Flat bands in periodic scattering networks

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Joint work with
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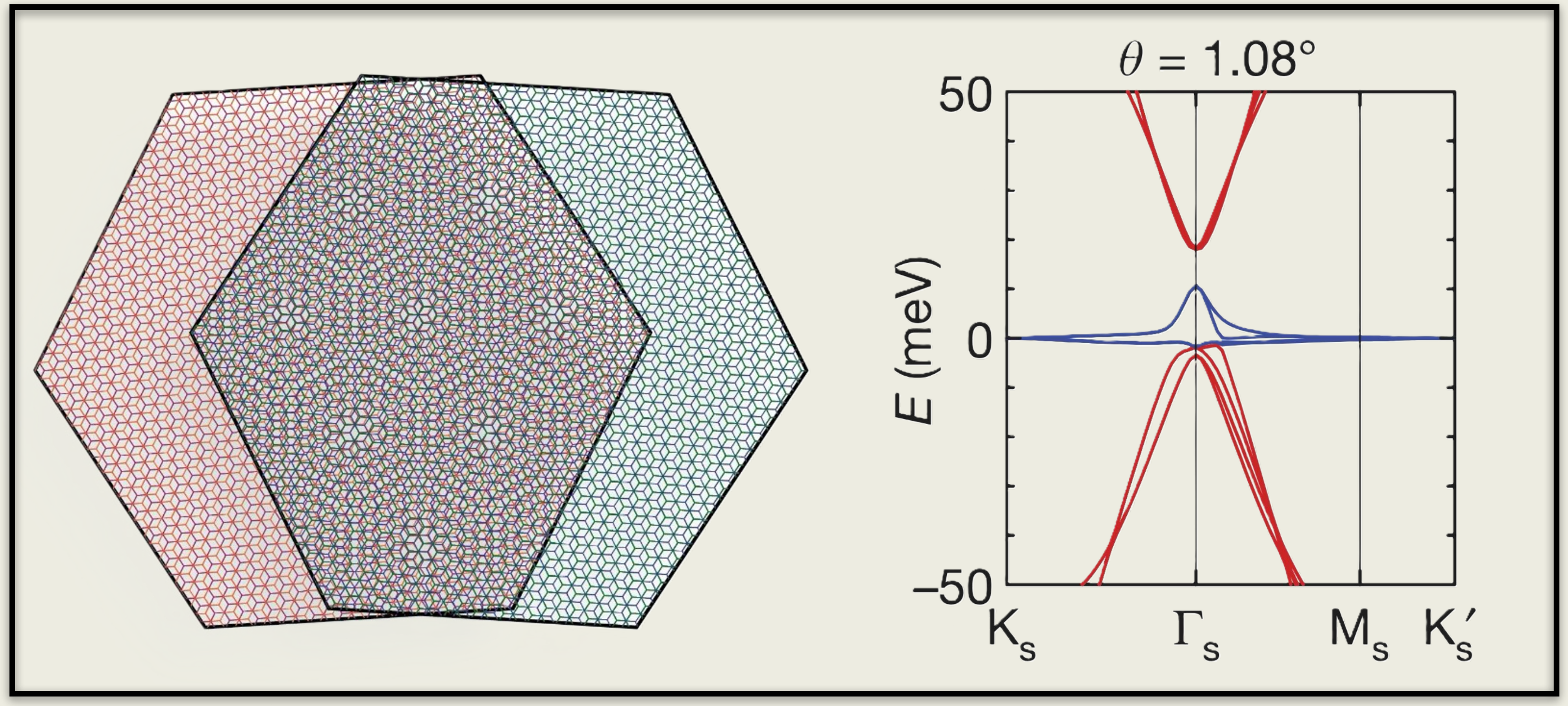
GDR Meeticc Banyuls 18 Nov 2025

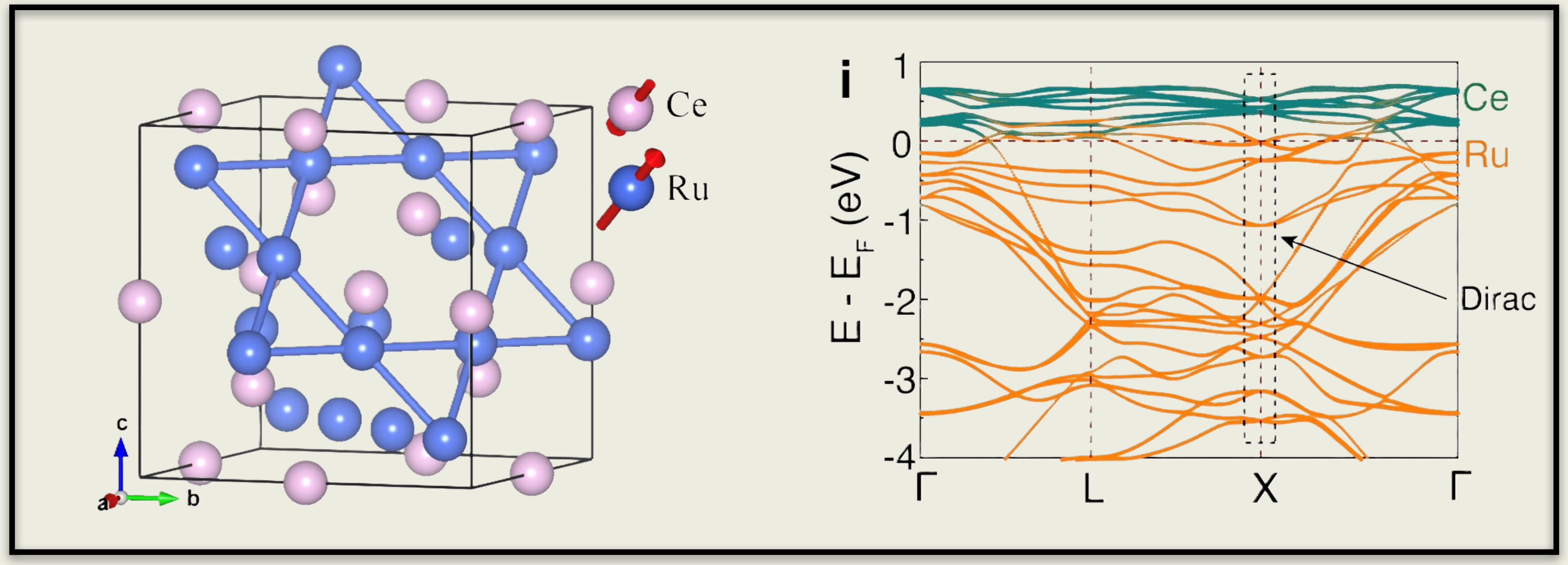
Flatbands (from TB models)

Experimentally, they appear in strongly correlated materials of interest:

Twisted bilayer graphene



Kagome material $CeRu_2$



See also Neves et al., Crystal net catalog of model flat band materials, npj Comp. Mat. '24

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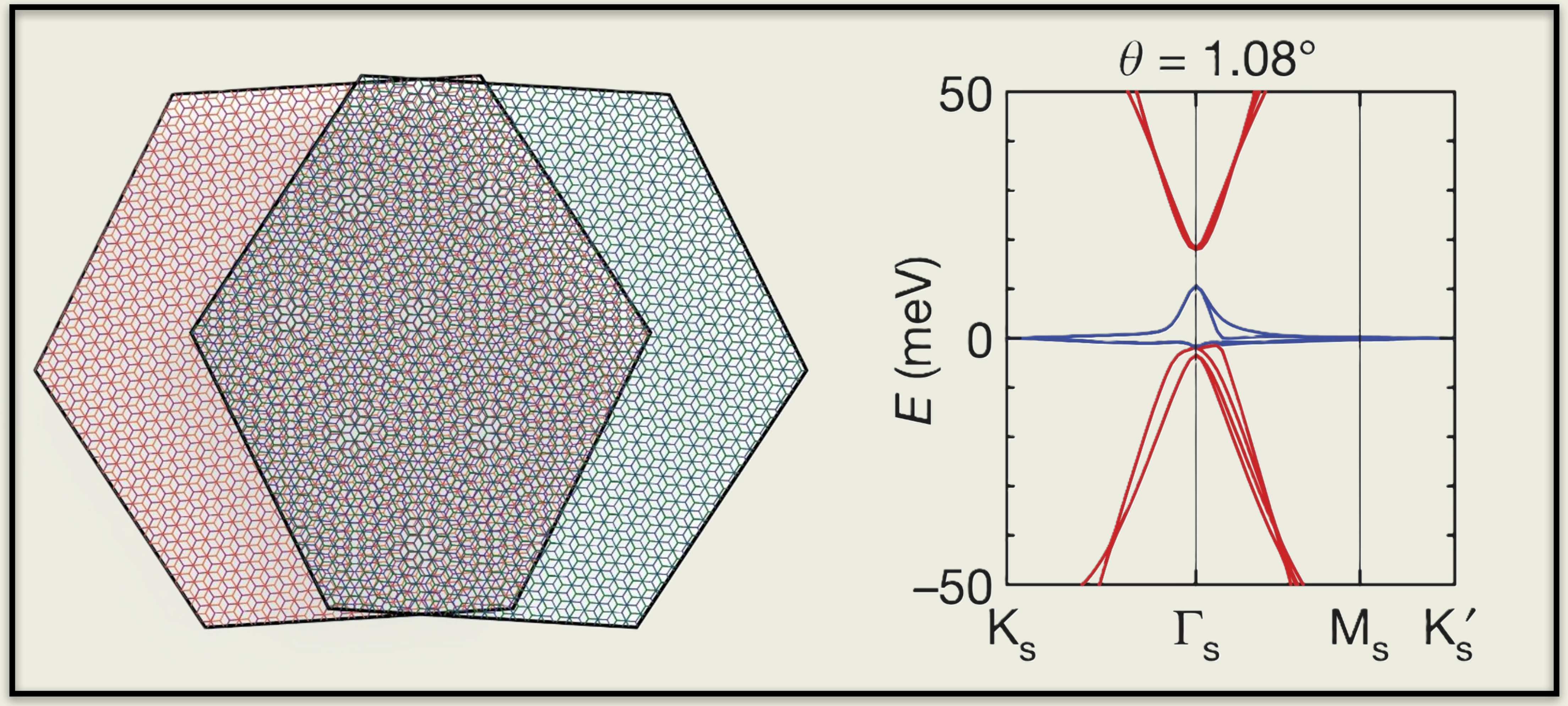

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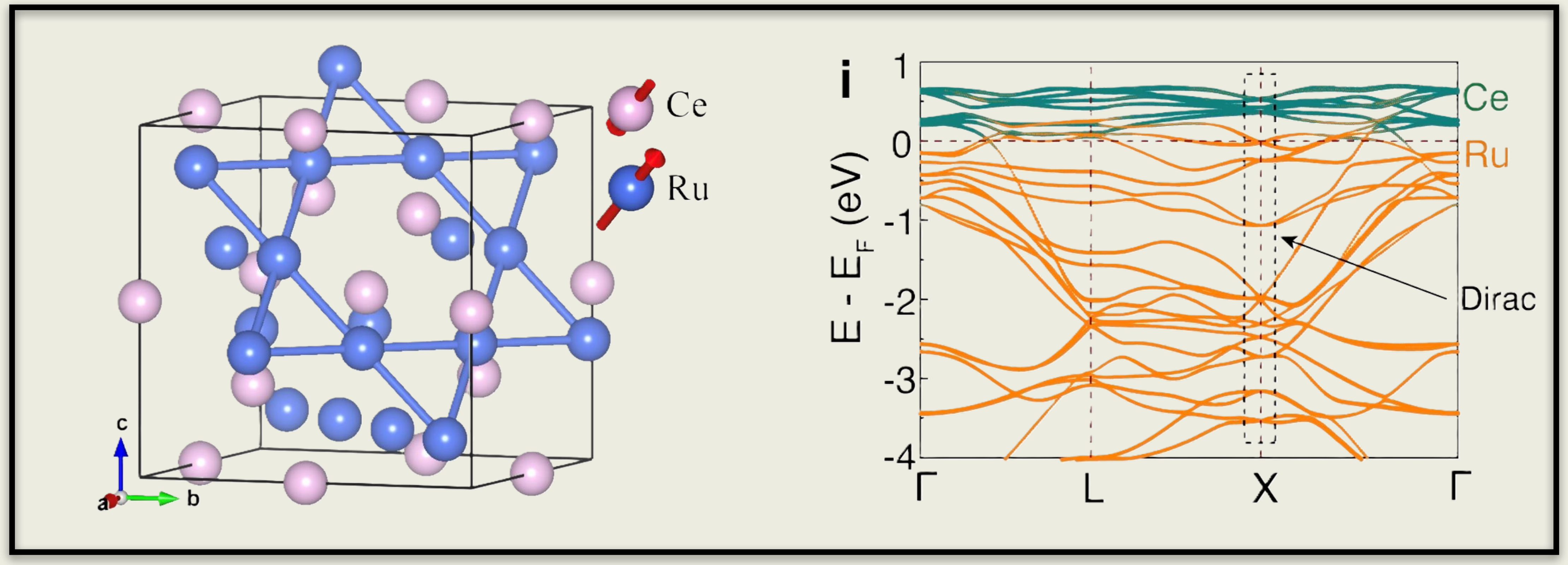


Theoretically, when interactions added, needs separate care!

1) Compact localized states (CLS)
2) Group velocity = 0
3) Kinetic energy = bandwidth = 0

See also Neves et al., Crystal net catalog of model flat band materials, npj Comp. Mat. '24

A MEET Apéro for ICC

Matériaux,
Etats ElecTroniques

Interactions et
Couplages
non-Conventionnels

Flat bands (FBs)

This talk, the apéro

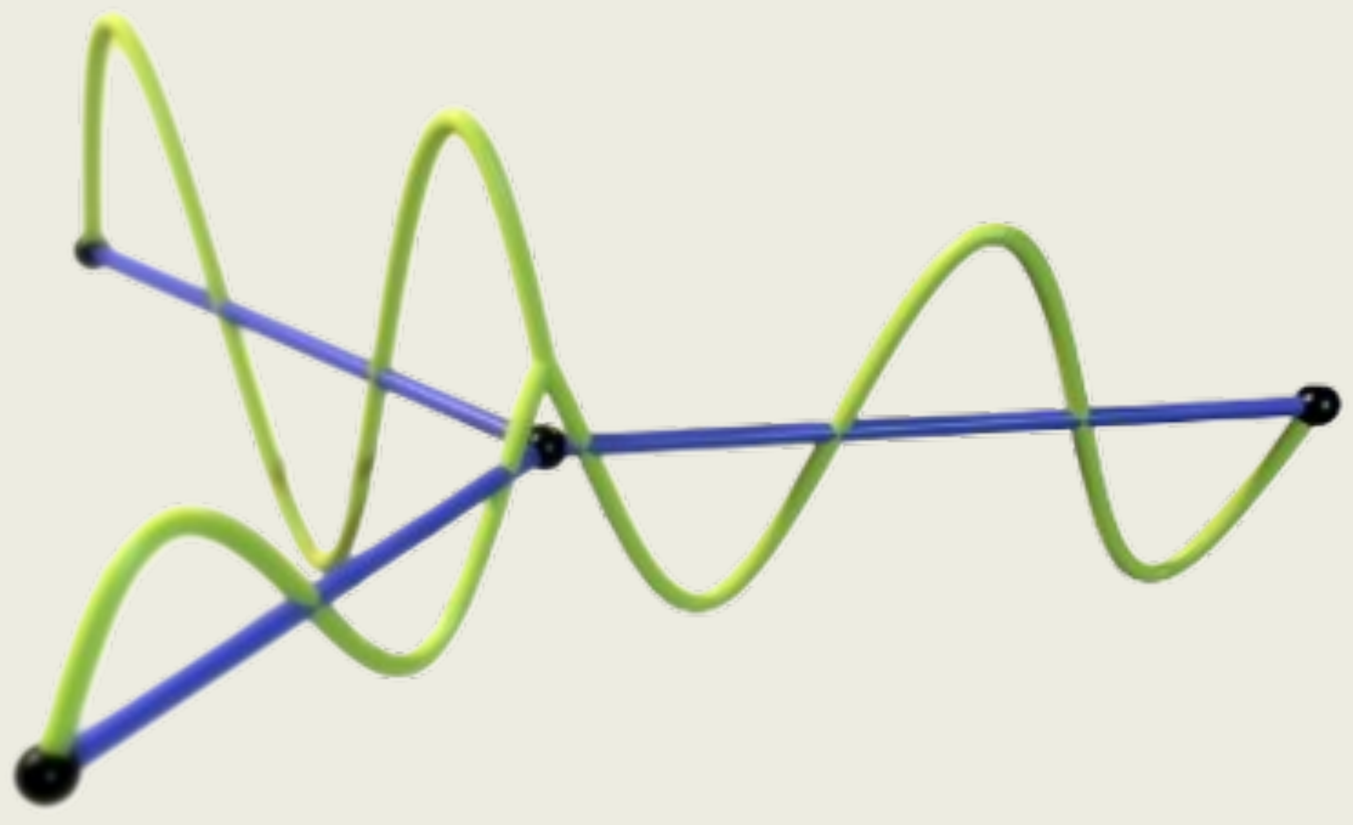
Strongly correlated phases

For future

Scattering network (aka Q graph)

Graph equipped with Schrödinger equation on bonds:

$$-\frac{d^2}{dx^2}\psi = E\psi, \quad E = k^2$$



Long history !

