \quad \bar{\psi}_{p,\uparrow}(\tau) = \chi_{p,\uparrow}(\tau), \quad \psi_{p,\downarrow}(\tau) = -\bar{\chi}_{-p,\downarrow}(\tau), \quad \bar{\psi}_{p,\downarrow}(\tau) = \bar{\chi}_{-p,\downarrow}(\tau),$$

$$\psi_{p,\uparrow}(\bar{\omega}) = \chi_{p,\uparrow}(\bar{\omega}), \quad \bar{\psi}_{p,\uparrow}(\bar{\omega}) = \chi_{p,\uparrow}(\bar{\omega}), \quad \psi_{p,\downarrow}(\bar{\omega}) = -\bar{\chi}_{-p,\downarrow}(-\bar{\omega}), \quad \bar{\psi}_{p,\downarrow}(\bar{\omega}) = \bar{\chi}_{-p,\downarrow}(-\bar{\omega}),$$

$$\psi(r, \tau) = \frac{1}{\beta} \sum_p \frac{1}{\sqrt{\text{vol}}} e^{i(\bar{p} \cdot \bar{r} - \bar{\omega} \tau)} \psi_p(\bar{\omega}), \quad \chi(r, \tau) = \frac{1}{\beta} \sum_p \frac{1}{\sqrt{\text{vol}}} e^{i(\bar{p} \cdot \bar{r} - \bar{\omega} \tau)} \chi_p(\bar{\omega}),$$

$$\bar{\psi}(r, \tau) = \frac{1}{\beta} \sum_p \frac{1}{\sqrt{\text{vol}}} e^{-i(\bar{p} \cdot \bar{r} - \bar{\omega} \tau)} \bar{\psi}_p(\bar{\omega}), \quad \bar{\chi}(r, \tau) = \frac{1}{\beta} \sum_p \frac{1}{\sqrt{\text{vol}}} e^{-i(\bar{p} \cdot \bar{r} - \bar{\omega} \tau)} \bar{\chi}_p(\bar{\omega}),$$

$$\widetilde{M}_{\bar{\omega}p, \bar{\omega}'p'} = \begin{bmatrix} [-i\bar{\omega} + \xi_p] \beta \delta_{\bar{\omega}\bar{\omega}'} \delta_{p,p'} & -\frac{\Delta_{p-p'}(\bar{\omega}-\bar{\omega}')}{\text{vol}} \\ -\frac{\Delta_{p'-p}^*(\bar{\omega}'-\bar{\omega})}{\text{vol}} & [-i\bar{\omega} - \xi_p] \beta \delta_{\bar{\omega}\bar{\omega}'} \delta_{p,p'} \end{bmatrix}$$

$$(\bar{\psi}_r(\tau), \psi_r(\tau)) \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] & \Delta_r(\tau) \\ \Delta_r^*(\tau) & \left[\partial_t + \frac{\nabla^2}{2m} + \mu \right] \end{bmatrix} \begin{pmatrix} \psi_r(\tau) \\ \bar{\psi}_r(\tau) \end{pmatrix}$$

$$Z = \int \mathcal{D}[\Delta, \Delta^*] \int \mathcal{D}[\psi, \bar{\psi}] e^{-\tilde{S}} = \int \mathcal{D}[\Delta, \Delta^*] e^{-\frac{1}{\beta} \sum_{\bar{\omega}} \frac{1}{\text{vol}} \sum_k \Delta_k^*(\bar{\omega}) \Delta_k(\bar{\omega})} \times$$

$$\int \mathcal{D}[\psi, \bar{\psi}] e^{-\frac{1}{\beta} \sum_{\bar{\omega}} \frac{1}{\beta} \sum_{\bar{\omega}'} \sum_p \sum_{p'} \bar{\chi} \tilde{M} \chi} = \text{const. det } \tilde{M} = \text{const. } e^{\text{tr} \ln \tilde{M}}$$

special case: uniform case. $\Delta(r, \tau) = \frac{1}{\beta} \sum_{\bar{\omega}} \frac{1}{\text{vol}} \sum_k e^{i(\vec{k} \cdot \vec{r} + \bar{\omega} \tau)} \Delta_k(\tau) = \Delta_0 = \text{const.}$, $\Delta_0 = \frac{\Delta_{k=0}(\bar{\Omega}=0)}{\beta \text{vol}}$