- Ruedenberg, Scherr, J. Chem. Phys. '53
- Montroll, J. Math. Phys. '70

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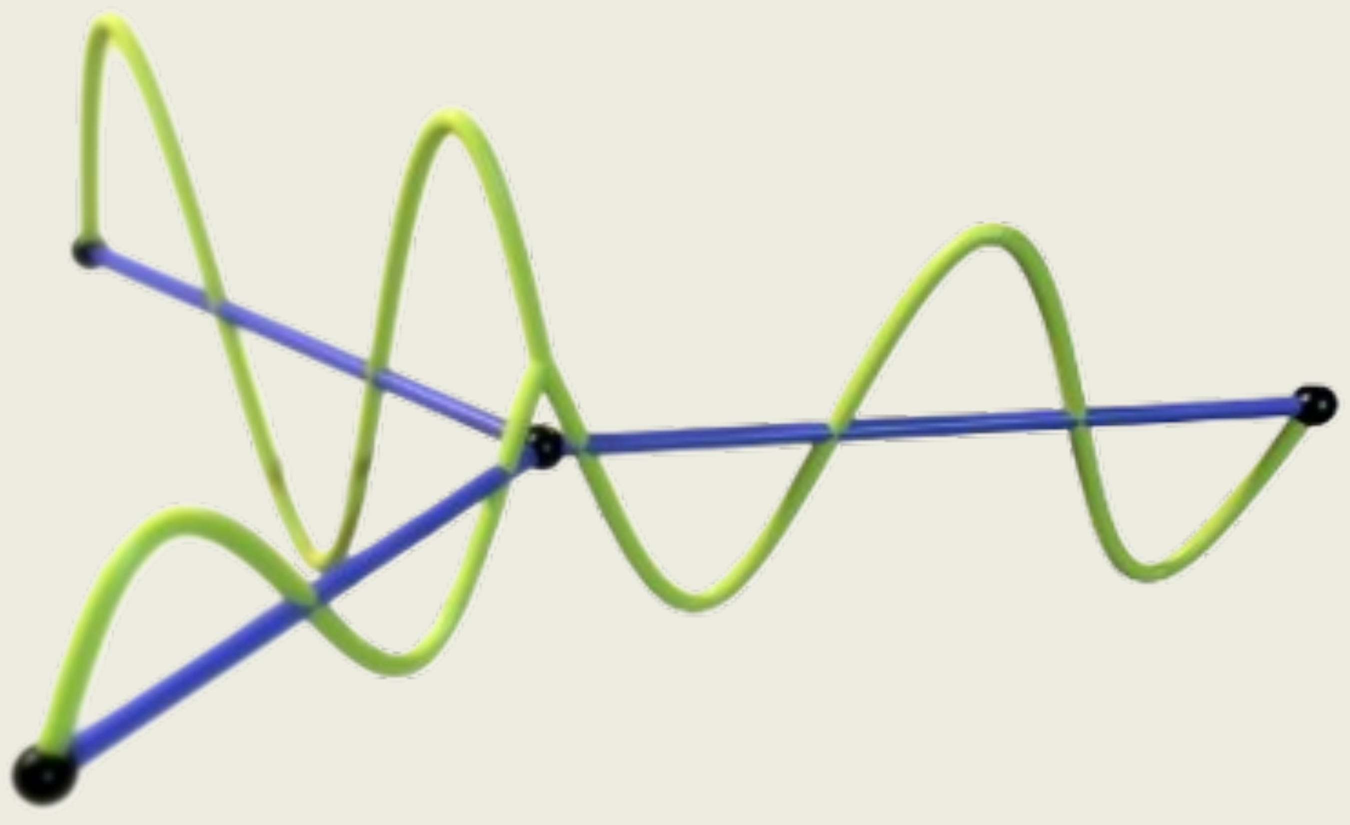

Long history !

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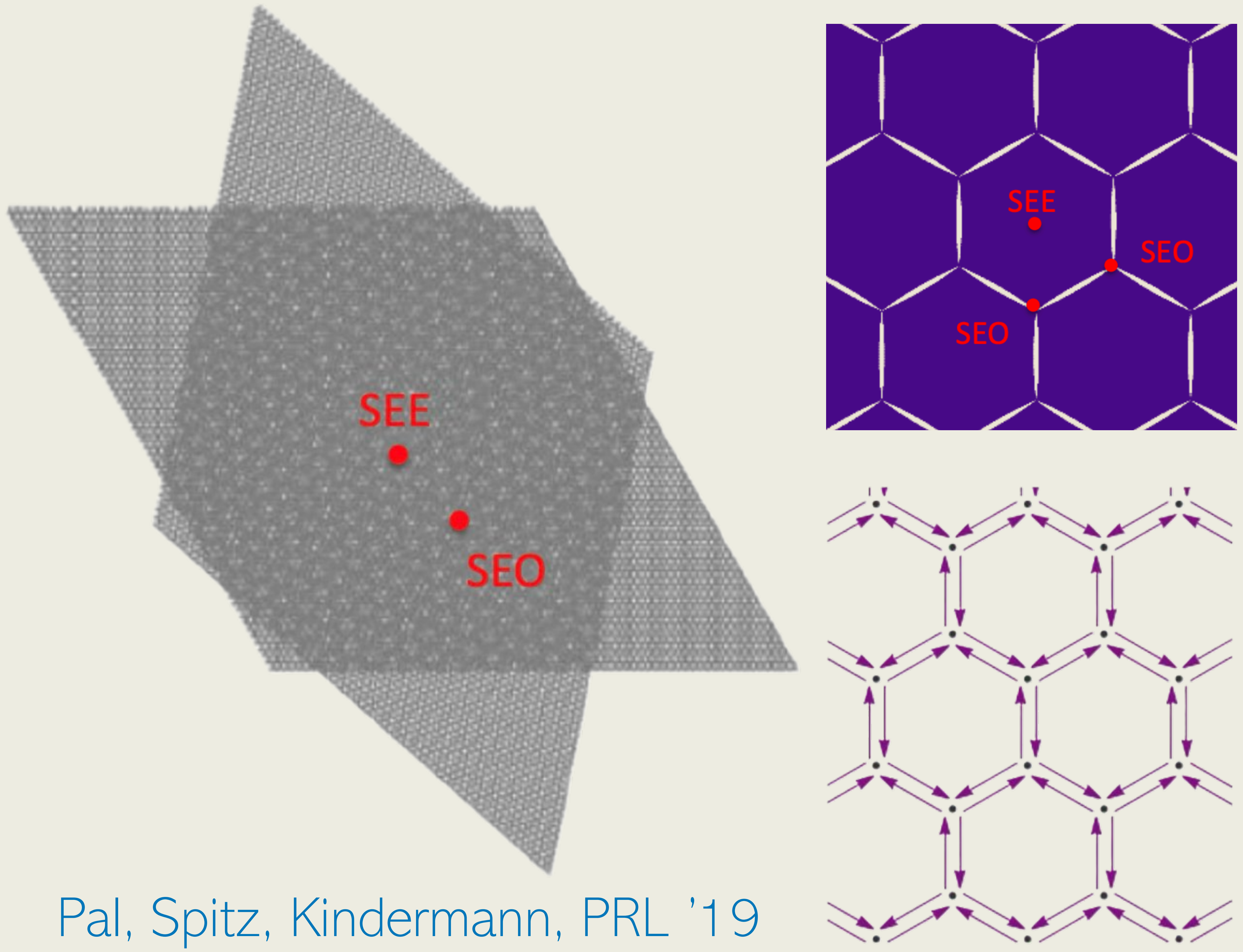
- Useful toy models for many systems: electron, phonon, photon, resonator, resistor…
- A complementary approach for TBG:



Pal, Spitz, Kindermann, PRL '19

TB vs Q graph

	TB models	**Scattering network models**
Degree of freedom	On sites	On bonds
Coupling	Hopping on bonds	Scattering at sites
Graph theory reference	Combinatorial / discrete graph	Quantum graph
Relation	Line graph map, Dirichlet-to-Neumann (DtN) map…	

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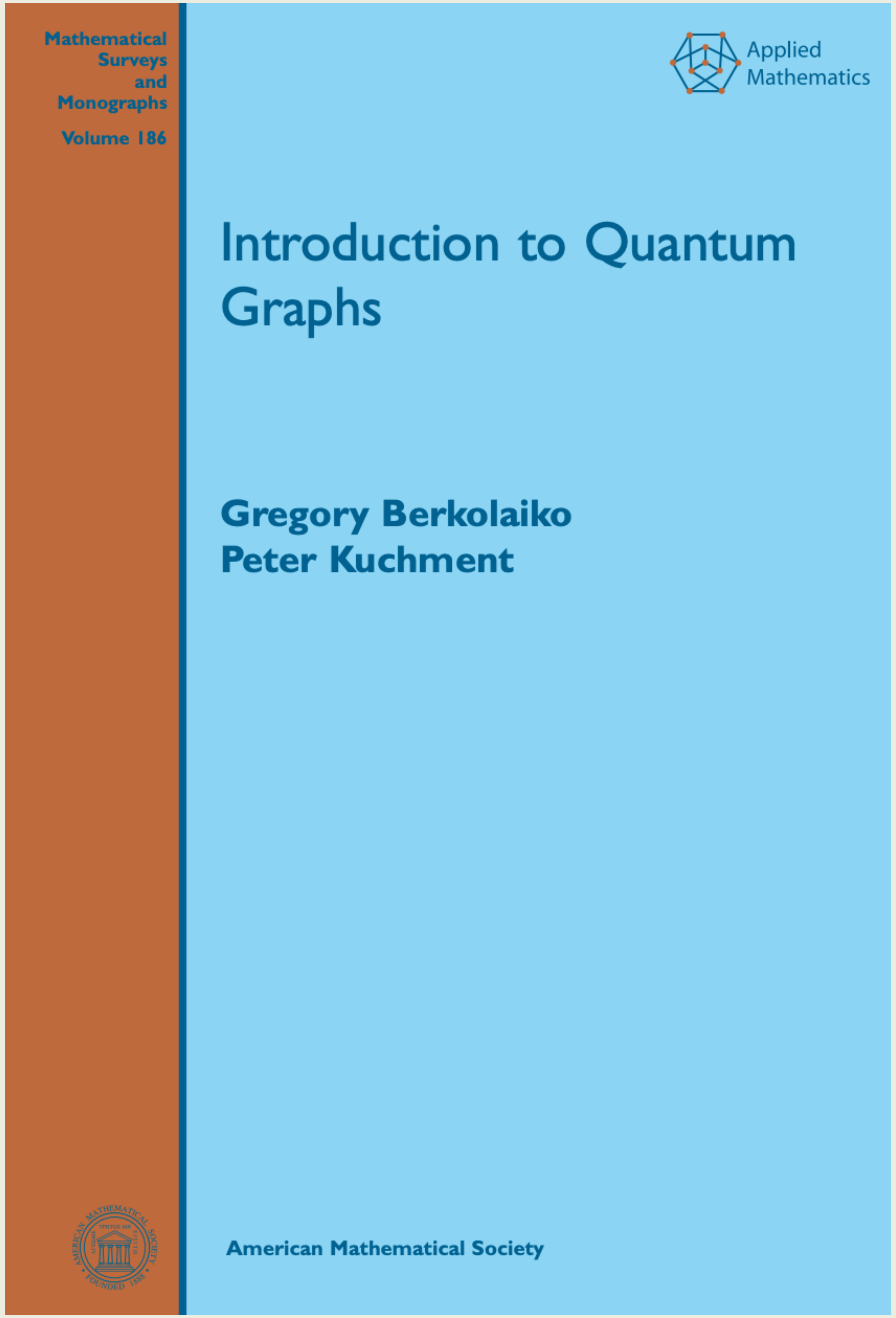

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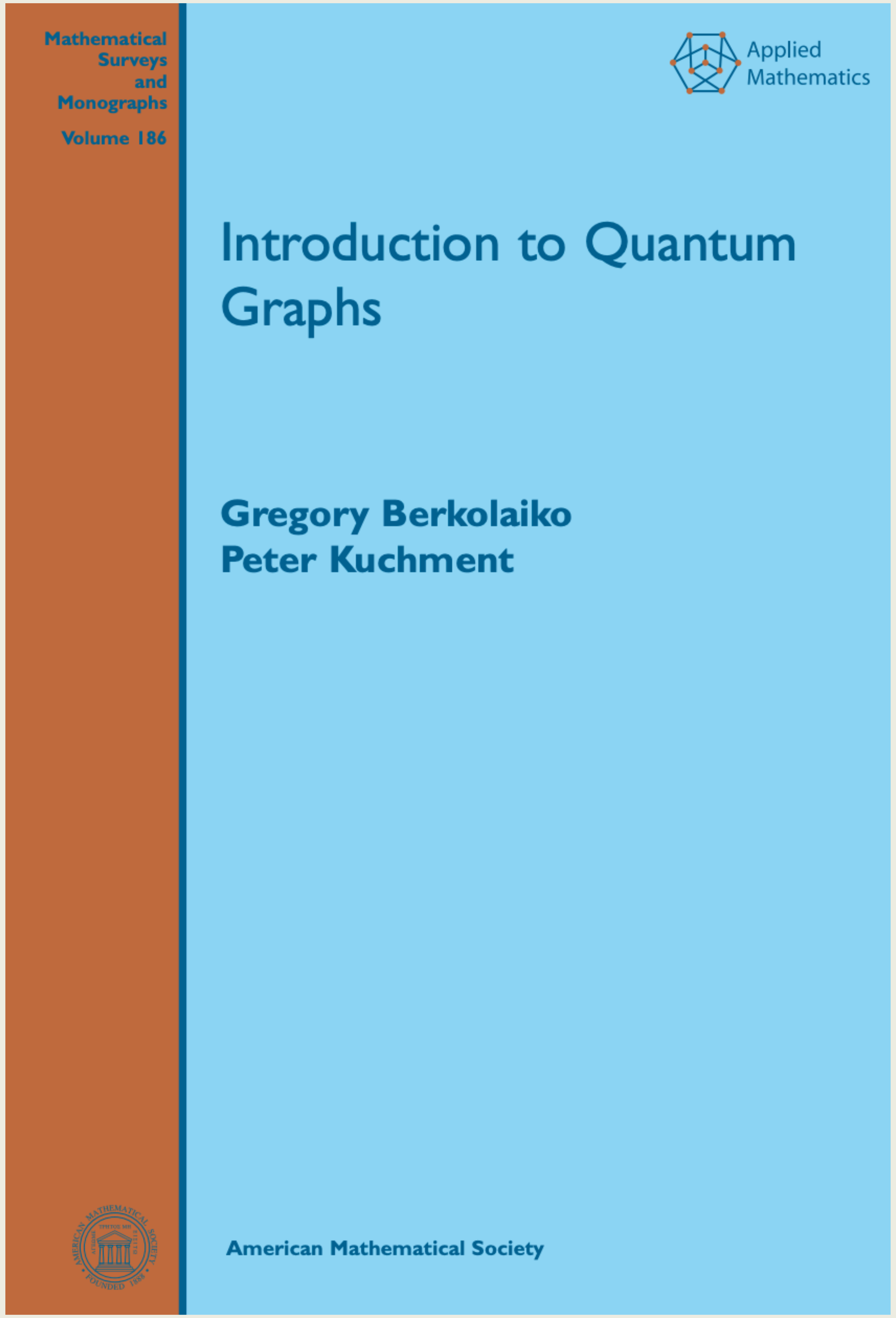

In this talk, always:

- Periodic Q graph
- No interactions, no disorder/randomness

FBs in periodic Q graph — Outline

- Spectrum of periodic Q graph
- FBs in honeycomb (single channel)
- FBs in “breathing” honeycomb (single channel)
- FBs in “breathing” honeycomb, multichannel