$$\Delta_k(\bar{\Omega}) \text{vol} = \left(\frac{\Delta_{k=0}(\bar{\Omega}=0)}{\beta \text{vol}} \right) \beta \delta_{\bar{\Omega}, 0} \delta_{k, 0}$$

$$M_{\bar{\omega}p, \bar{\omega}'p'} = \beta \begin{bmatrix} [-i\bar{\omega} + \xi_p] & -\Delta_0 \\ -\Delta_0^* & [-i\bar{\omega} - \xi_p] \end{bmatrix}$$

$$\det \left(\tilde{M}_{\bar{\omega}p, \bar{\omega}'p'} \right) = \prod_{\bar{\omega}} \prod_p \beta^2 \det \begin{bmatrix} [-i\bar{\omega} + \xi_p] & -\Delta_0 \\ -\Delta_0^* & [-i\bar{\omega} - \xi_p] \end{bmatrix} = \prod_{\bar{\omega}} \prod_p \beta^2 (-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)$$

$$= \text{const. exp} \left\{ \sum \sum \ln [(-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)] \right\}$$

$$\rightarrow \text{const. exp} \left\{ \beta \text{vol} \left(\frac{1}{\beta} \sum \right) \left(\frac{1}{\text{vol}} \sum \right) \ln [(-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)] \right\}$$

In the limit $\text{vol} \rightarrow \infty$, $\beta \rightarrow \infty$, we have integral $\int_{-\infty}^{\infty} \frac{d\bar{\omega}}{2\pi} \int \frac{d^k}{(2\pi)^d}$.

$$Z \sim \exp \left\{ -\beta \text{vol} \left[\frac{1}{\lambda} |\Delta_0|^2 - \left(\frac{1}{\beta} \sum \right) \left(\frac{1}{\text{vol}} \sum \right) \ln [(-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)] \right] \right\}$$

saddle point. extreme: $\frac{1}{\lambda} |\Delta_0|^2 - \left(\frac{1}{\beta} \sum \right) \left(\frac{1}{\text{vol}} \sum \right) \ln [(-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)] = W(|\Delta_0|^2) = \text{static potential.}$

Find $\frac{\partial}{\partial |\Delta_0|^2} W(|\Delta_0|^2) = 0$,

$$\frac{1}{\lambda} - \left(\frac{1}{\beta} \sum \right) \left(\frac{1}{\text{vol}} \sum \right) \frac{(-1)}{(-\bar{\omega}^2 - \xi_0^2 - |\Delta_0|^2)} = 0,$$

solution $\Delta_0(T=0) = 2\Lambda e^{-\frac{1}{v(\bar{0})\lambda}}$.

Now we will derive effective action W for the Goldstone mode.

$$e^{-W(\Delta, \Delta^*)} = e^{-S \frac{1}{\lambda} \int_r \int_{\tau} \Delta^*(r\tau) \Delta(r\tau)} \det(\tilde{M})$$

We will call the determinant $\det \tilde{M}$ as the one calculated in the fourier space.

$$\det \tilde{M} = \text{Det} M = e^{-S_F[\Delta, \Delta^*]},$$

$$\tilde{M}_{\bar{\omega}p, \bar{\omega}'p'} = \beta \delta_{\bar{\omega}, \bar{\omega}'} \delta_{p, p'} \begin{bmatrix} [-i\bar{\omega} + \xi_p] & -\Delta_0 \\ -\Delta_0^* & [-i\bar{\omega} - \xi_p] \end{bmatrix}$$

$$M_{r\tau, r'\tau'} = \delta^{(1)}(\tau - \tau') \delta^{(d)}(r - r') \begin{bmatrix} \left[\partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right] & \Delta_r(\tau) \\ \Delta_r^*(\tau) & \left[\partial_t + \frac{\nabla^2}{2m} + \mu \right] \end{bmatrix}$$

Let us first switch off the temporal dependence, only add spatial dependence of the form $\Delta_r(\tau) = \Delta_0 e^{2i\vec{Q}\cdot\vec{r}}$, $\Delta(r, \tau) = \Delta_0 e^{i\theta(r, \tau)}$, to diagonalize \widetilde{M} , we use U

$$U^+ \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] & \Delta_0 e^{2i\vec{Q}\cdot\vec{r}} \\ \Delta_0^* e^{-2i\vec{Q}\cdot\vec{r}} & \left[\partial_t + \frac{\nabla^2}{2m} + \mu \right] \end{bmatrix} U$$

$U = \text{diag}(e^{i\vec{Q}\cdot\vec{r}}, e^{-i\vec{Q}\cdot\vec{r}})$ set $\Delta_0 = |\Delta_0| > 0$,

$$e^{-i\vec{Q}\cdot\vec{r}} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] e^{i\vec{Q}\cdot\vec{r}} = \left[\partial_\tau - \frac{1}{2m} \left(e^{-i\vec{Q}\cdot\vec{r}} \nabla e^{i\vec{Q}\cdot\vec{r}} \right) \left(e^{-i\vec{Q}\cdot\vec{r}} \nabla e^{i\vec{Q}\cdot\vec{r}} \right) - \mu \right] = \left[\partial_\tau - \frac{1}{2m} \left(-\nabla + i\vec{Q} \right) \left(-\nabla + i\vec{Q} \right) - \mu \right]$$

$$U^+ \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] & \Delta_0 e^{2i\vec{Q}\cdot\vec{r}} \\ \Delta_0^* e^{-2i\vec{Q}\cdot\vec{r}} & \left[\partial_t + \frac{\nabla^2}{2m} + \mu \right] \end{bmatrix} U = \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} + i\frac{\vec{Q}\cdot\nabla}{m} - \left(\mu - \frac{Q^2}{2m} \right) \right] & \Delta_0 \\ \Delta_0 & \left[\partial_\tau + \frac{\nabla^2}{2m} - i\frac{\vec{Q}\cdot\nabla}{m} + \left(\mu - \frac{Q^2}{2m} \right) \right] \end{bmatrix}$$

Fourier transform

$$\widetilde{N}_1 \rightarrow (\widetilde{M}_1)_{\bar{\omega}p, \bar{\omega}'p'} = \beta \delta_{\bar{\omega}\bar{\omega}'} \delta_{p,p'} \begin{bmatrix} \left[-i \left(\bar{\omega} - i\frac{\vec{Q}\cdot\vec{p}}{m} \right) + \frac{p^2}{2m} - \left(\mu - \frac{Q^2}{2m} \right) \right] & -\Delta_0 \\ -\Delta_0 & -i \left(\bar{\omega} - i\frac{\vec{Q}\cdot\vec{p}}{m} \right) - \frac{p^2}{2m} + \left(\mu - \frac{Q^2}{2m} \right) \end{bmatrix}$$

$$\begin{aligned} -S_F[\Delta = \Delta_0 e^{2i\vec{Q}\cdot\vec{r}}, \Delta^* = \Delta_0 e^{-2i\vec{Q}\cdot\vec{r}}] &= \ln \det(\widetilde{M}_1) = \\ &= \beta \text{vol} \int_{\bar{\omega}} \int_p \ln \left[- \left(\bar{\omega} - i\frac{\vec{Q}\cdot\vec{p}}{m} \right) - \left(\frac{p^2}{2m} - \left(\mu - \frac{Q^2}{2m} \right) \right) - |\Delta_0|^2 \right] \end{aligned}$$

let $\bar{\omega}' = \bar{\omega} - i\frac{\vec{Q}\cdot\vec{p}}{m}$, it is below the real axis of $\bar{\omega}$, and $|\vec{Q}\vec{p}/m| < |\Delta_0| = 2\Lambda e^{-\frac{1}{v(0)\lambda}}$, $|\vec{p}| < \Lambda$, the integral is not changed. Thus the only change is $\mu \rightarrow \mu' = \mu - \frac{Q^2}{2m}$.