Our work

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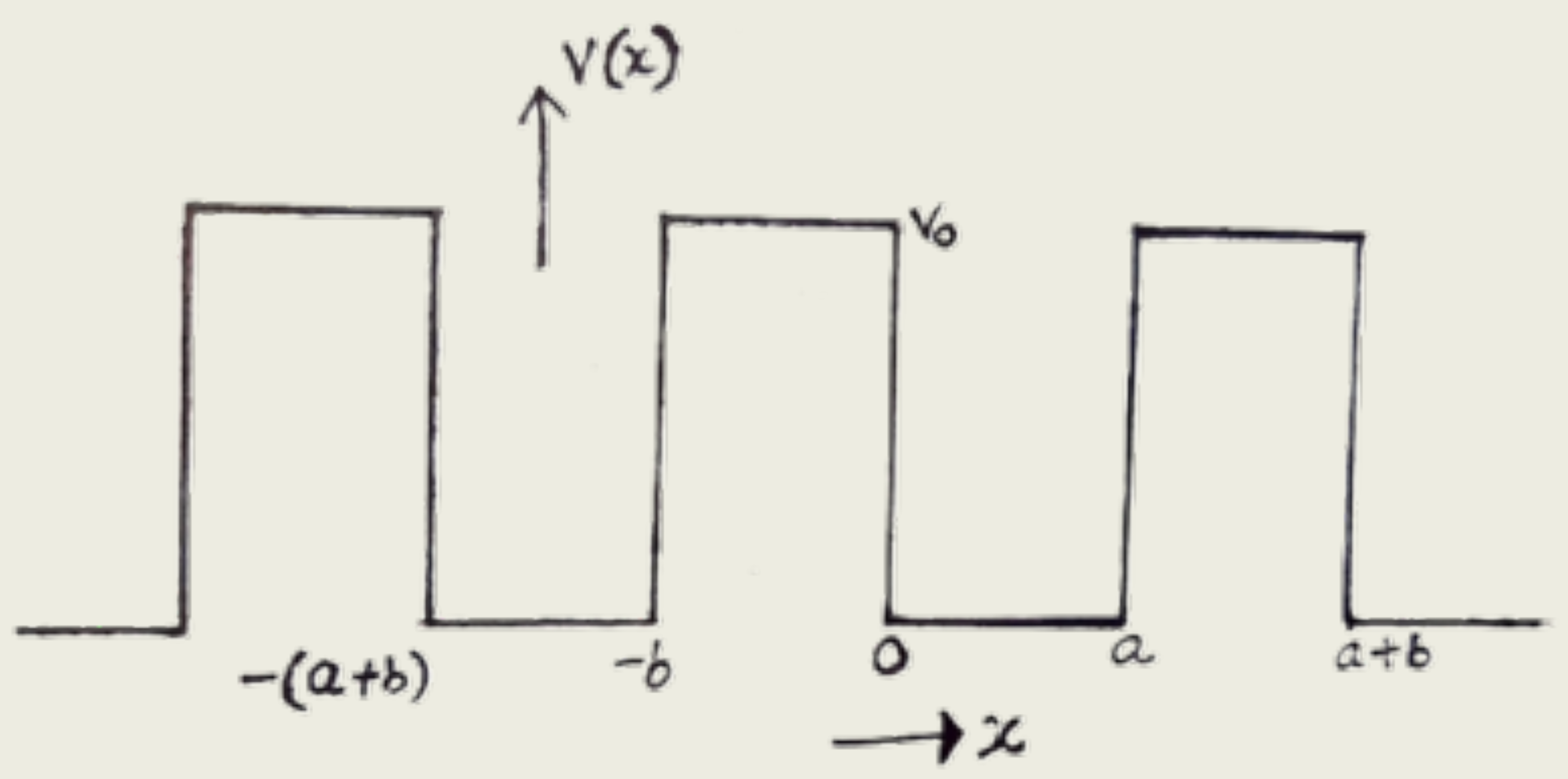
Our work

FBs survive many nontrivial generalizations, as long as lattice symmetry present !

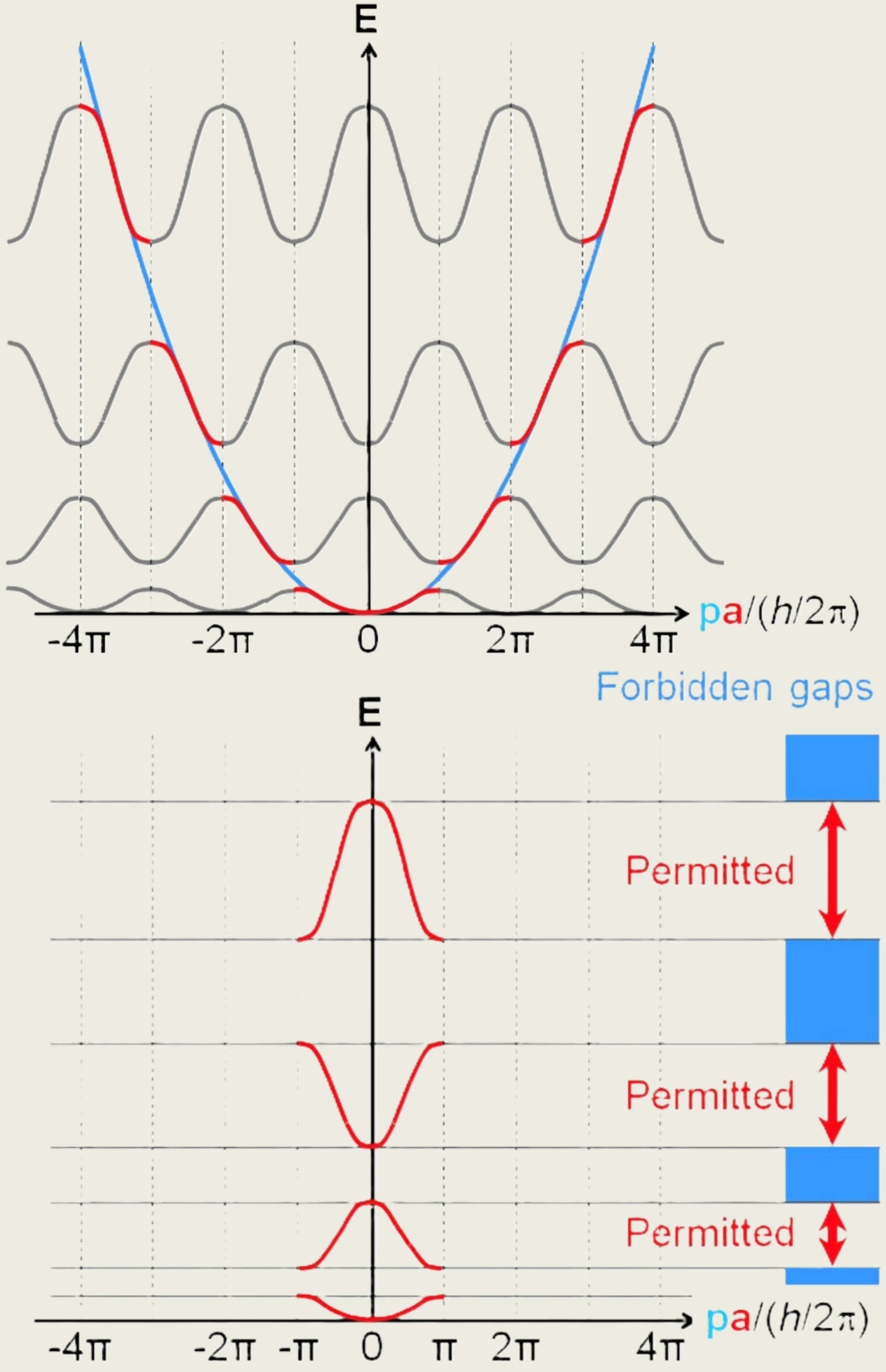
Simplest periodic Q graph (1D; no FB)

Kronig-Penney model in 1D :

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$$

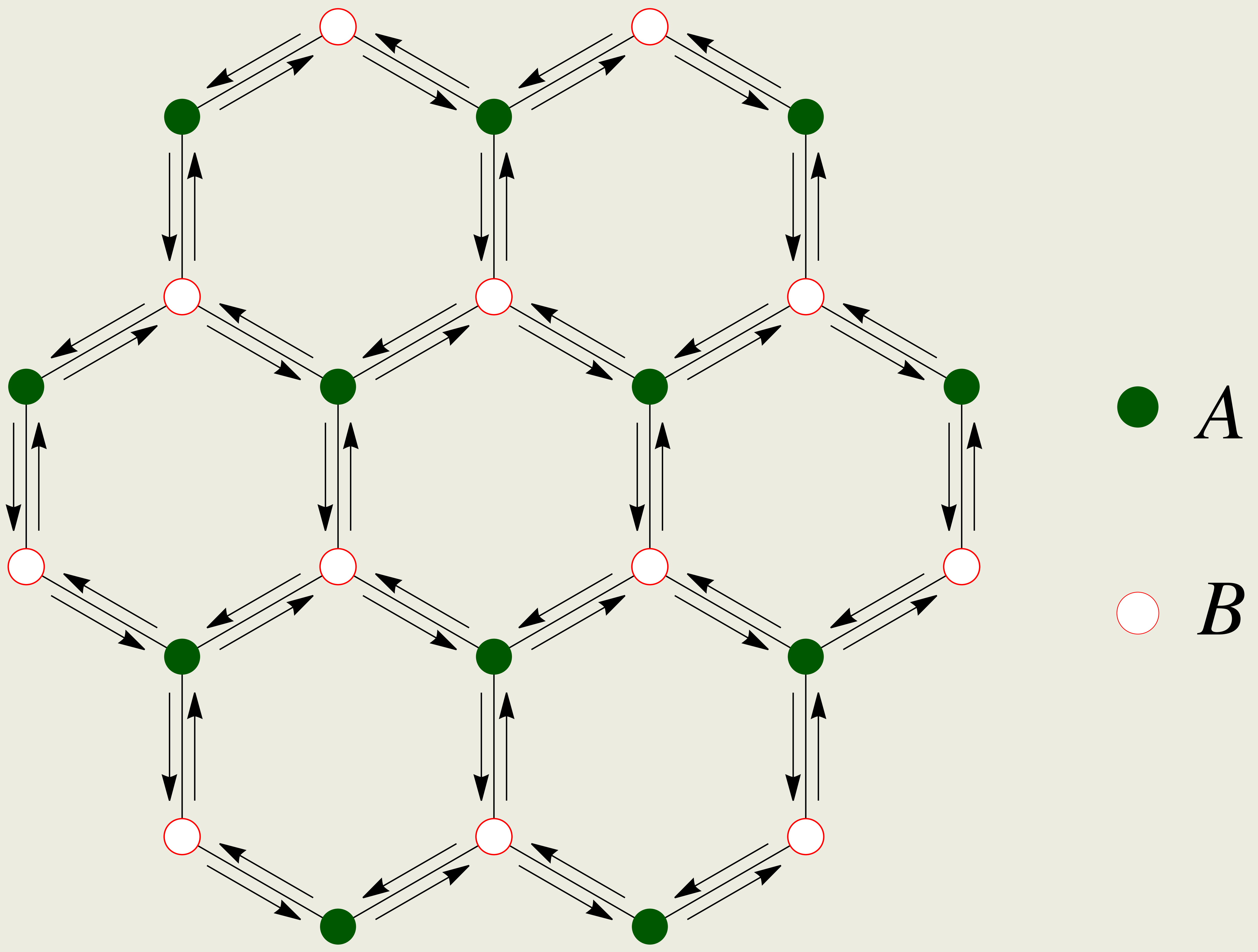


Cottam, Ransonm Vounckx, Towards Cross-Modeling between Life and Solid State Physics



2D honeycomb as a Q graph

Kuchment, Post, Comm. Math. Phys. '07



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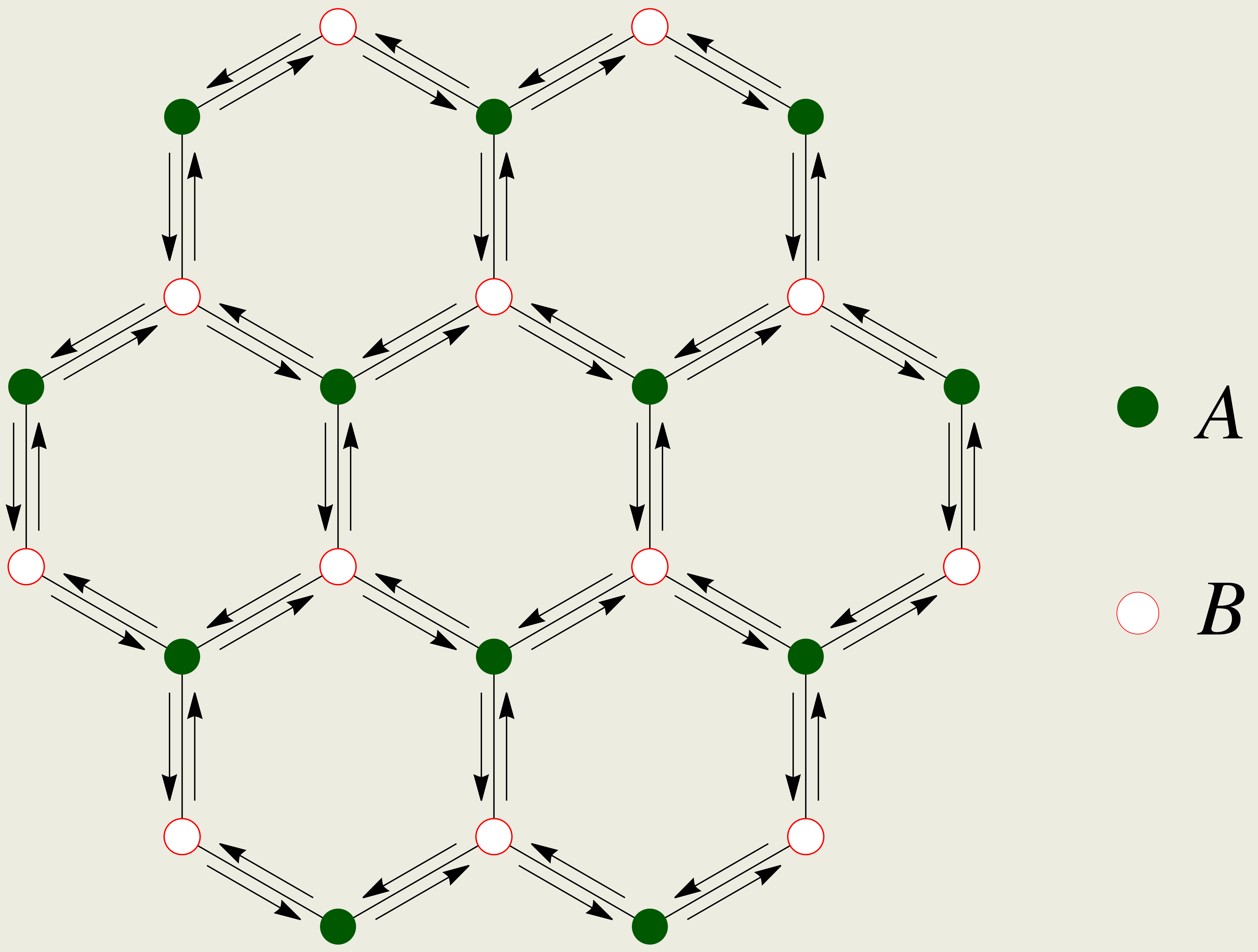

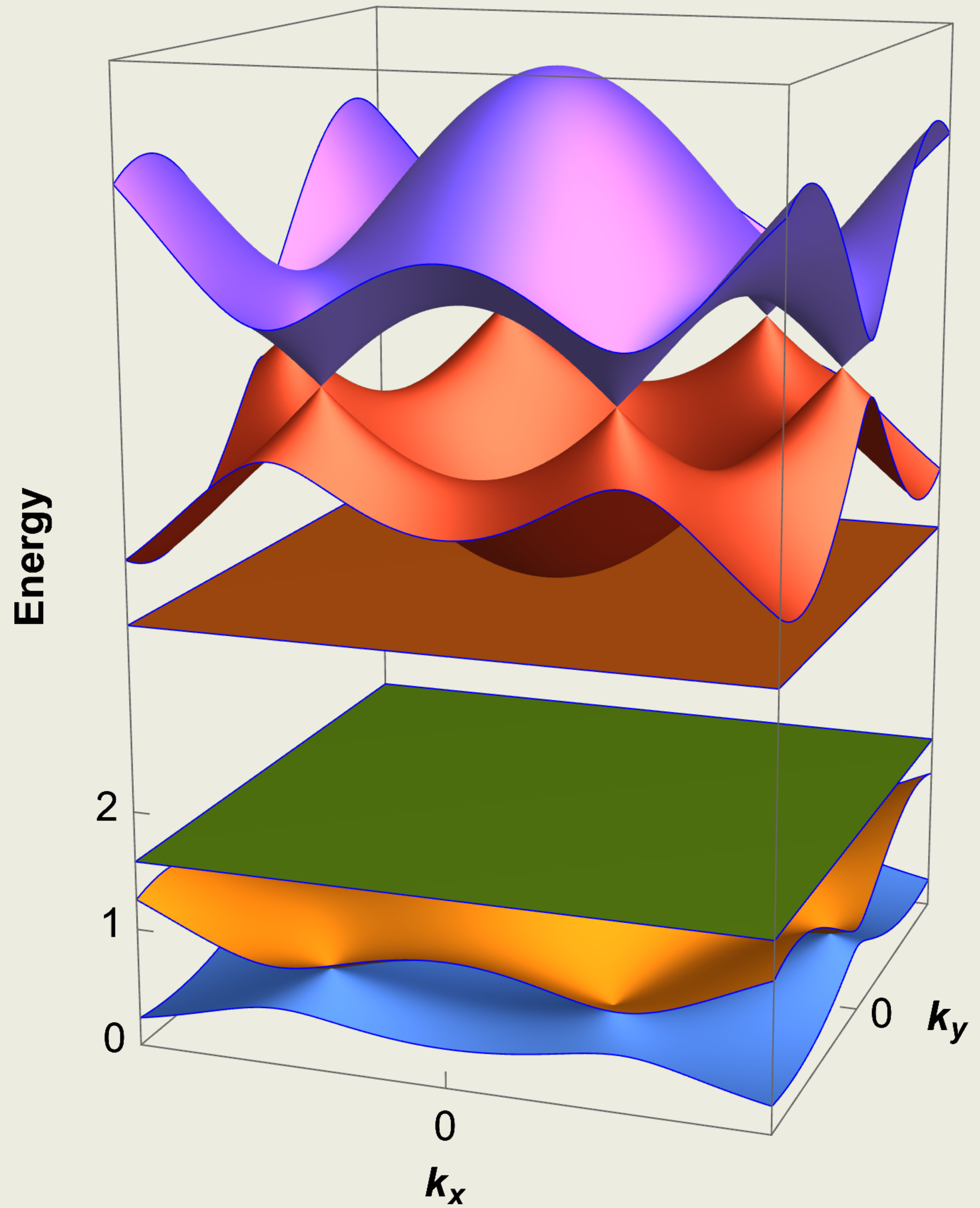

Honeycomb Q graph — a history

- Amovilli, Leys, March, J. Math. Chem. (2004), "Electronic Energy Spectrum of Two-Dimensional Solids and a Chain of C Atoms from a Quantum Network Model"

Solved the dispersive part only;

- Kuchment, Post, Com. Math. Phys. (2007), "On the spectra of carbon nano-structures"

Treated FB separately;
(Many names for FB: point spectrum, scar…)

- Pal, Spitz, Kindermann, PRL (2019), "Emergent Geometric Frustration and Flat Band in Moiré Bilayer Graphene"

Scattering matrix;

Dispersive + FB

Periodic Q graph: physics vs math

Introduces Bloch quasi-momentum $\mathbf{q}$

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$$

+ Vertex conditions

	Physics	**Math**
Potential V	V=0 or δ-potential	Any V(x)
Vertex conditions	S matrix	Neumann / Dirichlet
Dispersive bands	Secular equation of S	Matching equation
Flat bands	Secular equation of S	Constructive

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$$S(k,\mathbf{q})\,|\,\mathbf{q}\rangle = |\,\mathbf{q}\rangle$$

determines a relation $\mathbf{q}$ vs k;

spectrum is $E(k(\mathbf{q})) = k^2(\mathbf{q})$.

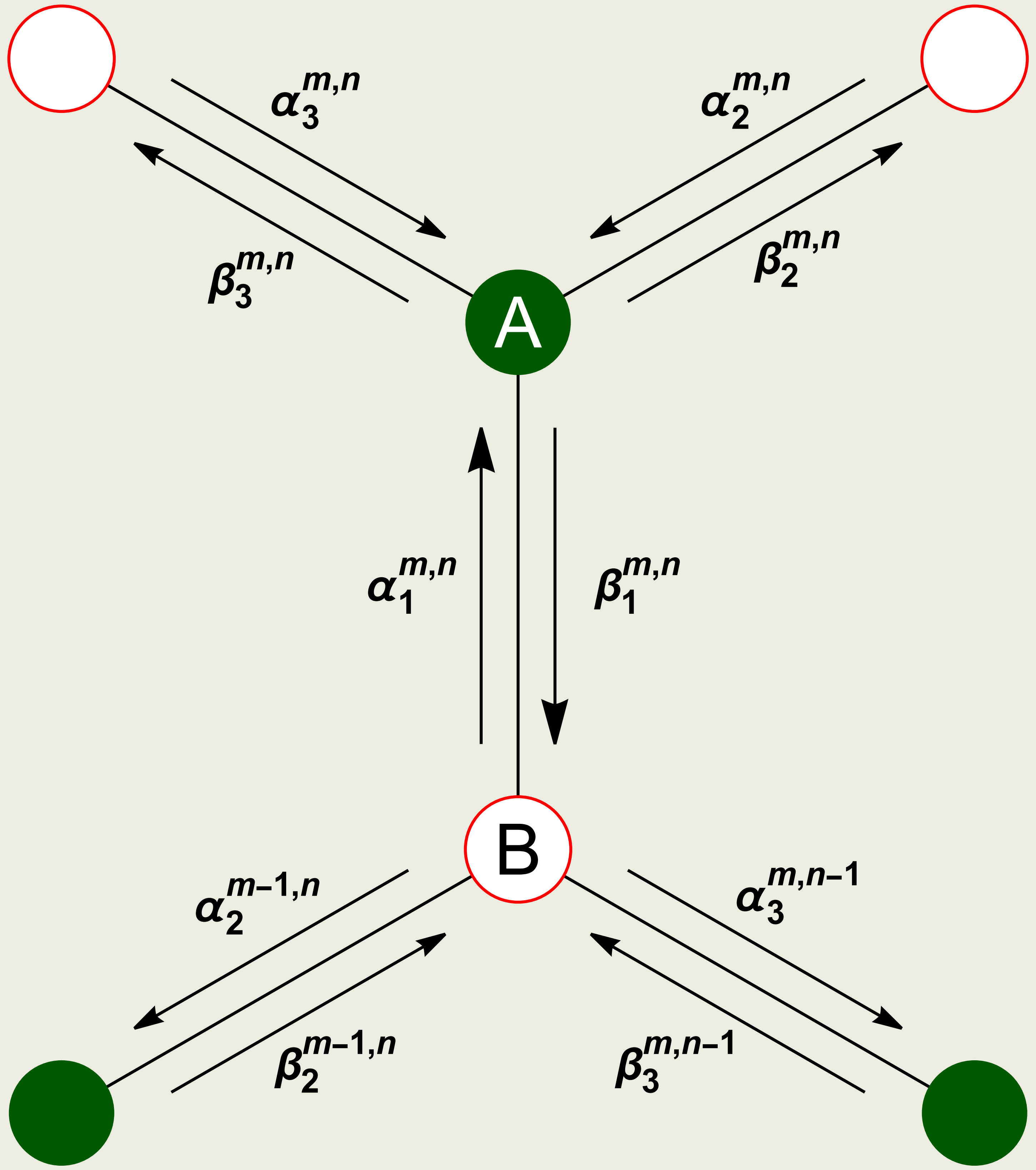
We will be using physics approach !

Q honeycomb: Scattering approach

Vertex condition given by S matrices at A and B :

$$S_{A/B} = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}$$

Symmetry enforced !



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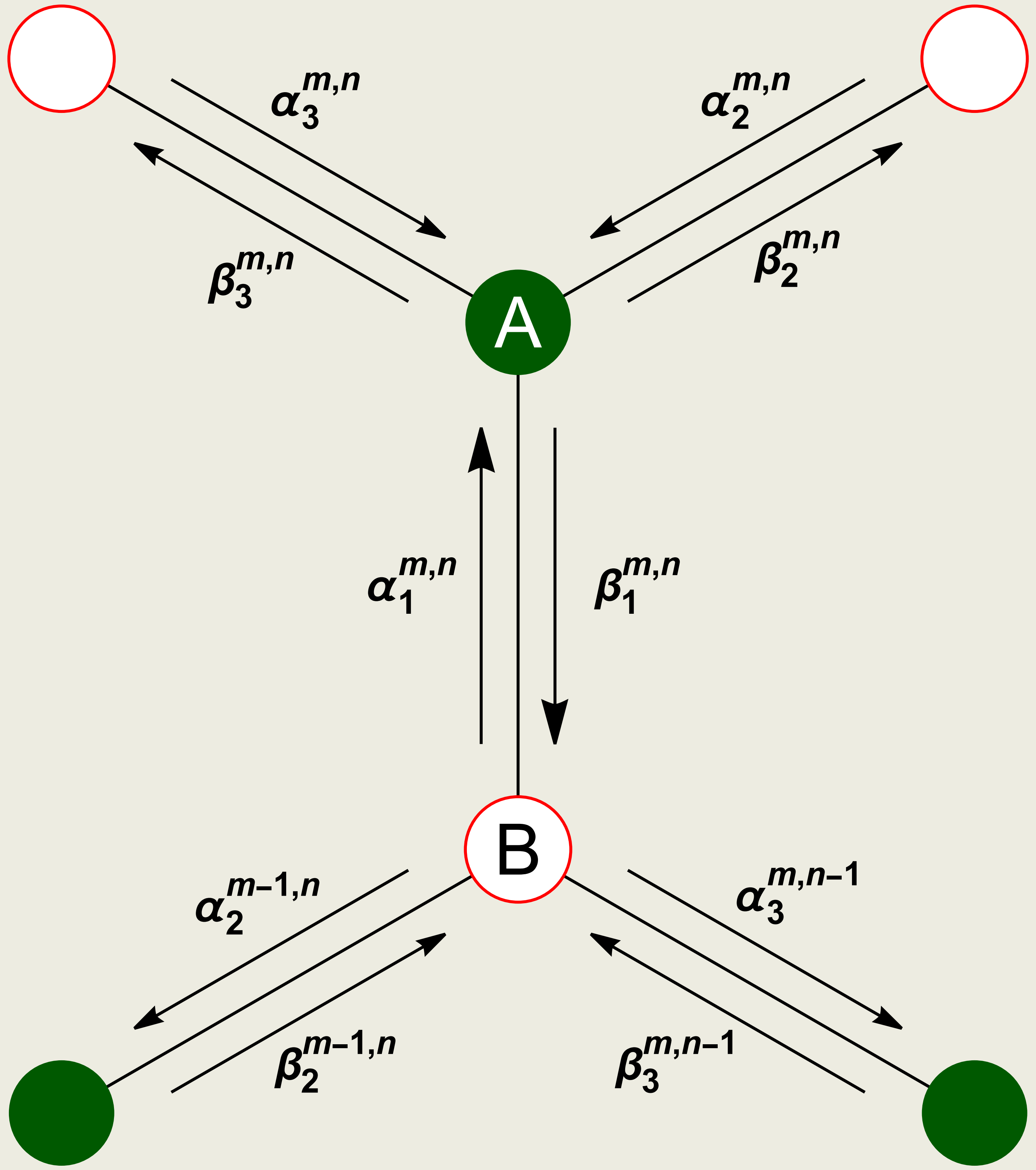


Bands obtained from secular equation for $S_{A/B}$:

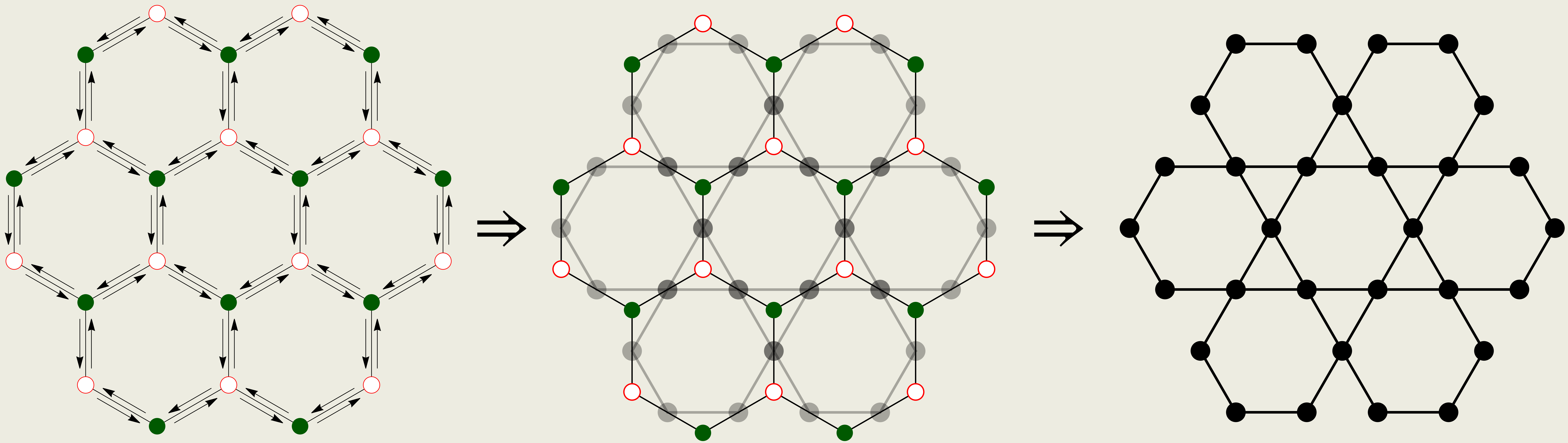
$$\begin{pmatrix} 0 & e^{2ika_0} S_A \\ D_{\mathbf{q}}^{\dagger} S_B D_{\mathbf{q}} & 0 \end{pmatrix} \begin{pmatrix} |\alpha_{\mathbf{q}}\rangle \\ |\beta_{\mathbf{q}}\rangle \end{pmatrix} = \begin{pmatrix} |\alpha_{\mathbf{q}}\rangle \\ |\beta_{\mathbf{q}}\rangle \end{pmatrix}$$

$$D_{\mathbf{q}} = \mathrm{Diag}(1, e^{i\mathbf{q}\cdot\mathbf{e}_1}, e^{i\mathbf{q}\cdot\mathbf{e}_2})$$

Implies a relation $k(\mathbf{q})$; can be solved analytically

FB in Q honeycomb — heuristic argument 1

Line graph transformation $\mathscr{L}(G)$:

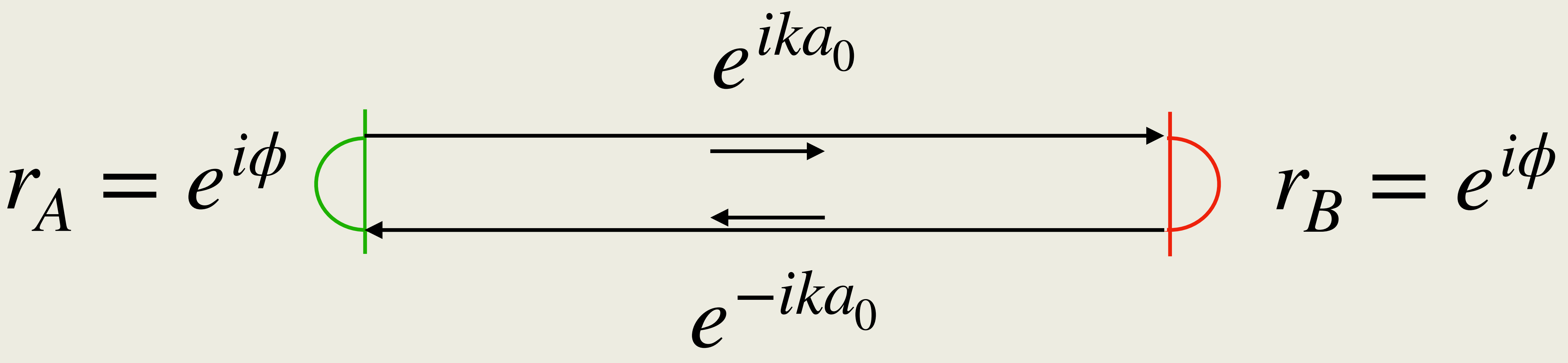


FB in Q(G) $\Leftarrow$ **FB in TB**$(\mathscr{L}(G))$

FB in Q honeycomb — heuristic argument 2

Bohr-Sommerfeld condition

Assume **perfect** **mirrors at all nodes**
(isolated/decoupled bond states) :



Consistency requires

$$2ka_0 + 2\phi \in 2\pi\mathbb{Z}$$

FB in Q honeycomb — heuristic argument 2

Bohr-Sommerfeld condition **carries through to honeycomb:**

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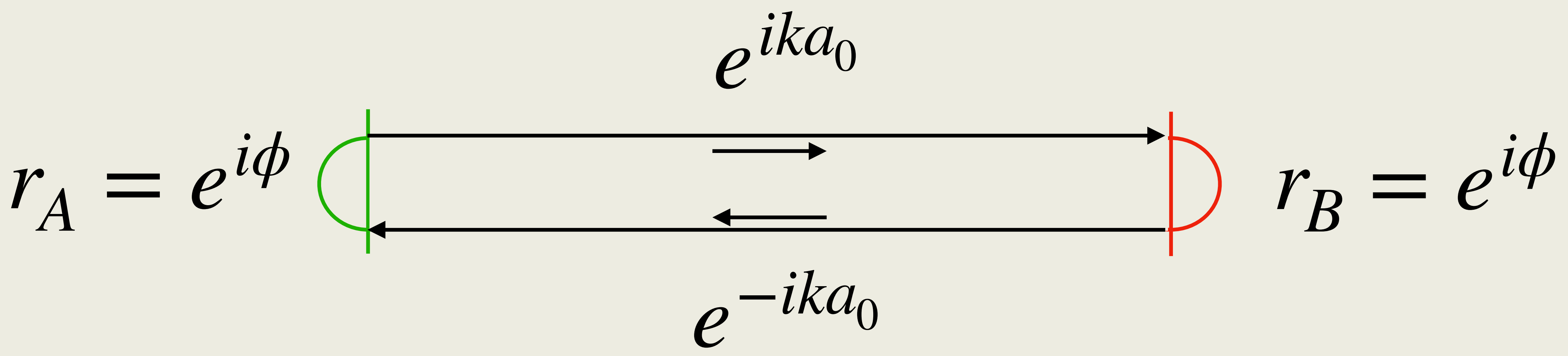