$-S_F = [\Delta = \Delta_0 e^{2i\vec{Q}\cdot\vec{r}}, \Delta^* = \Delta_0 e^{-2i\vec{Q}\cdot\vec{r}}] = \text{Taylor in } Q^2/2m$, we get

$$-S_F[\Delta_0, \Delta_0] + \beta \text{vol} \frac{\partial S_F}{\partial \mu} (-1) \frac{Q^2}{2m} + O(Q^2/2m)^2$$

differentiate with respect to the chemical potential we get the number density of fermions.

The calculation tell us we can calculate the fermion determinant even if its not a delta in space.

$$-S_F\{\Delta = \Delta_0 e^{2i\vec{Q}\cdot\vec{r}}\} = S_F[\Delta_0] + \beta \text{vol} \rho \frac{Q^2}{2m}$$

temporal dependence: assume $\Delta_r(\tau) = \Delta_0 e^{2i\vec{\Omega}\tau}$. ($|\Delta_0| = \Delta_0$), choose $U = \text{diag}[e^{i\vec{\Omega}\tau}, e^{-i\vec{\Omega}\tau}]$
Thus

$$U^+ \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu \right] & \Delta_0 e^{2i\vec{\Omega}\tau} \\ \Delta_0^* e^{-2i\vec{\Omega}\tau} & \left[\partial_t + \frac{\nabla^2}{2m} + \mu \right] \end{bmatrix} U = \begin{bmatrix} \left[\partial_\tau - \frac{\nabla^2}{2m} - \mu' \right] & \Delta_0 \\ \Delta_0^* & \left[\partial_t + \frac{\nabla^2}{2m} + \mu' \right] \end{bmatrix}$$

$\mu \rightarrow \mu' = \mu - i\vec{\Omega}$. Thus

$$\begin{aligned}
-S_F &= [\Delta = \Delta_0 e^{2i\bar{\Omega}\tau}, \Delta^* = \Delta_0 e^{-2i\bar{\Omega}\tau}] = -S_F[\Delta_0, \Delta_0] + \beta \text{vol} \frac{\partial F}{\partial \mu} (-i)\bar{\Omega} + \text{higer term} \\
&= -S_F[\Delta = \Delta^* = \Delta] + \beta \text{vol} \rho i \bar{\Omega} + \dots
\end{aligned}$$

This is just a way to calculate some initially complicated looking determinant of fermions.
conclusion: if we can write $\Delta_r(\tau) = \Delta_0 e^{2iv(r,\tau)}$, where all the time and spatial dependence of $\Delta_r(\tau)$ is in the phase, if $\vec{Q} \rightarrow (\nabla\theta)$, $\bar{\Omega} \rightarrow (\partial_\tau\theta)$, then for general variation:

$$S_F[\Delta = \Delta_0 e^{2i\theta(r,\tau)}, \Delta^* = \Delta_0 e^{-2i\theta(r,\tau)}] = S_F[\Delta_0] + \int_\tau \int_r [i\rho \partial_\tau \theta(r,\tau) + \frac{\rho}{2m} (\nabla\theta)^2]$$

Note: compare with $S[\phi, \phi^*] = \int_\tau \int_r \left\{ \phi^* (\partial_\tau \phi) - \frac{1}{2m^*} (\nabla\phi^*) \cdot (\nabla\phi) + \frac{g}{2} (|\phi|^2 - \rho_0)^2 \right\}$,

we see that the fermion SC low energy theory is the same as the low energy theory of bosons.

$$S[\phi = \sqrt{\rho} e^{iv}] = \int_\tau \int_r \left\{ i\rho (\partial_\tau \theta) + \frac{\rho}{2m} (\nabla\theta)^2 + \frac{g}{2} (\rho - \rho_0)^2 + \frac{1}{8m\rho} (\nabla\rho)^2 \right\}$$

Conclusion: SC is really boson fluid, the goldstone mode in SC isn't really there, because you couple to electromagnetism, which is a gauge theory (minimal coupling), and the Higgs mechanism gets rid of the goldstone mode, because the fermions are charged, and there is a charge associated with the phase and couple to electromagneticsn. On the otheh hand helium-3 doesn't have charge, thus have goldstone mode.

Everything we talked about W can also acquire angular dependence, and we just have to consider each wave mode (s,p,d) wave....

The end of the class for the 2016W.