Consistency requires

$$2ka_0 + 2\phi \in 2\pi\mathbb{Z}$$

$$e^{2ika_0} S_A D_{\mathbf{q}}^{\dagger} S_B D_{\mathbf{q}} |\alpha_{\mathbf{q}}\rangle = |\alpha_{\mathbf{q}}\rangle$$

$$S_{A/B} = e^{i\phi} e^{i\theta |u\rangle\langle u|}, \quad |u\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

There's one eigenvector satisfying

$$|\alpha_{\mathbf{q}}\rangle \perp |u\rangle \qquad |\alpha_{\mathbf{q}}\rangle \perp D_{\mathbf{q}}^{\dagger} |u\rangle$$

which reduces the secular eq. to

$$e^{i(2ka_0 + 2\phi)} |\alpha_{\mathbf{q}}\rangle = |\alpha_{\mathbf{q}}\rangle$$

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Our work

Symmetry of Q honeycomb

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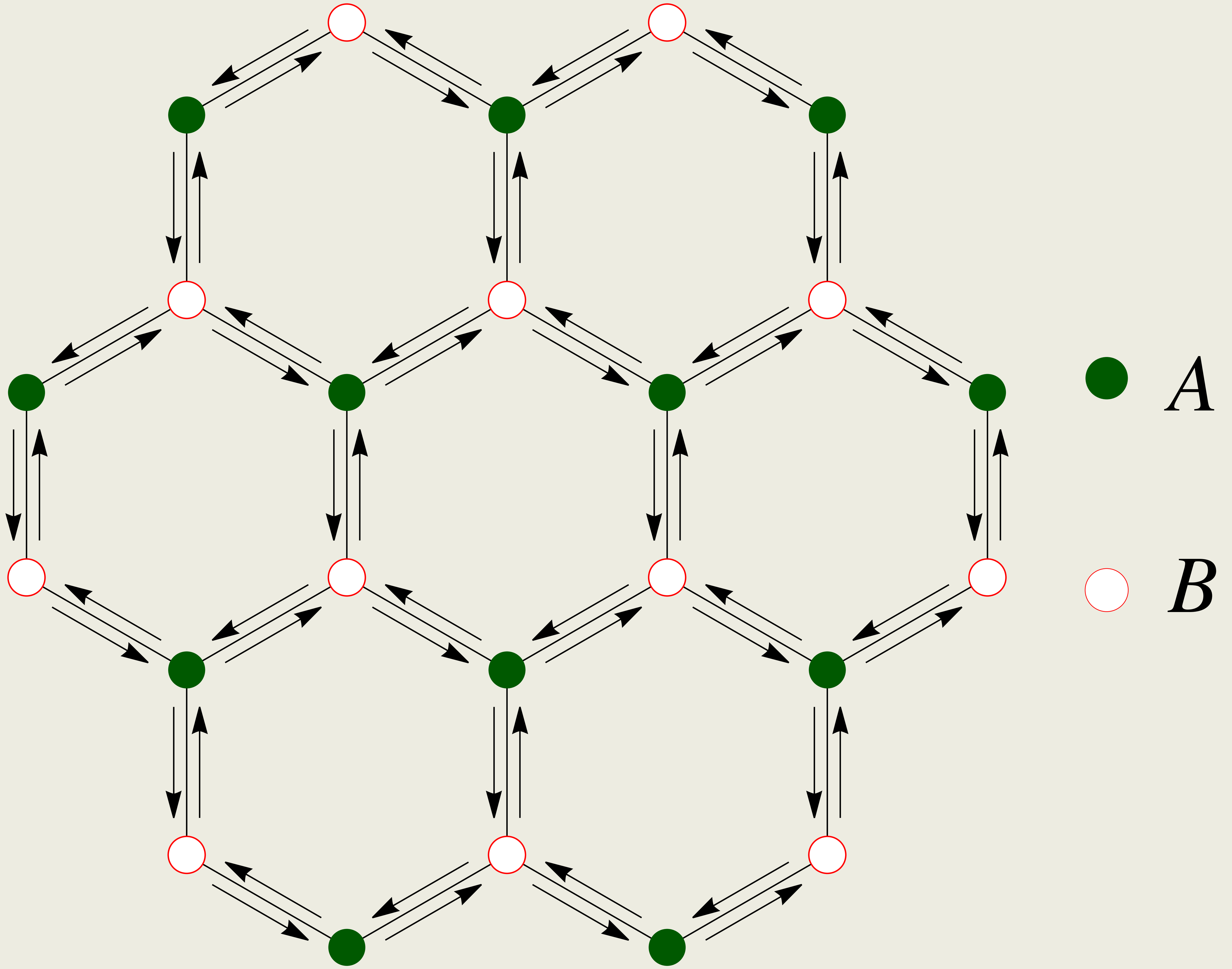

- Translation
- Sublattice exchange $\mathscr{M}$ ($A \leftrightarrow B$)
- Onsite symmetry (dihedral D_3)

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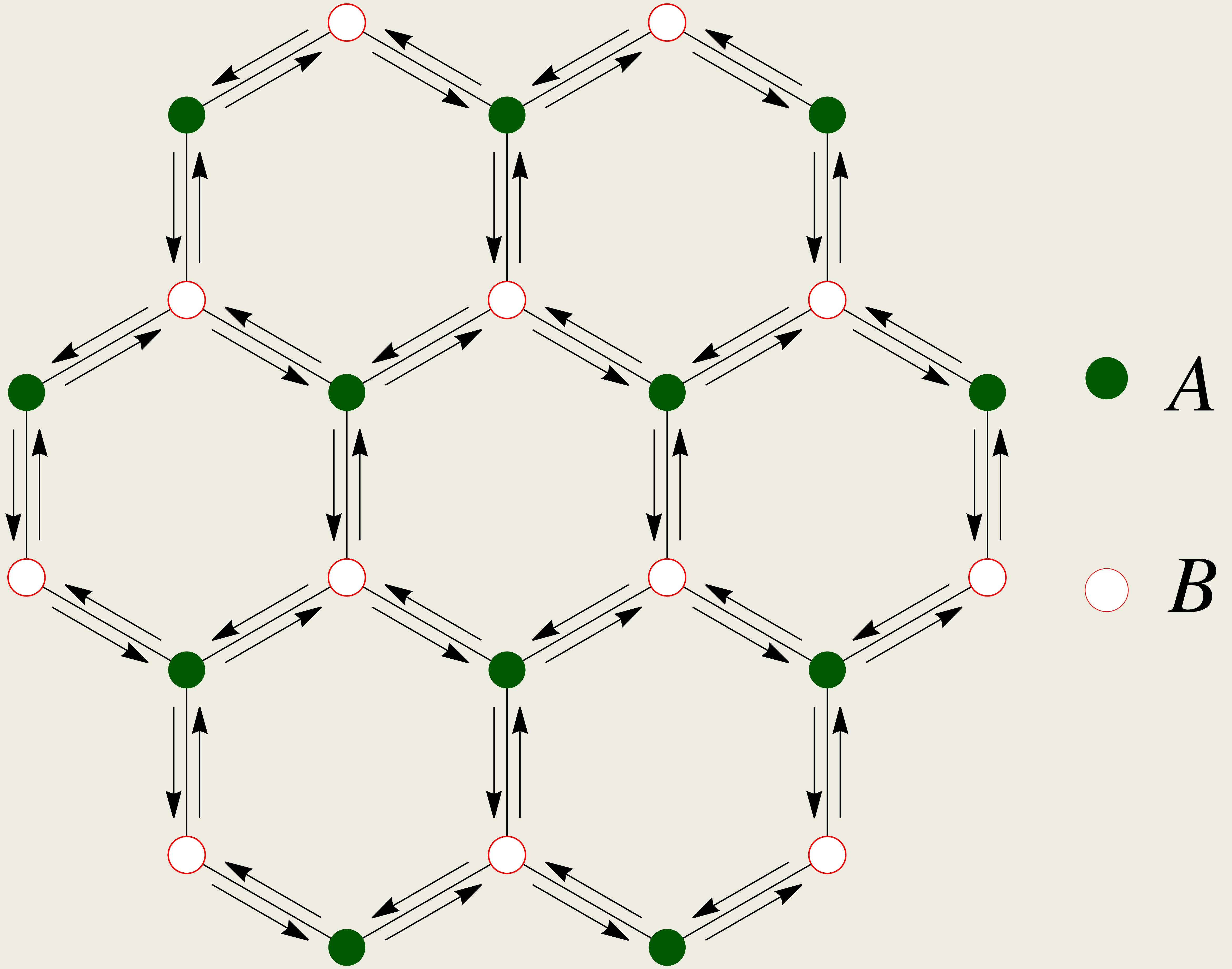


- Translation
- Sublattice exchange $\mathscr{M}$ ($A \leftrightarrow B$)
- Onsite symmetry (dihedral D_3)

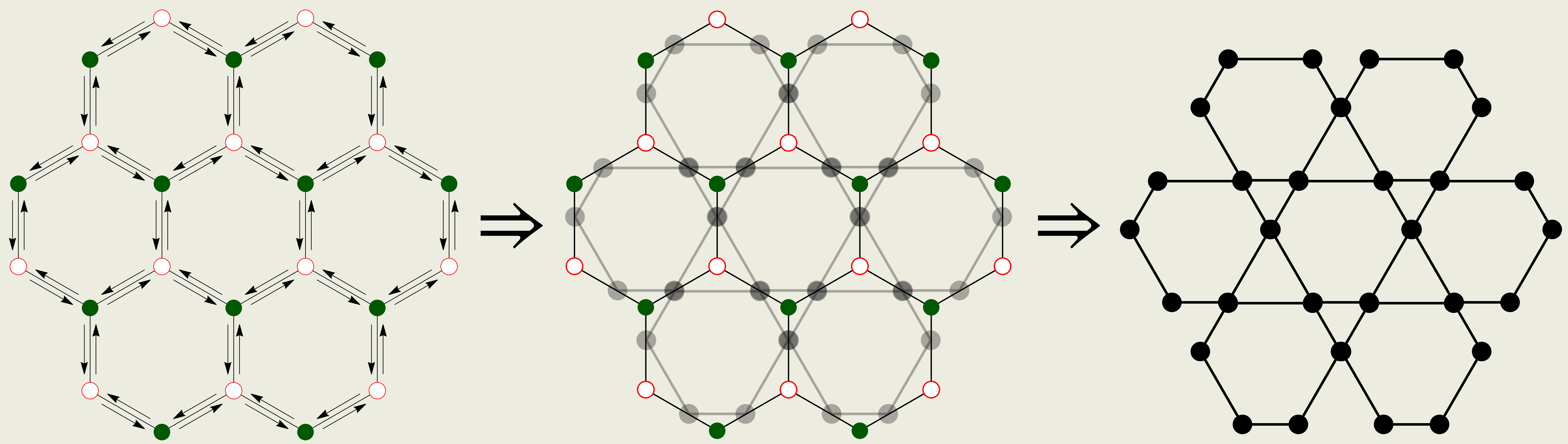
FBs persist even if $\mathscr{M}$ is broken!

FBs in the Q breathing honeycomb

Break sublattice exchange $\mathscr{M}$ ($A \leftrightarrow B$)

$$S_A \neq S_B$$

$$S_s = e^{i\phi_s} e^{i\theta_s |u\rangle\langle u|}, \quad s = A, B$$



"Breathing honeycomb"

Breathing kagome

FBs in the Q breathing honeycomb

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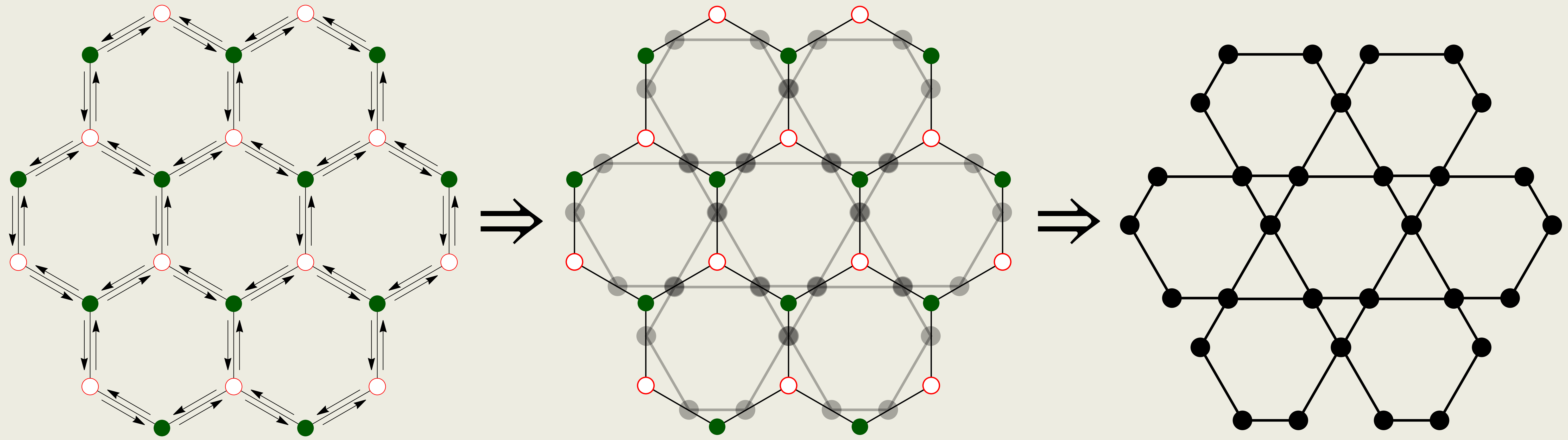
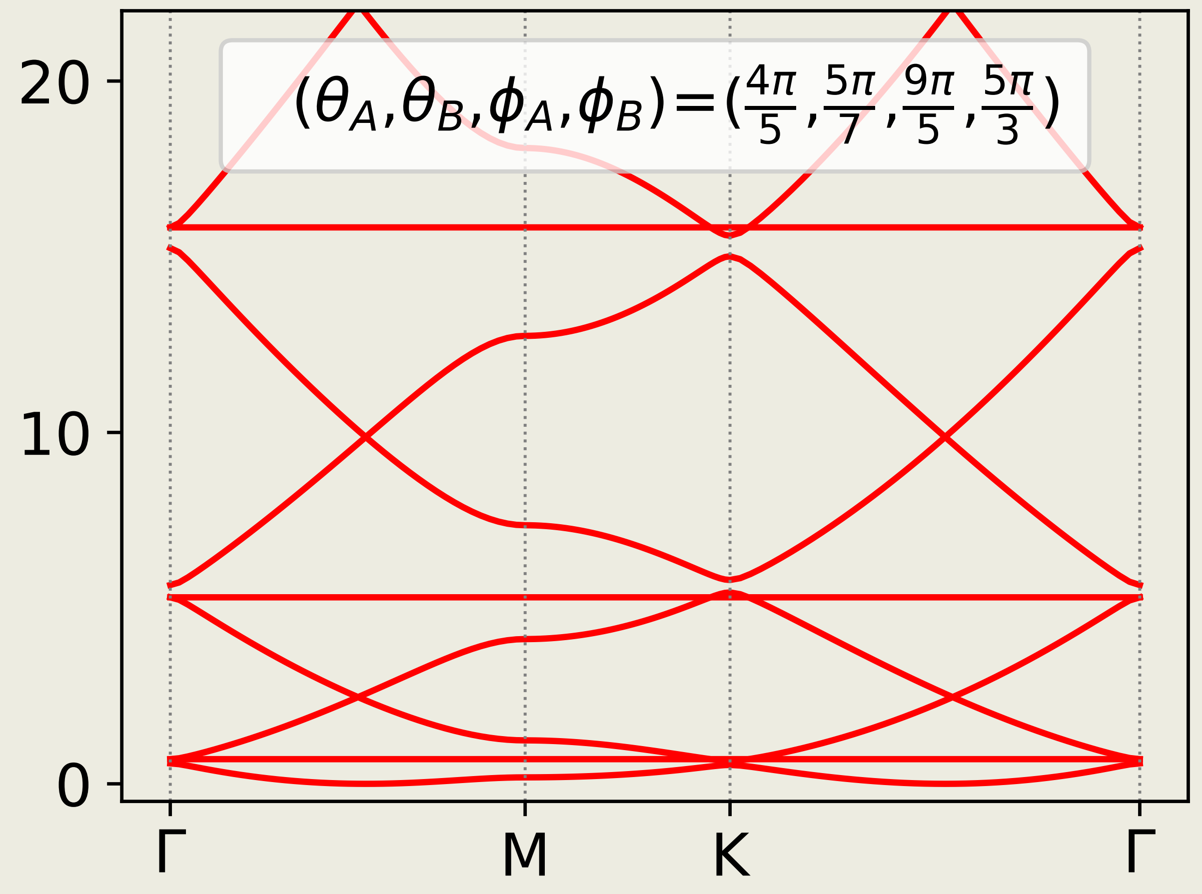
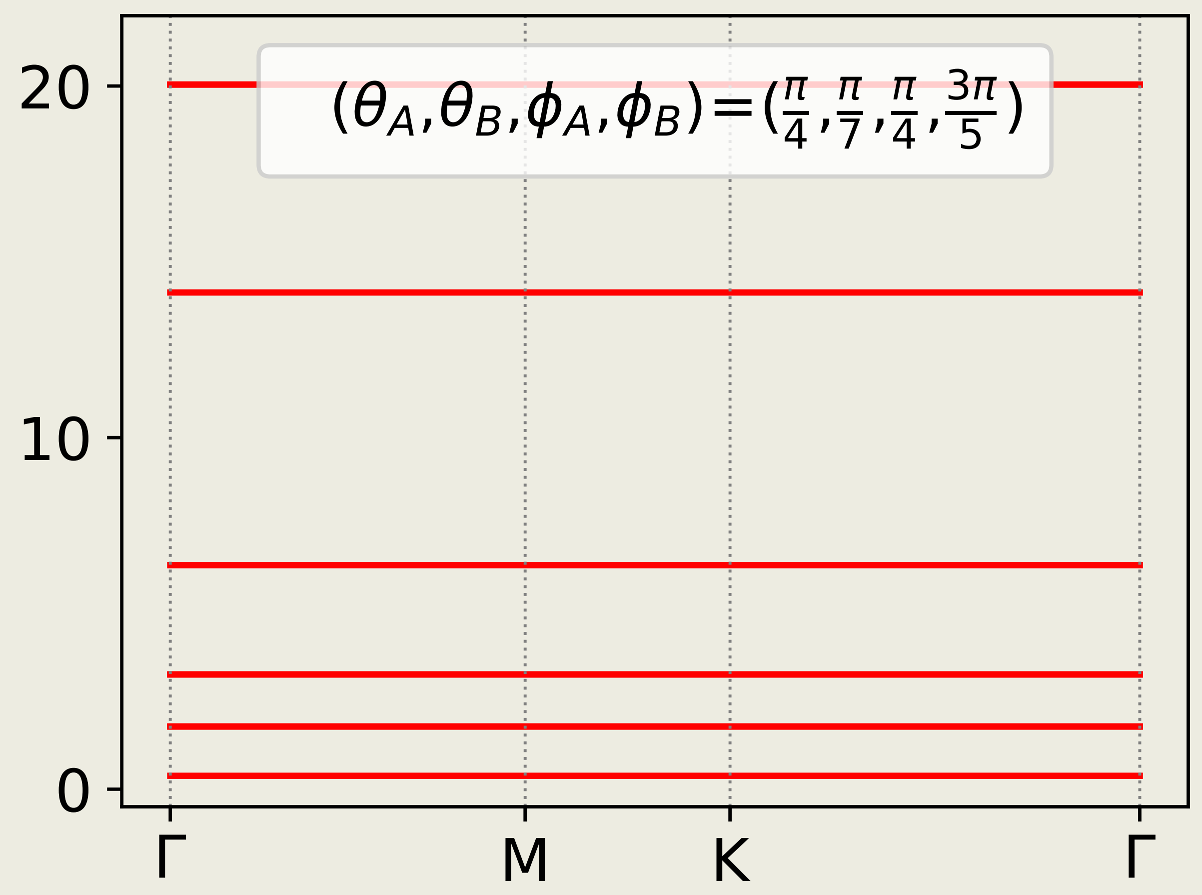


“Breathing honeycomb” Breathing kagome



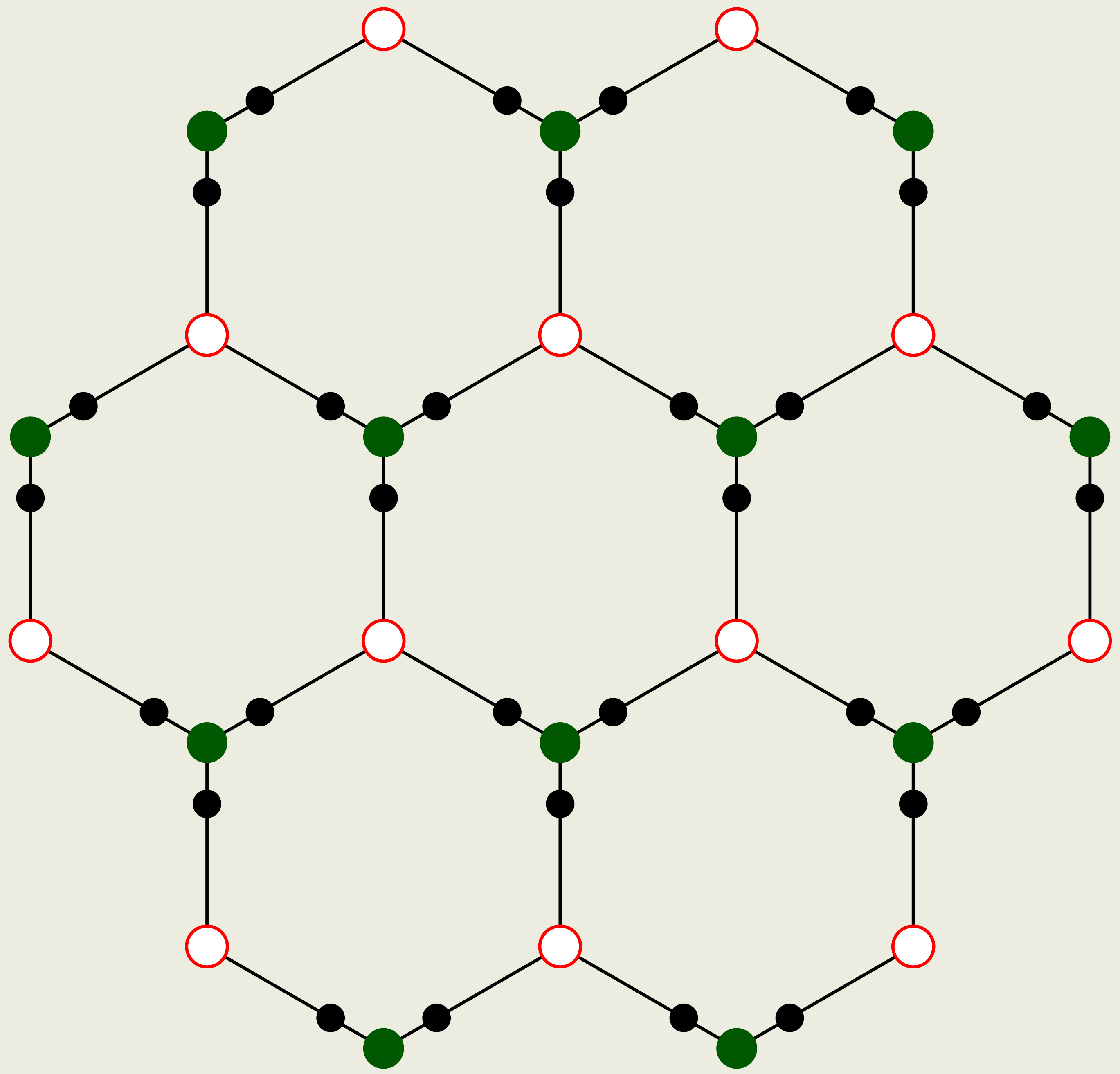


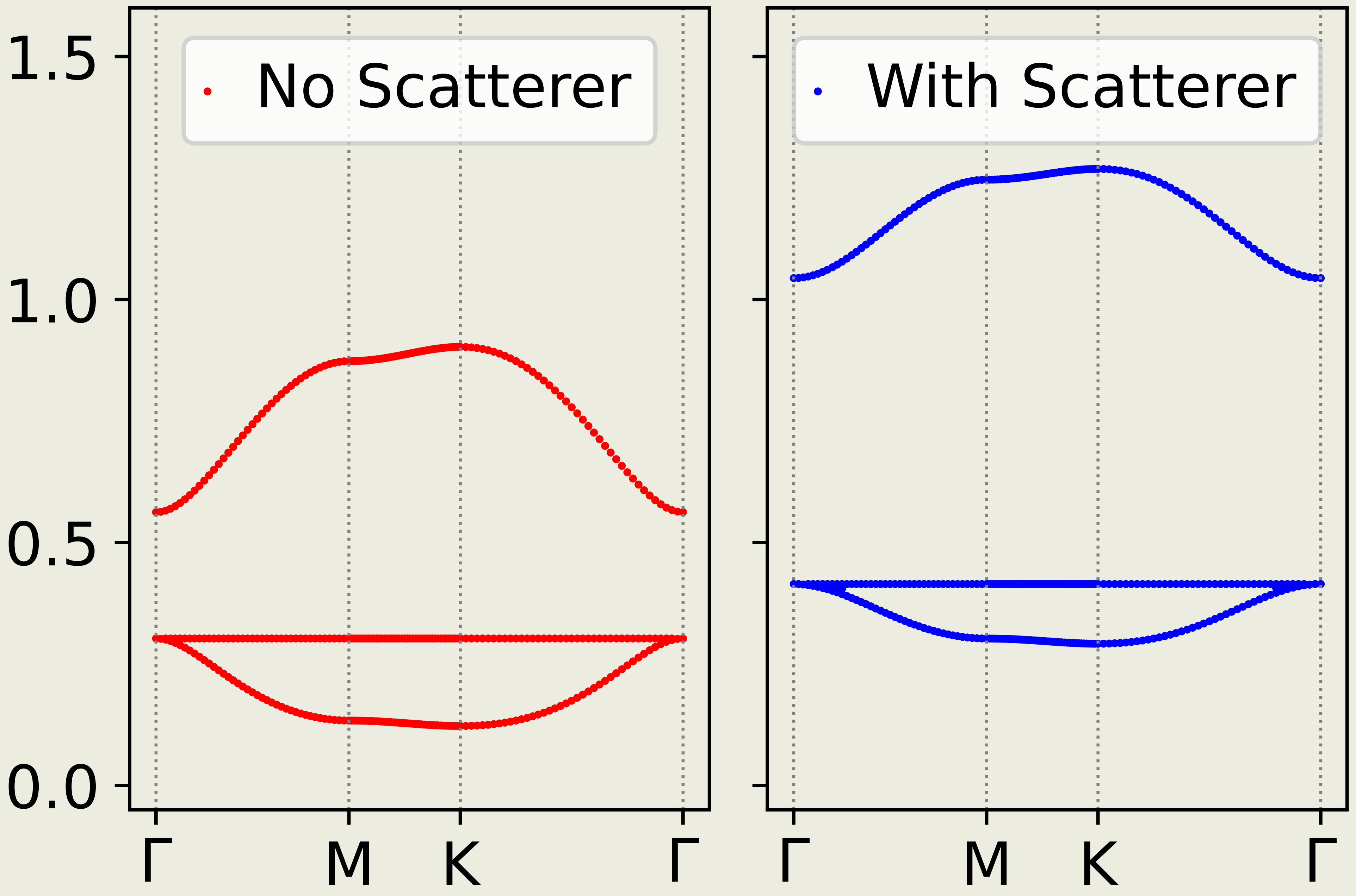
Accidental FB

Breathing by artificial decoration

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$$

$$V(x) = V_0\delta(x - d)$$





$$S_{\bullet} = e^{i\phi}\begin{pmatrix} e^{i\delta}r & \sqrt{1-r^2} \\ \sqrt{1-r^2} & -e^{-i\delta}r \end{pmatrix}$$

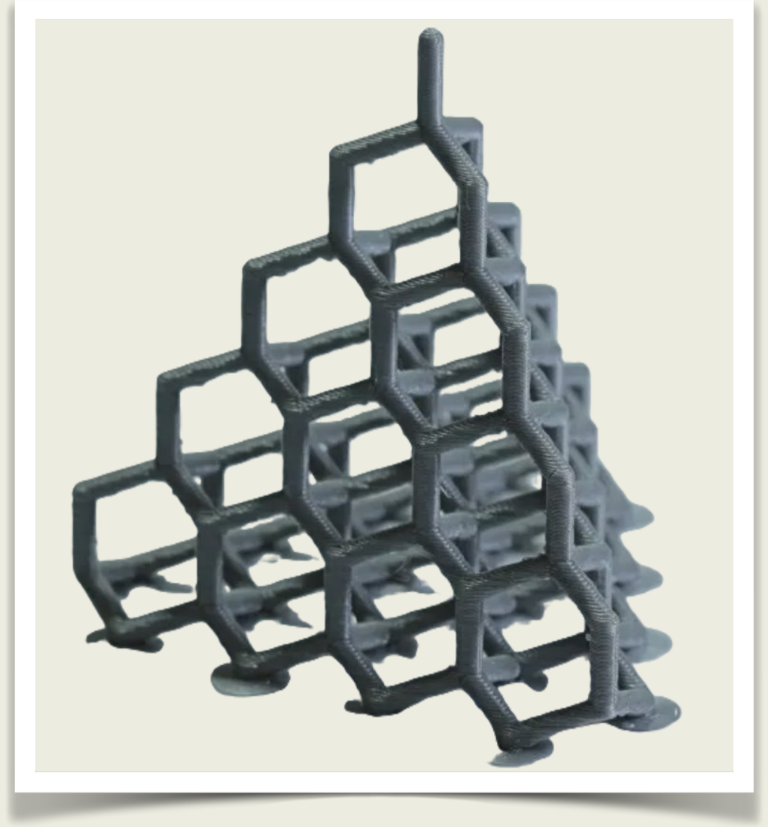
$$(d, \phi, \delta, r) = \left(0.3, \frac{\pi}{20}, \frac{\pi}{7}, 0.4\right)$$

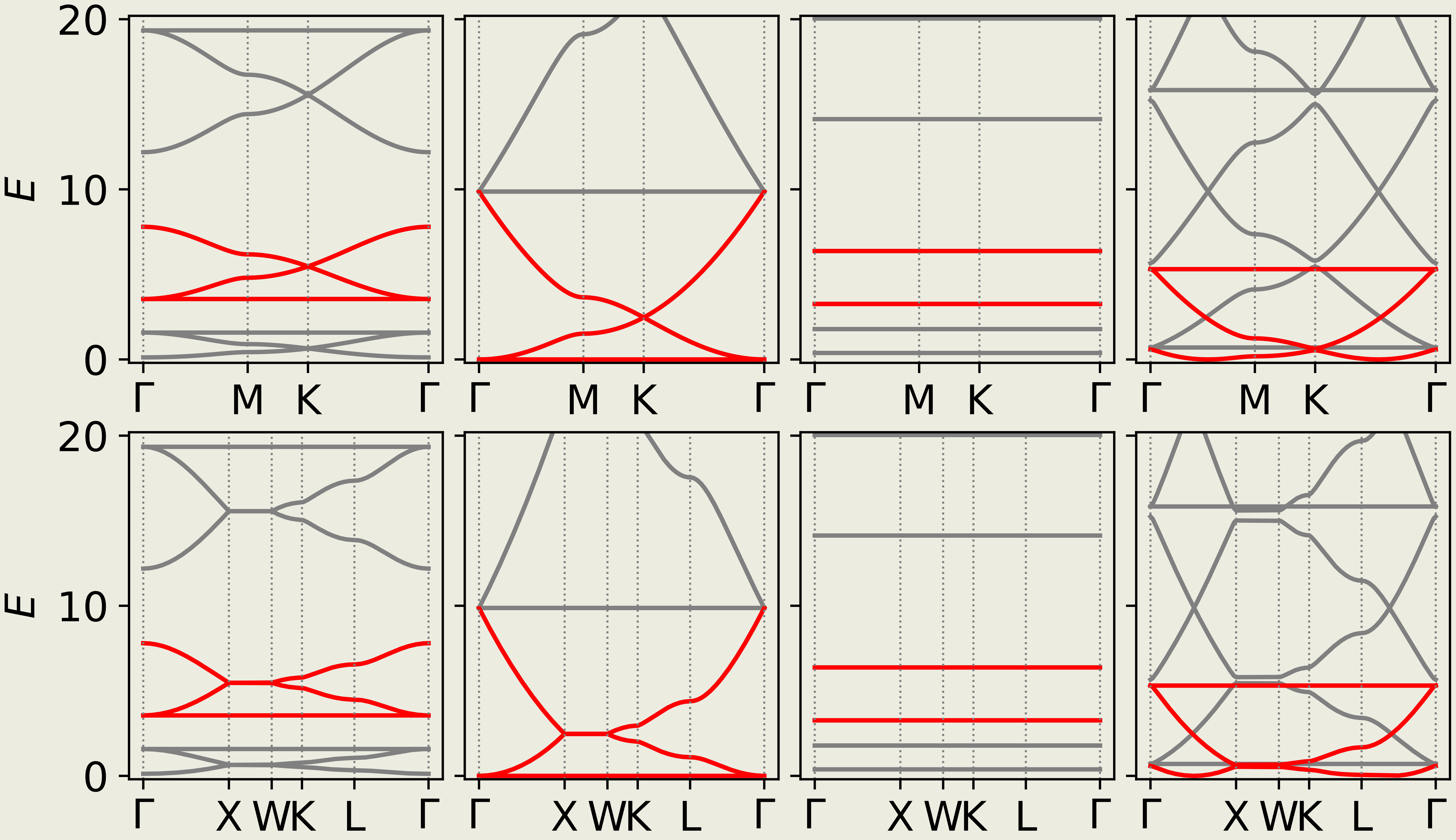
FBs: 2D honeycomb vs 3D diamond

$$(\theta_A, \theta_B, \phi_A, \phi_B) = \left(\frac{\pi}{9}, \frac{\pi}{9}, \frac{2\pi}{5}, \frac{2\pi}{5}\right) \quad (0, 0, \pi, \pi) \quad \left(\frac{\pi}{4}, \frac{\pi}{7}, \frac{\pi}{4}, \frac{3\pi}{5}\right) \quad \left(\frac{4\pi}{5}, \frac{5\pi}{7}, \frac{9\pi}{5}, \frac{5\pi}{3}\right)$$

2D Honeycomb

3D Diamond





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Our work

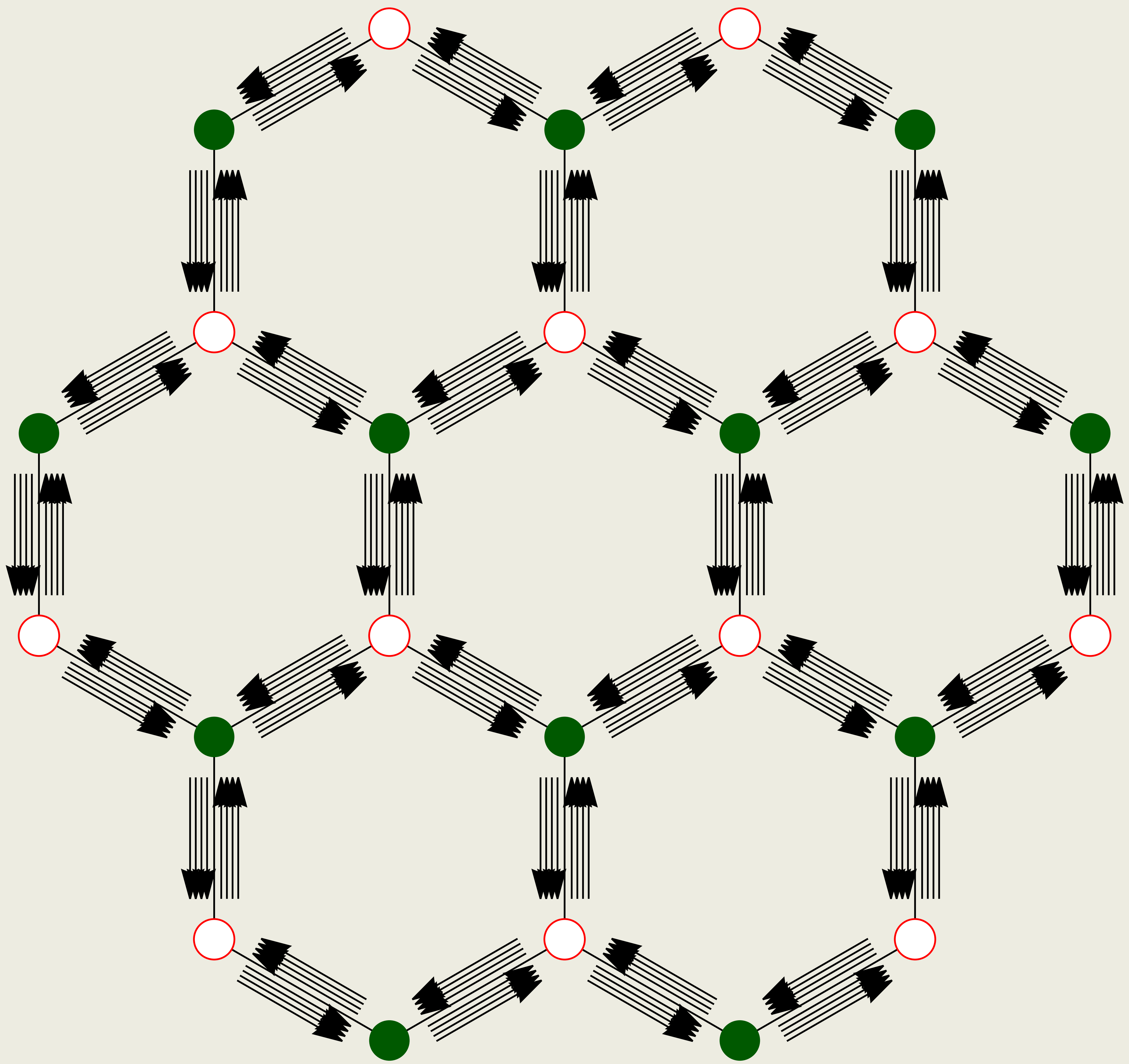
Towards realistic wires

Experimental implementation using nano fabrication techniques

- Metallic wires : different elements
- Semiconducting wires

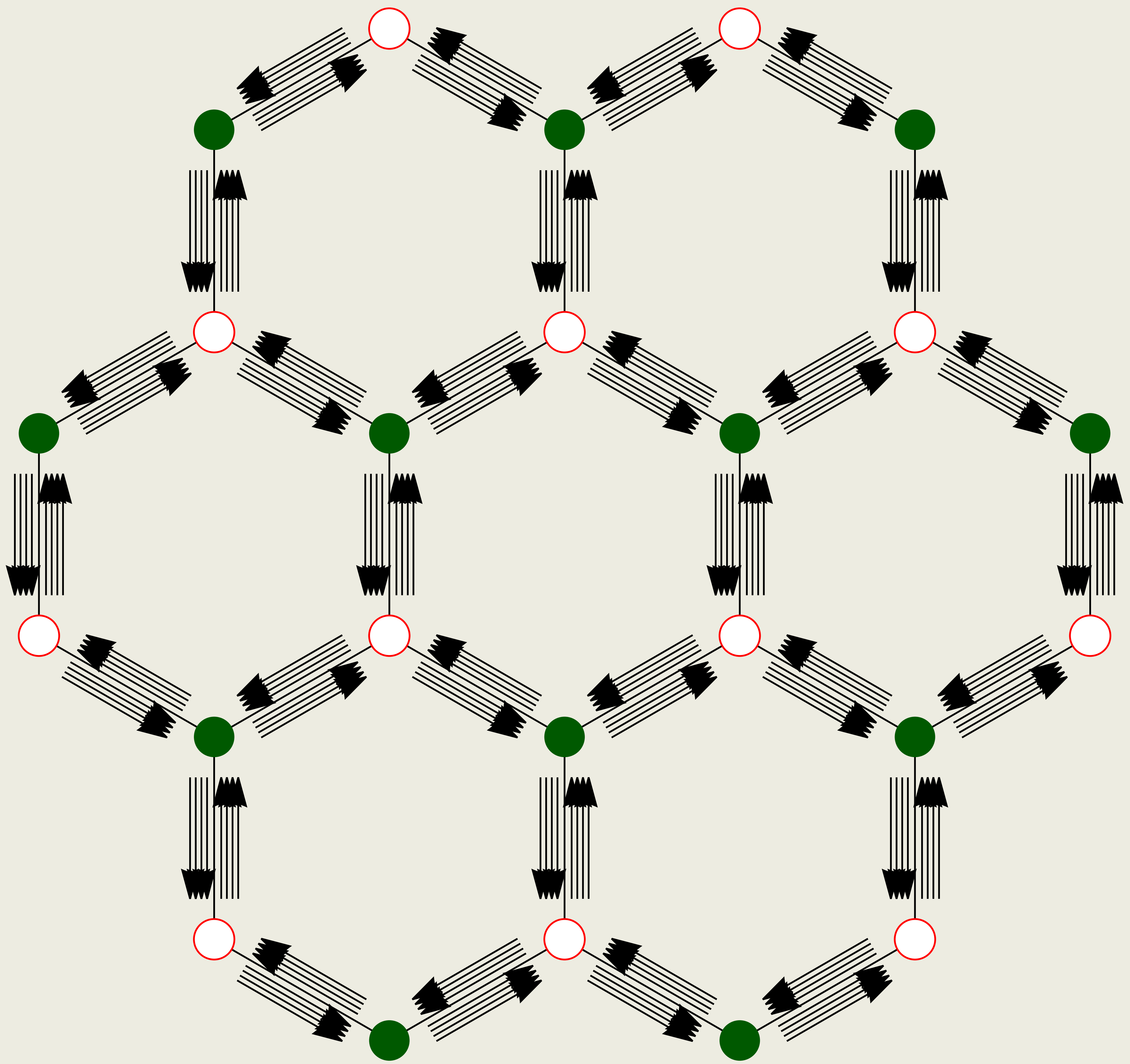
Collaboration with
Cécile Naud & Florence Levy-Bertrand
(Néel Institute Grenoble)

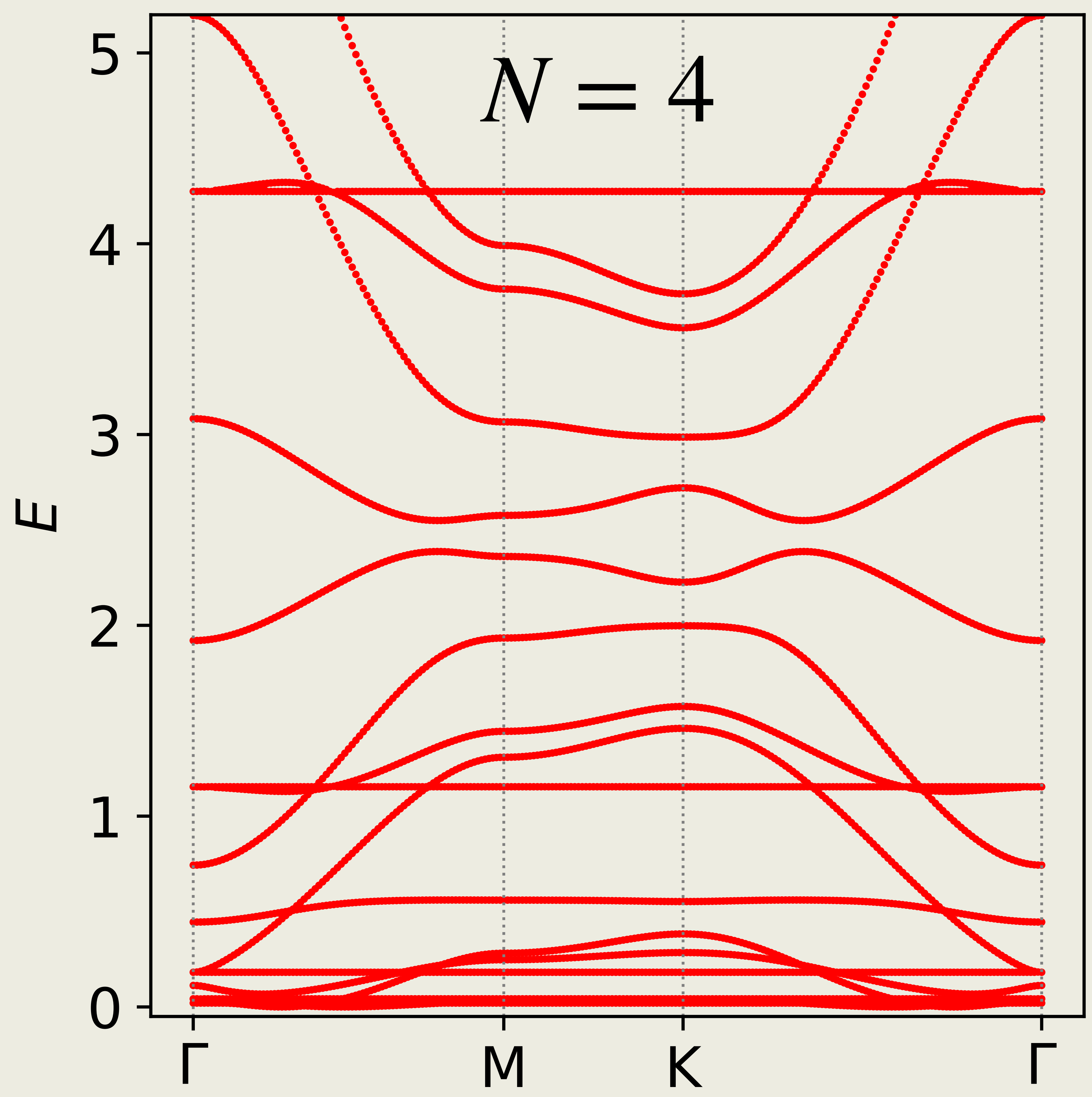
FBs in the N-channel Q honeycomb



$N = 4$

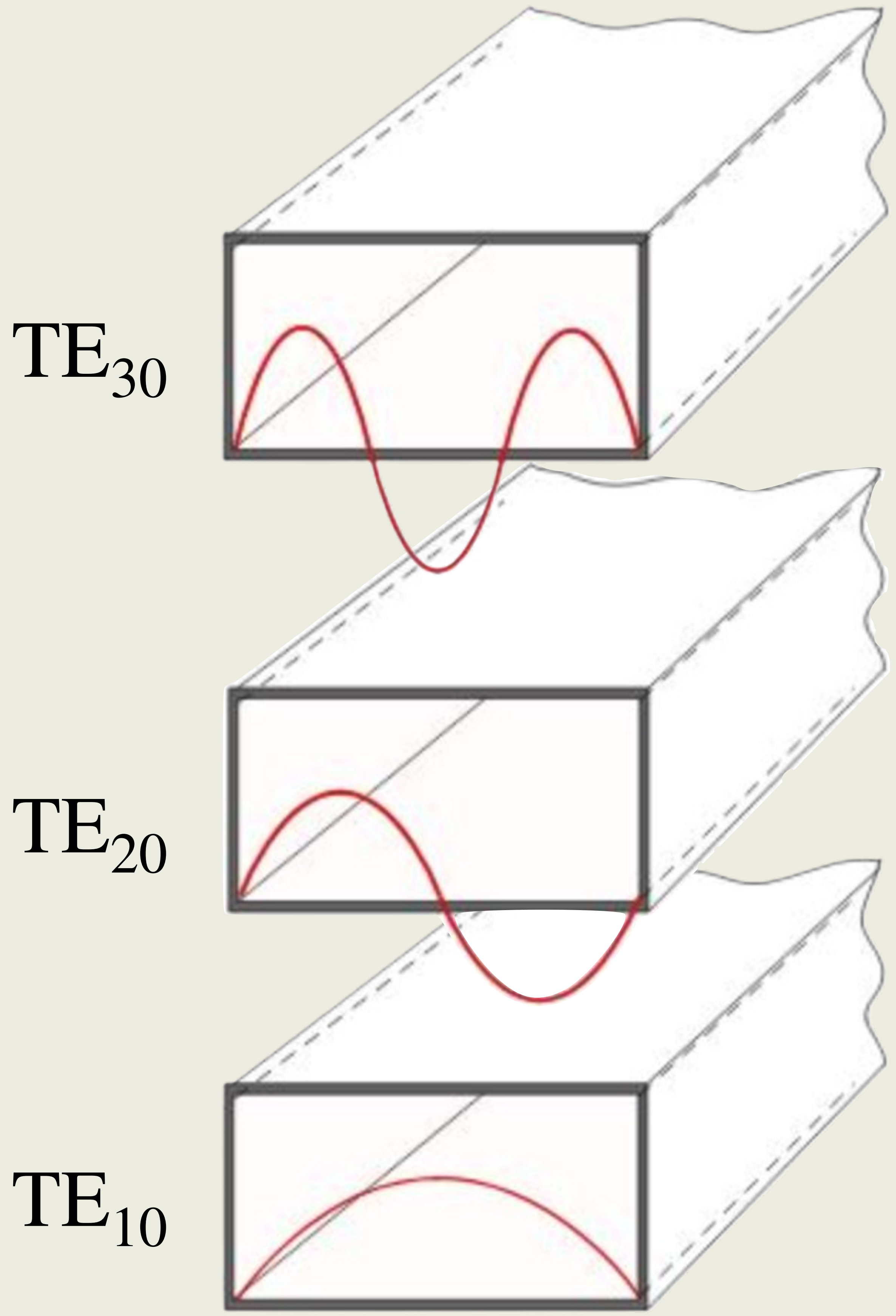
FBs in the N-channel Q honeycomb

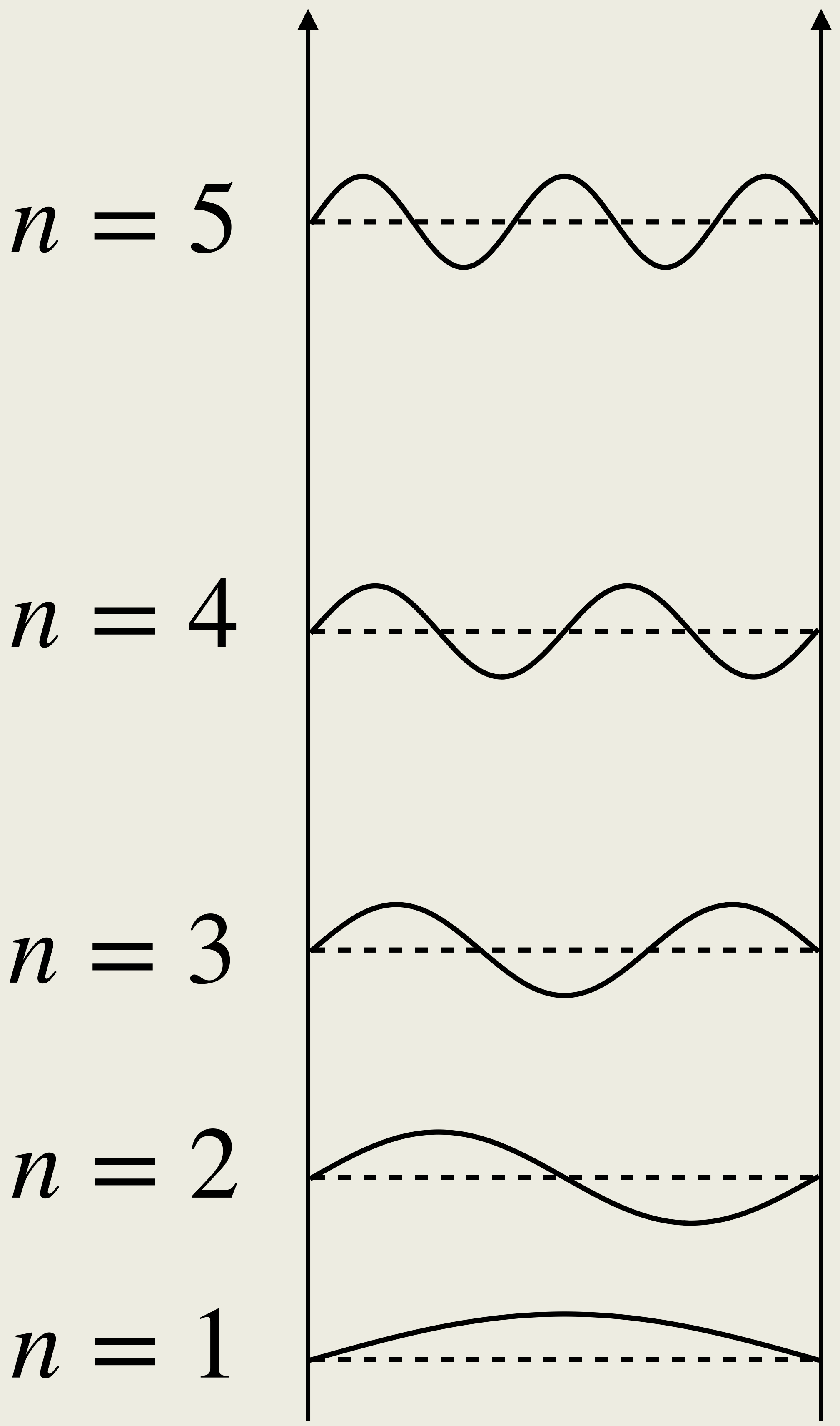




FBs persist upon adding more channels !

Channel parity





$N = N_+ + N_-$: Total # of channels

N_+: # of parity **even** channels

N_-: # of parity **odd** channels

Symmetry enforced S matrix

Scattering matrices at nodes :

$$S_{A,B} \in U(3N)/D_3$$

- Translation symmetry
- Onsite symmetry (dihedral D_3)
- Other symmetries **NOT** enforced

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Scattering matrices at nodes :

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Explicit parametrization (some group theory analysis involved) :

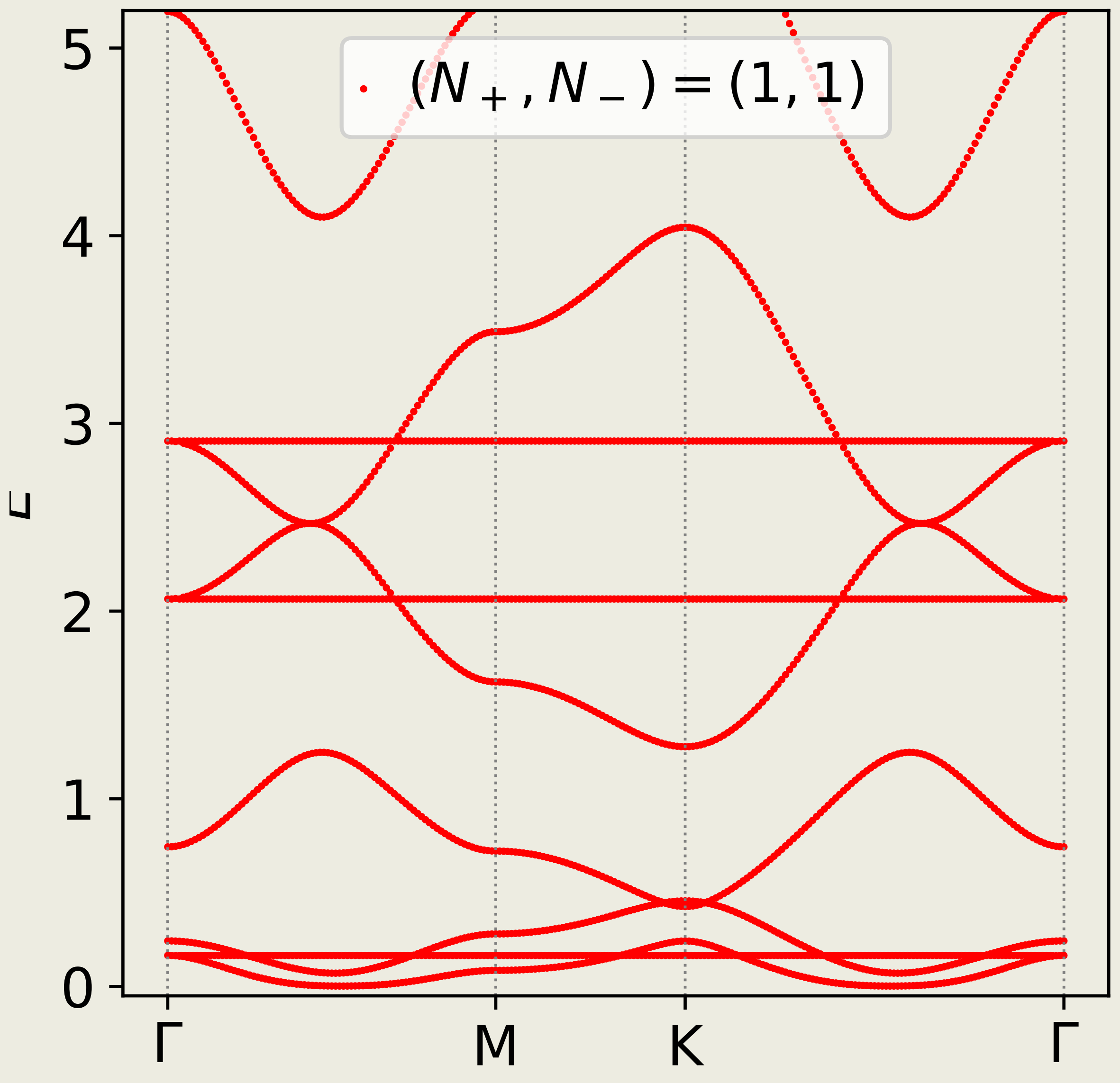
$$S_{A,B} = S_0 \otimes (\mathbb{I}_3 - P_u) + \mathrm{diag}(S_+, S_-) \otimes P_u$$

Channel : $S_0 \in U(N), \quad S_\pm \in U(N_\pm),$

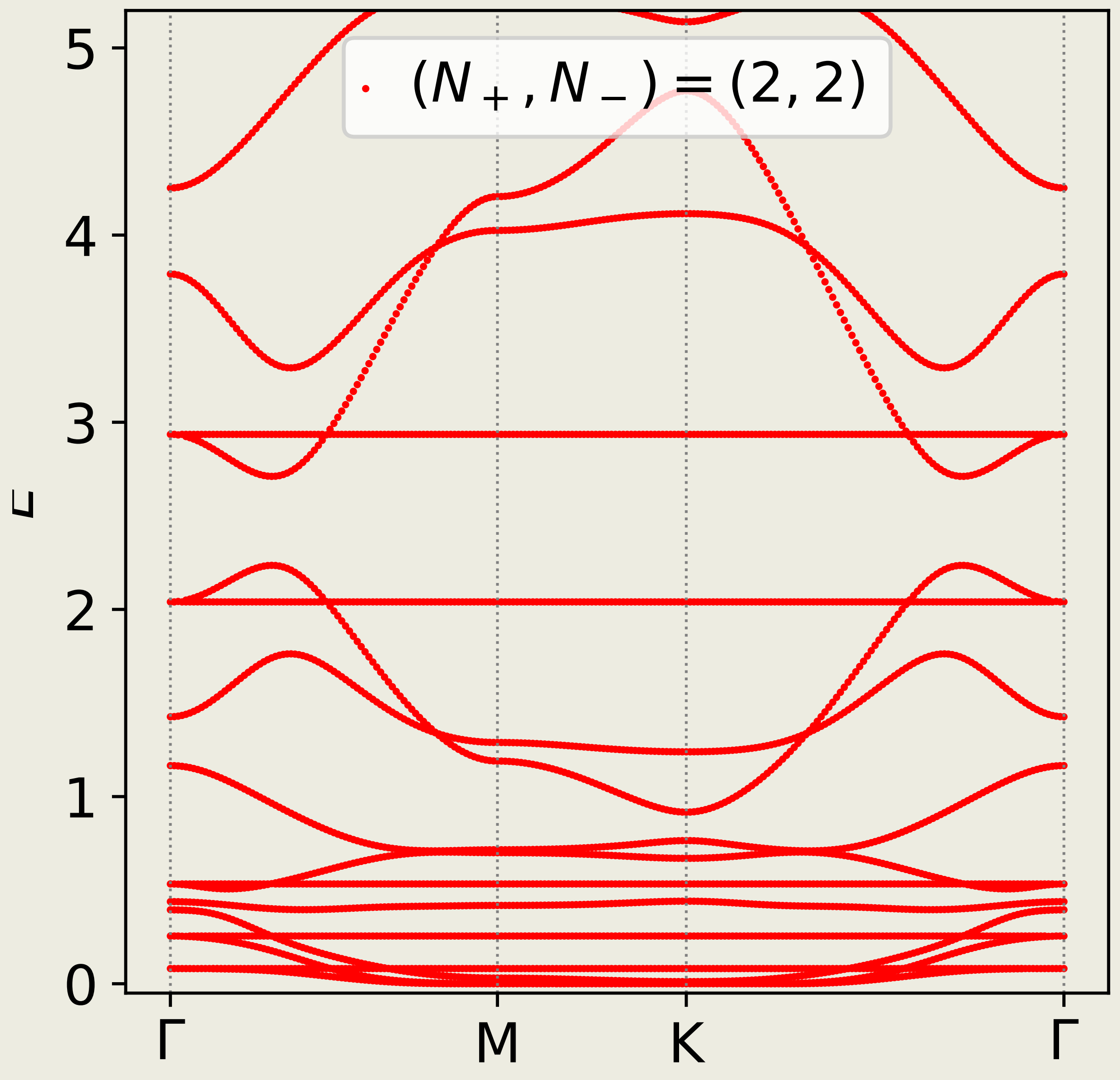
Leg : $P_u = |u\rangle\langle u|, \quad |u\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

Similar symmetry analysis exists for the multi-channel 3D diamond.

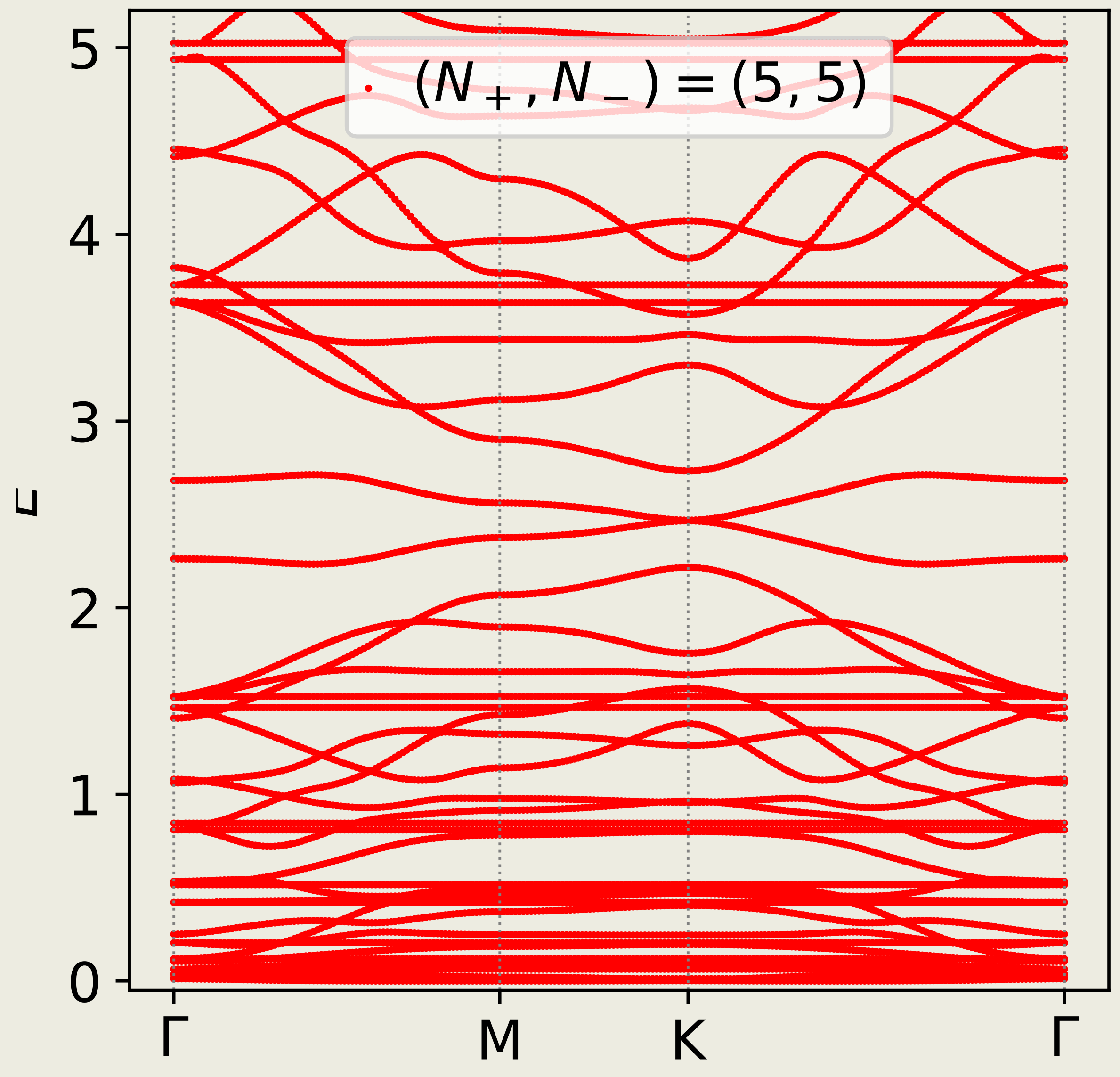
FBs in the multiband Q honeycomb



$(N_+, N_-) = (1,1)$



$(N_+, N_-) = (2,2)$



$(N_+, N_-) = (5,5)$

1/3 of bands are flat — for arbitrary N_+ and N_- !

Conclusion

- (Periodic) **Q graph**: rich playground for physics and math ideas
- **Q honeycomb** is very special : we found (and were surprised by) **robust FBs**, which persist to the **multichannel** case!
- Applicable to many physical systems (**quantum and classical**) : photonic wave guides, arrays of cavities, classical waveguides, etc…

Perspectives

- **Transport** theory on Q graph with FB
- **Topology** of FB?
- **Interaction** effects in quantum graph with FB: Luttinger liquid, superconductivity…
- **Disorder** on FB (especially the multichannel case)

Akkermans, Comtet, Desbois, Montambaux, Gand, Texier, Ann. Phys. '00

References by Christophe Texier in '08, '10, '12